ON THRESHOLD AMPLITUDES III: $2 \to n$ PROCESSES

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The $2 \to n$ scattering with final particles at rest is discussed. The comparison with purely soft processes allows to identify symmetries responsible for vanishing of certain $2 \to n$ amplitudes. Some examples are given.

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1. Introduction

In the present paper, third of the series (*cf.* [1, 2]), we consider $2 \to n$ processes: two particles of opposite momenta scatter to produce n bosons at rest. This process has been already considered, on the tree level, using propagator approach [3–6], diagrammatics [7, 8] or Feynman wave function method [1, 9]. Last method seems to be the most efficient. In Sec. 2 we present a very simple and straightforward derivation of the Brown–Zhai [9] algorithm. We give a general procedure to construct $2 \to n$ amplitudes which, in particular, allows us, for a wide class of theories, to compare $2 \to n$ amplitude with a purely soft one. By using the results concerning the relations between symmetries and vanishing of purely soft amplitudes [2, 10] we are able to identify, in certain cases, the symmetries underlying nullification in $2 \to n$ scattering. In Sec. 3 we discuss some examples involving only bosons while Sec. 4 is devoted to fermion–antifermion scattering. The latter is discussed in more detail and the complete expression for the amplitude is given and compared, for $n = 2$, with diagrammatic result. Sec. 5 contains some conclusions.

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2. Calculating $2 \to n$ amplitudes

Let us first remind the general procedure for calculating the matrix elements of field operators in the tree approximation [1,9]. Our starting point is a Lagrangian

$$L = \frac{1}{2}\sum_{i=1}^{N}(\partial_\mu \Phi_i \partial^\mu \Phi_i - m_i^2 \Phi_i^2) - V(\underline{\Phi})\,, \tag{1}$$

where $\{\Phi_i\}_{i=1}^N$ is a multiplet of scalar fields; higher-spin fields can be also included.

Let $\{f_n^{(i)}\}_{n=0}^\infty$ be a complete set of normalized positive-energy solutions to the Klein–Gordon equation of mass m_i; in what follows we shall use momentum basis, $f_{\vec{p}}^{(i)}(x) = \frac{1}{\sqrt{(2\pi)^3 2E_p}}\mathrm{e}^{-ipx}$, $p^2 = m_i^2$. Define the classical free field by

$$\Phi_{0i}(x) = \sum_n \left(\beta_n^{(i)} f_n^{(i)}(x) + \bar{\beta}_n^{(i)}\overline{f_n^{(i)}(x)}\right). \tag{2}$$

Consider the set of integral equations for $\Phi_i(x)$,

$$\Phi_i(x) = \Phi_{0i}(x) - \int d^4y \Delta_{F_{ij}}(x-y)\frac{\partial V}{\partial \Phi_j(y)}\,, \tag{3}$$

where $\Delta_{\mathrm{F}_{ij}}(x) = \Delta_{\mathrm{F}}(x;\ m_i^2)\delta_{ij}$ is the Feynman propagator. Under some assumptions Eq. (3) admits unique solution, at least at the formal perturbative level. Once Eq. (3) is solved we obtain $\Phi_i(x)$ as a function of $\beta_n^{(i)}$ and $\bar{\beta}_n^{(i)}$,

$$\Phi_i(x) \equiv \Phi_i(x \mid \beta,\, \bar{\beta})\,. \tag{4}$$

Taking succesive derivatives with respect to β's and $\bar{\beta}$'s at $\beta = \bar{\beta} = 0$ one obtains the matrix elements of the field operator $\hat{\Phi}_i(x)$ between in- and out states in the tree-level approximation [1,9]; for example,

$$\left.\frac{\partial^2 \Phi_i(x \mid \beta, \bar{\beta})}{\partial \beta_n^{(k)} \partial \bar{\beta}_m^{(l)}}\right|_{\beta=\bar{\beta}=0} = \langle (ml); \mathrm{out} \mid \hat{\Phi}_i(x) \mid (nk); \mathrm{in}\rangle\Big|_{\mathrm{tree}}\,. \tag{5}$$

In practice, it is more convenient to work with differential equations instead of integral ones. Eqs. (2), (3) imply

$$(\Box + m_i^2)\Phi_i(x \mid \beta, \bar{\beta}) + \frac{\partial V(\Phi)}{\partial \Phi_i(x \mid \beta, \bar{\beta})} = 0\,, \tag{6}$$

$$\Phi_i(x \,\big|\, \beta, \bar{\beta})\,\big|_{V=0} = \Phi_{0i}(x)\,. \tag{7}$$

The converse is, in general, not true. Indeed, β's and $\bar{\beta}$'s enter $\Phi_i(x \mid \beta, \bar{\beta})$ as arbitrary constants; they can be replaced, without alterning (6) and (7), by any coupling constant dependent functions $\beta(\lambda)$, $\bar{\beta}(\lambda)$ such that $\beta(0) = \beta$, $\bar{\beta}(0) = \bar{\beta}$. The solution to (6), (7) can be made unique by adding further constraints following from the tree-graph interpretation. Consider Φ^4-theory as an example,

$$V(\Phi) = \frac{\lambda}{4!}\Phi^4 \,.$$

Topological relations for tree graphs imply that β, $\bar{\beta}$ and λ enter the matrix elements of field operator in the combination $\lambda^n \beta^k \bar{\beta}^l$ with $k + l = 2n + 1$; in other words, the contribution of the λ^n-order is a homogeneous polynomial in β and $\bar{\beta}$ of degree $2n+1$. This additional condition makes (6), (7) equivalent to (3).

Let us now apply the above formalism to the problem of $2 \to n$ scattering. Assume we have two initial particles of the same kind, carried, say, by Φ_1, with the momenta $\vec{p}$ and $-\vec{p}$; we want to compute the amplitude for producing n particles at rest, the final particles being of different kind than the initial ones.

Differentiating (6), (7) with respect to $\beta^{(1)}_{\vec{p}}$ we arrive at the following set of equations

$$(\Box + m_i^2)\frac{\partial \Phi_i(x)}{\partial \beta^{(1)}_{\vec{p}}} + \frac{\partial^2 V}{\partial \Phi_i(x) \partial \Phi_j(x)} \frac{\partial \Phi_j(x)}{\partial \beta^{(1)}_{\vec{p}}} = 0 \,, \tag{8}$$

$$\left.\frac{\partial \Phi_i(x)}{\partial \beta^{(1)}_{\vec{p}}}\right|_{V=0} \equiv \delta_{i1} f^{(1)}_{\vec{p}}(x) \,. \tag{9}$$

Now, $\frac{\partial \Phi_j(x)}{\partial \beta^{(1)}_{\vec{p}}}$ generates, in the tree approximation, the matrix elements of i-th field operator with one "1" particle carrying the momentum $\vec{p}$ already present in the initial state. The second initial particle is obtained by applying LSZ reduction procedure to the matrix elements of $\hat{\Phi}_1(x)$. Therefore, we have only to generate final particles at the threshold and the only parameters we need to keep nonvanishing are $\overline{\beta}^{(i)}_0$, $i = 2, \ldots, N$. This allows us to simplify the whole algorithm. First of all note that all matrix elements generated by the solutions to (6) are now $\vec{x}$ -independent and Eqs. (6), (7) can be rewritten as

$$(\partial_t^2 + m_i^2)\Phi_i(t \mid \beta, \overline{\beta}) + \frac{\partial V(\Phi)}{\partial \Phi_i(t \mid \beta, \overline{\beta})} = 0 \,, \tag{10}$$

$$\Phi_i(t \,\big|\, \beta, \overline{\beta})\big|_{V=0} = \frac{\overline{\beta}^{(i)}_0}{\sqrt{(2\pi)^3 2m_i}} \mathrm{e}^{im_i t} \,. \tag{11}$$

Denote

$$\psi_i(x \mid \beta, \overline{\beta}) \equiv \frac{\partial \Phi_i(x \mid \beta, \overline{\beta})}{\partial \beta^{(1)}_{\vec{p}}} \tag{12}$$

again with all β's and $\overline{\beta}$'s, except $\overline{\beta}^{(i)}_0$, $i \neq 1$, vanishing: $\psi_i(x \mid \beta, \overline{\beta})$ generate matrix elements with one particle, carrying the momentum $\vec{p}$, in the initial state and arbitrany number of particles at rest in the final one. Therefore, due to the translational invariance,

$$\psi_i(x \mid \beta, \overline{\beta}) = \tilde{\psi}_i(t \mid \beta, \overline{\beta}) \mathrm{e}^{i\vec{p}\vec{x}}\,, \tag{13}$$

and Eqs. (8), (9) take form

$$(\partial_t^2 + M_i^2)\tilde{\psi}_i(t \mid \beta, \overline{\beta}) + \frac{\partial^2 V}{\partial \Phi_i(t) \partial \Phi_j(t)} \tilde{\psi}_j(t \mid \beta, \overline{\beta}) = 0\,, \tag{14}$$

$$\tilde{\psi}_i(t)\Big|_{V=0} = \delta_{i1} \frac{1}{\sqrt{(2\pi)^3 2M_1}} \mathrm{e}^{-iM_1 t}\,, \tag{15}$$

where $M_i^2 \equiv m_i^2 + \vec{p}^2$.

Having solved (14), (15) one can compute the relevant amplitude from LSZ formalism:

$$\frac{i(2\pi)^3 \delta^{(3)}(\vec{p} + \vec{q})}{\sqrt{(2\pi)^3 2M_1}} \int\limits_{-\infty}^{\infty} dt \mathrm{e}^{-iM_1 t}(\partial_t^2 + M_1^2)\tilde{\psi}_1(t \,|\, \beta, \overline{\beta}) \tag{16}$$

generates all $2 \to n$ amplitudes.
Eqs. (13)–(16) provide the general solution to our problem.

Particulary interesting situation emerges if the theory exhibits additional symmetry

$$\Phi_1 \longrightarrow \Phi_1' = -\Phi_1\,, \quad \Phi_i \longrightarrow \Phi_i' = \Phi_i\,, \quad i \neq 1\,. \tag{17}$$

Consider first Eqs. (6), (7); once we put $\beta^{(1)}_{\vec{q}} = 0 = \overline{\beta}^{(1)}_{\vec{q}}$ both Eq. (6) and the initial condition (7) are invariant under (17). By uniqueness of the solution we find

$$\Phi_1(x \,|\beta, \overline{\beta})\big|_{\beta^{(1)}_{\vec{q}} = 0 = \overline{\beta}^{(1)}_{\vec{q}}} = 0\,. \tag{18}$$

This conclusion is obviously equivalent to the statement that only graphs with even number of external "1"-lines are allowed. The counterparts of (10), (11) read now

$$(\partial_t^2 + m_i^2)\Phi_i(t) + \frac{\partial V_1(\Phi)}{\partial \Phi_1(t)} = 0\,, \qquad i = Z, \ldots, N \tag{19}$$

$$\Phi_i(t)|_{V=0} = \overline{\beta}^{(i)}_0 \mathrm{e}^{im_i t}\,, \quad i = 2, \ldots, N \tag{20}$$

$$V_1(\underline{\Phi}) = V(\underline{\Phi})|_{\Phi_1 = 0}\,, \tag{21}$$

which is a set of coupled dynamical equations for the reduced system obtained by ignoring Φ_1. Let us further consider Eq. (14). We show that

$$\left.\frac{\partial \Phi_j(x \mid \beta, \overline{\beta})}{\partial \beta^{(1)}_{\vec{p}}}\right|_{\beta^{(1)}_{\vec{q}}=0=\overline{\beta}^{(1)}_{\vec{q}}} = 0\,, \quad j \neq 1\,. \tag{22}$$

This is, again, a consequence of the fact that only even number of external "1"-lines is admitted. The formal proof is based on symmetry (17): from the uniqueness of the solution to (6), (7) it follows then that $\Phi_i(x \mid \beta, \overline{\beta})$, $i \neq 1$, are even functions of $\beta^{(1)}_{\vec{q}}$, $\overline{\beta}^{(1)}_{\vec{q}}$ which implies (22).

Taking into account the above property we can rewrite (14) as

$$(\partial_t^2 + M_1^2)\tilde{\psi}_1(t \mid \beta, \overline{\beta}) + \left.\frac{\partial^2 V}{\partial \Phi_1(t)^2}\right|_{\Phi_1=0} \cdot \tilde{\psi}_1(t \mid \beta, \overline{\beta}) = 0\,, \tag{23}$$

$$\tilde{\psi}_1(t)\,|_{V=0} = \frac{1}{\sqrt{(2\pi)^3 2M_1}} \mathrm{e}^{-iM_1 t}\,. \tag{24}$$

We see that in the presence of the symmetry (17) the problem gets simplified: first we solve $N-1$ equations for $\Phi_i(t \mid \beta, \overline{\beta})$, $i = 2, \ldots, N$ and then one equation (23) for $\tilde{\psi}_1(t \mid \beta, \overline{\beta})$.

Let us now make the following important observation. Take the modified theory given by the Lagrangian

$$\tilde{L} = \frac{1}{2}\left(\partial_\mu \Phi_1 \partial^\mu \Phi_1 - M_1^2 \Phi_1^2\right) + \frac{1}{2}\sum_{i=2}^{N}\left(\partial_\mu \Phi_i \partial^\mu \Phi_i - m_i^2 \Phi_i^2\right) - V(\underline{\Phi}) \tag{25}$$

with $M_1^2 \equiv m_1^2 + \vec{p}^{\,2}$, and consider the same process $2 \to n$ except that now <u>all</u> particles, both initial as well as final, are at rest. Repeating the procedure outlined above we conclude that the amplitudes for both processes coincide, up to irrelevant delta function expressing momentum conservation. This result can again be easily understood on the diagrammar level. Symmetry (17) implies that there is a unique chain of Φ_1-lines connecting both initial particles and the three-momentum flows only through lines of this chain.

The observation made above allows us to explain, at least in some cases, the origin of amplitudes nullification in $2 \to n$ processes. In fact, for purely threshold amplitudes their nullification may result from symmetry properties of the reduced Hamiltonian system obtained by ignoring space-dependence in the original problem [9].

3. Some examples

3.1. Quartic interaction: 0(2)-theory

Let us start with the model considered in [9]:

$$L = \frac{1}{2}\sum_{i=1}^{2}\left(\partial_\mu \Phi_i \partial^\mu \Phi_i - m_i^2 \Phi_i^2\right) - \frac{\lambda}{4!}\left(\Phi_1^2 + \Phi_2^2\right) . \tag{26}$$

The reduced theory is integrable and separable in elliptic coordinates [11]; in fact, it is an example of the so-called Garnier system [11]. In order to calculate amplitudes "1" → "2" at the threshold we look for the solution to the field equations obeying

$$\Phi_1(t)\mid_{\lambda=0} \;=\; \frac{\beta_1 \mathrm{e}^{-im_1 t}}{\sqrt{(2\pi)^3 2m_1}} \equiv z_1(t)\,, \tag{27}$$

$$\Phi_2(t)\mid_{\lambda=0} \;=\; \frac{\overline{\beta}_2 \mathrm{e}^{im_2 t}}{\sqrt{(2\pi)^3 2m_2}} \equiv z_2(t)\,. \tag{28}$$

They read [12]

$$\Phi_1 \;=\; z_1\left(1-\frac{\lambda\kappa}{48m_2^2}z_2^2\right)\left(1-\frac{\lambda}{48m_1^2}z_1^2-\frac{\lambda}{48m_2^2}z_2^2+\frac{\lambda^2\kappa^2}{48^2 m_1^2 m_2^2}z_1^2 z_2^2\right)^{-1}, \tag{29}$$

$$\Phi_2 \;=\; z_2\left(1+\frac{\lambda\kappa}{48m_1^2}z_1^2\right)\left(1-\frac{\lambda}{48m_1^2}z_1^2-\frac{\lambda}{48m_2^2}z_2^2+\frac{\lambda^2\kappa^2}{48^2 m_1^2 m_2^2}z_1^2 z_2^2\right)^{-1}, \tag{30}$$

where $\kappa \equiv \frac{m_1+m_2}{m_1-m_2}$.

In order to calculate $2 \to n$ amplitude, with all particles at rest, we compute

$$\frac{\partial \Phi_1}{\partial \beta_1}\mid_{\beta_1=0} = \left(1-\frac{\lambda\kappa z_2^2}{48m_2^2}\right)\left(1-\frac{\lambda}{48m_2^2}z_2^2\right)^{-1}\frac{\mathrm{e}^{-im_1 t}}{\sqrt{(2\pi)^3 2m_1}}\,. \tag{31}$$

On the other hand consider $2 \to n$ process with hard initial particles. The relevant amplitude can be computed according to the scheme presended in [9]. The appropriate wave function obeys Eq. (3.3) of [9] with $a = 2$ and is given by Eq. (3.8) therein

$$\psi(x) \;=\; \mathrm{e}^{-ipx} F\left(1,-1,1-\frac{E_p}{m_2}\,;y\right), \tag{32}$$

$$y \;\equiv\; -\frac{\lambda}{48m_2^2}\,\frac{z_2^2}{1-\frac{\lambda}{48m_2^2}z_2^2}\,, \tag{33}$$

or, explicitly,

$$\psi(x) = \mathrm{e}^{-ipx} \left(\frac{1 - \frac{\lambda}{48m_2^2}(\frac{E_p+m_2}{E_P-m_2})z_2^2}{1 - \frac{\lambda}{48m_2^2}z_2^2} \right) . \tag{34}$$

Eqs. (31) and (34) can be now compared. According to the general reasoning presented in Sec. 2 one should identify $E_p \leftrightarrow m_1$ and neglect space-dependent factor in (34). Then we see that (31) and (34) indeed coincide (one should also take into account the difference in wave function normalization).

In particular, the nuliffication of $2 \to n$ hard-soft amplitudes results from Ward identities for purely soft ones [2].

3.2. Henon–Heiles system

Consider the theory given by the Lagrangian

$$\begin{aligned} L \;=\; & \frac{1}{2}\left(\partial_\mu \Phi_1 \partial^\mu \Phi_1 - m_1^2 \Phi_1^2\right) + \frac{1}{2}\left(\partial_\mu \Phi_2 \partial^\mu \Phi_2 - m_2^2 \Phi_2^2\right) \\ & + - \frac{g}{2!}\Phi_1^2 \Phi_2 - \frac{\lambda}{3!}\Phi_2^3 . \end{aligned} \tag{35}$$

It is symetric under $\Phi_1 \to -\Phi_1$, $\Phi_2 \to \Phi_2$. There exists no stable ground state here but on the perturbative level theory is perfectly well defined. We shall analyse its threshold behaviour in few steps.

(A) $2 \to n$ amplitudes: Φ_1-particles

We assume that both initial particles are carried by Φ_1 field. According to the procedure outlined in Sec. 2 we have to solve first the field equation for Φ_2 assuming $\Phi_1 \equiv 0$ and no space-dependence,

$$\left(\partial_t^2 + m_2^2\right)\Phi_2 + \frac{\lambda}{2}\Phi_2^2 = 0 , \tag{36}$$

$$\Phi_2 \,|_{\lambda=0} = z\mathrm{e}^{im_2 t} \equiv \frac{\overline{\beta}}{\sqrt{(2\pi)^3 2m_2}}\, \mathrm{e}^{im_2 t} . \tag{37}$$

The proper solution (*i.e.* the one reproducing tree-graph expansion) is obtained by assuming the total energy to vanish; it reads

$$\Phi_2(t) = \frac{z\mathrm{e}^{im_2 t}}{\left(1 - \frac{z\lambda}{12m_2^2}\mathrm{e}^{im_2 t}\right)^2} . \tag{38}$$

This is the generating function for Φ^3-theory. The counterpart of Eq. (23) takes the form

$$(\Box + m_1^2 + g\Phi_2(t))\psi(x) = 0 . \tag{39}$$

Puting

$$\psi(x) = \frac{1}{\sqrt{(2\pi)^3 2E_p}} \mathrm{e}^{-ipx} \tilde{\psi}(t), \;\; p^2 = m_1^2, \;\; E_p \equiv \sqrt{\vec{p}^2 + m_1^2} \tag{40}$$

we arrive at

$$\left(\partial_t^2 - 2iE_p\partial_t + g\Phi_2(t)\right)\tilde{\psi}(t) \;=\; 0\,, \tag{41}$$

$$\tilde{\psi}(t)|_{g=0} \;=\; 1\,. \tag{42}$$

In order to solve (41) we make a change of variable [9]

$$y = \frac{\frac{-z\lambda}{12m_2^2}\mathrm{e}^{im_2 t}}{1 - \frac{z\lambda}{12m_2^2}\mathrm{e}^{im_2 t}}\,, \tag{43}$$

which converts (41) into

$$\left(y(1-y)\frac{d^2}{dy^2} + \left(\left(1 - \frac{2E_p}{m_2}\right) - 2y\right)\frac{d}{dy} + \frac{12g}{\lambda}\right)\tilde{\psi}(y) = 0\,. \tag{44}$$

The relevant solution reads

$$\tilde{\psi}(t) = F\left(a_1, a_2, \;\; 1 - \frac{2E_p}{m_2}; y\right), \tag{45}$$

where F is the hypergeometric function and

$$a_{1,2} = \frac{1 \pm \sqrt{1 + \frac{48g}{\lambda}}}{2}\,. \tag{46}$$

Let us now apply the LSZ-reduction to produce the second initial particle:

$$\begin{aligned} &\frac{i}{\sqrt{(2\pi)^3 2E_q}} \int d^4x \mathrm{e}^{-iqx}(\Box + m_1^2)\psi(x) \\ &= \frac{i\delta^{(3)}(\vec{p} + \vec{q})}{2E_p} \int\limits_{-\infty}^{\infty} dt \mathrm{e}^{-iE_p t}(\partial_t^2 + E_p^2)(\mathrm{e}^{-iE_p t}\tilde{\psi}(t))\,. \end{aligned} \tag{47}$$

Now, from (47) we see that $\tilde{\psi}(t)$ has the following exspansion

$$\tilde{\psi}(t) = \sum_{n \geq 0} \alpha_n \mathrm{e}^{inm_2 t}\,. \tag{48}$$

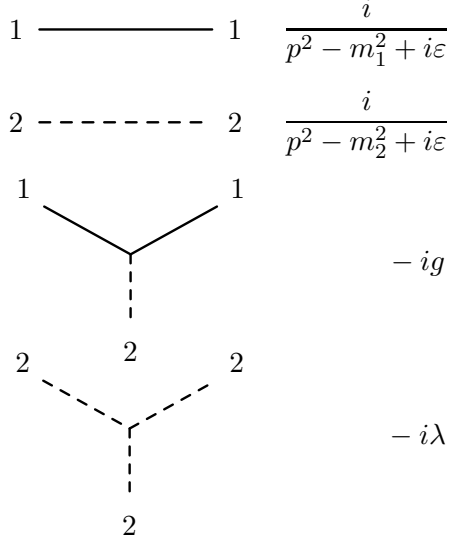
Taking into account the contribution from n-th term one gets

$$\frac{i}{\sqrt{(2\pi)^3 2E_q}} \int d^4x \mathrm{e}^{-iqx}(\Box + m_1^2)\psi(x) \mid_n$$
$$= 2i\pi\delta^{(3)}(\vec{p}+\vec{q})\delta(2E_p - nm_2)(2E_p - nm_2)\alpha_n \; ; \tag{49}$$

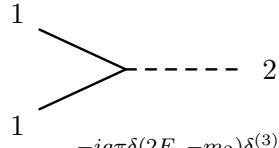
the conribution is nonvanishing only provided α_n has a pole at $2E_p = nm_2$. This pole term is easily identified from the solution (45). The result reads

$$A(2 \to n) \; = \; \frac{4i\pi\delta^{(3)}(\vec{p}+\vec{q})\delta(2E_p - nm_2)E_p}{((2\pi)^3 2m_2)^{\frac{n}{2}}\, n!} \left(\frac{\lambda}{12m_2^2}\right)^n$$
$$\times \frac{\Gamma(a_1+n)\Gamma(a_2+n)}{\Gamma(a_1)\Gamma(a_2)} \, . \tag{50}$$

Eq. (50) can be checked by Feynman-graph method. The relevant Feynman rules read



For $n = 1$ there is only one contribution coming from the elementary vertex



which contributes an amount $\frac{-ig\pi\delta(2E_p - m_2)\delta^{(3)}(\vec{p}+\vec{q})}{\sqrt{(2\pi)^3 2m_2 E_p}}$, in agreement with (50) for $n = 1$.

The case $n = 2$ is slightly more complicated. There are three graphs contributing

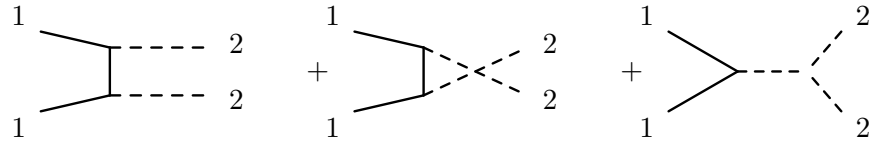


Fig. 1

and they sum up to

$$\frac{i\delta^{(3)}(\vec{p}+\vec{q})\delta(2E_p - 2m_2)g}{(2\pi)^2 2m_2^4}\left(g - \frac{\lambda}{6}\right), \tag{51}$$

which again agrees with general formula (50).

Let us now note the following: for

$$\frac{g}{\lambda} = \frac{N(N+1)}{12} \tag{52}$$

one finds $a_2 = -N$. Therefore, due to $\Gamma(a_2+n)/\Gamma(a_2) = (-N)(1-N)\ldots(n-1-N)$, all amplitudes $A(2 \to n)$, $n \geq N+1$, vanish. In particular, for $N = 1$ the only nonvanishing amplitude corresponds to $n = 1$. This latter result can be understood by invoking the equivalence with purely threshold amplitudes, as discussed in Sec. 2. In fact, $N = 1$ implies, through (52), $g = \frac{\lambda}{6}$; in this case the Henon–Heiles system becomes integrable and separable in parabolic coordinates [11]. Let us consider the relevant solution in more detail.

(B) Henon–Heiles model: integrable case

If $g = \frac{\lambda}{6}$ the reduced dynamics becomes separable in parabolic coordinates

$$\begin{aligned} \Phi_1^2 &= -4\zeta\eta\,, \\ \kappa &\equiv \frac{3}{\lambda}(4m_1^2 - m_2^2)\,, \\ \Phi_2 &= \zeta + \eta + \kappa\,. \end{aligned} \tag{53}$$

The additional integral of motion resulting from the separation of variables reads

$$\begin{aligned} F &= \frac{\lambda}{2}\dot{\Phi}_1(\Phi_1\dot{\Phi}_2 - \Phi_2\dot{\Phi}_1) + \frac{\lambda\kappa}{2}(\dot{\Phi}_1^2 + m_1^2\Phi_1^2) + \frac{\lambda m_1^2}{2}\Phi_1^2\Phi_2 \\ &\quad + \frac{\lambda^2}{24}\Phi_1^2\left(\frac{\Phi_1^2}{4} + \Phi_2^2\right). \end{aligned} \tag{54}$$

F generates the following symmetry transformation

$$\begin{aligned} \Phi_1 \to \Phi_1' &= \Phi_1 + \varepsilon\lambda\left(\kappa\dot{\Phi}_1 + \frac{1}{2}(\Phi_1\dot{\Phi}_2 - 2\Phi_2\dot{\Phi}_1)\right), \\ \Phi_2 \to \Phi_2' &= \Phi_2 + \frac{\varepsilon\lambda}{2}\Phi_1\dot{\Phi}_1 . \end{aligned} \tag{55}$$

We are looking for the solution obeying

$$\begin{aligned} \Phi_1 \mid_{\lambda=0} &= z_1 \mathrm{e}^{-im_1 t}, \\ z_i &\equiv \frac{\beta_i}{\sqrt{(2\pi)^3 2m_i}}, \\ \Phi_2 \mid_{\lambda=0} &= z_2 \mathrm{e}^{im_2 t}. \end{aligned} \tag{56}$$

It corresponds to $E = 0$ and $F = 0$ and reads

$$\Phi_1 = \frac{2m_1(4m_1^2 - m_2^2)(1-x)y}{\lambda(2m_1(1-x)(1-y^2) + m_2(1+x)(1+y^2))}, \tag{57}$$

$$\Phi_2 = \frac{4\kappa(m_2^2 x(1+y^2)^2 + 4m_1^2 y^2(1-x)^2)}{(2m_1(1-x)(1-y^2) + m_2(1+x)(1+y^2))^2}, \tag{58}$$

where

$$x = \frac{z_2\lambda}{m_2^2}\left(\frac{2m_1 + m_2}{2m_1 - m_2}\right)\mathrm{e}^{im_2 t}, \tag{59}$$

$$y = \frac{z_1\lambda \mathrm{e}^{-im_1 t}}{2m_1(2m_1 - m_2)}. \tag{60}$$

Let us analyse this solution in some detail. For $2m_1 \neq m_2$ there are no resonances so the relevant threshold amplitudes vanish. On the other hand,

$$\frac{\partial\Phi_1}{\partial z_1} \mid_{z_1=0} = \frac{1 - \left(\frac{2m_1+m_2}{2m_1-m_2}\right)\frac{z_2\lambda}{m_2^2}\mathrm{e}^{im_2 t}}{1 - \frac{z_2\lambda}{m_2^2}\mathrm{e}^{im_2 t}} \simeq 1 - \frac{2\lambda z_2 \mathrm{e}^{im_2 t}}{m_2(2m_1 - m_2)} + O\left(z_2^2\right), \tag{61}$$

which gives nonzero amplitude $2 \to 1$ for $2m_1 = m_2$. This is in agreement with general arguments concerning the relation between symmetries and amplitude nullification [2,10]; in fact, for $2m_1 = m_2$ the relevant linear part in the transformation formulae (55) is absent.

By replacing m_1 by $E_p \equiv \sqrt{\vec{p}^2 + m_1^2}$ our expression (61) coincides with (45) calculated for $N = 1$. Therefore, the symmetry (55), which is valid for arbitrary masses, inplies the vanishing of $2 \to n$ hard-soft amplitudes.

4. Fermions

It is straightforward to include higher-spin bosons into our scheme. As far as fermions are concerned the only but crucial modification is to replace the parameters β, $\bar{\beta}$ entering (2) by anticommuting Grassman variables; this allows to implement Pauli exclusion principle.

As an example we shall compute the threshold amplitudes for n-boson production by fermion-antifermion pair. The theory we are considering provides a toy model for fermionic mass generation is the standard model via spontaneous symmetry breaking. It has been considered from the point of view of amplitudes nullification in Refs. [4, 5]; diagrammatic approach through recurrence relations has been developed in [7].

Our model contains massless fermionic field coupled by Yukawa term to the Higgs field. The relevant Lagrangian reads

$$L = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi + \frac{1}{2}\left(\partial_{\mu}\Phi\partial^{\mu}\Phi + m^2\Phi^2\right) - \frac{\lambda}{4!}\Phi^4 - g\Phi\psi\bar{\psi}\,. \tag{62}$$

Due to the wrong sign of mass term the vacuum solution is nontrivial

$$\psi = 0\,, \quad \bar{\psi} = 0\,, \quad \Phi = v \equiv \sqrt{\frac{3!m^2}{\lambda}}\,. \tag{63}$$

Define the physical field ρ by

$$\rho \equiv \Phi - v\,. \tag{64}$$

In terms of this new field the Lagrangian reads

$$\begin{aligned} L \;=\; & \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - M)\psi \\ & + \frac{1}{2}(\partial_{\mu}\rho\partial^{\mu}\rho - m_{\rho}^2\rho^2) - \frac{\lambda v}{3!}\rho^3 - \frac{\lambda}{4!}\rho^4 + -g\rho\bar{\psi}\psi\,; \end{aligned} \tag{65}$$

here $M = gv$, $m_{\rho} = \sqrt{2}m$ are fermionic and bosonic masses, respectively.

L is invariant under $\psi \to -\psi$, $\bar{\psi} \to -\bar{\psi}$, $\rho \to \rho$; therefore, one can apply the strategy of Sec. 2. First, we solve field equations for ρ under the condition $\psi = \bar{\psi} = 0$ and

$$\rho\,|_{\lambda=0} = \frac{\beta \mathrm{e}^{im_{\rho}t}}{\sqrt{(2\pi)^3 2m_{\rho}}} \equiv z_0 \mathrm{e}^{im_{\rho}t} \equiv z(t)\,. \tag{66}$$

The solution reads [9]

$$\rho(t) = \frac{z(t)}{1 - \frac{z(t)}{2v}}\,, \tag{67}$$

or

$$\Phi(t) = v\frac{1 + \frac{z(t)}{2v}}{1 - \frac{z(t)}{2v}}\,. \tag{68}$$

The counterpart of (23) takes now the form

$$(i\gamma^\mu\partial_\mu - g\Phi(t))\psi_{ps}(x) \;=\; 0\,, \tag{69}$$

$$\psi_{ps}(x)|_{\substack{g=0\\ gv=M}} \;=\; \sqrt{\frac{M}{(2\pi)^3 E_p}}u(p,s)\mathrm{e}^{-ipx}\,. \tag{70}$$

The same reasoning as in bosonic case suggests the following Ansatz for ψ_{ps}:

$$\psi_{ps}(x) = \begin{pmatrix}\zeta F(t)\\ \eta\tilde{F}(t)\end{pmatrix}\mathrm{e}^{ip^k x^k}\mathrm{e}^{-iE_p t}\,, \tag{71}$$

where

$$\begin{pmatrix}\zeta\\ \eta\end{pmatrix} = \sqrt{\frac{M}{(2\pi)^3 E_p}}u(p,s) \tag{72}$$

is the standard solution to Dirac equation [13]. Inserting the Ansatz (71), (72) into Eq. (69) we arrive at

$$(E_p + i\partial_t - g\Phi(t))F(t)\zeta + \tilde{F}(t)p_k\sigma^k\eta \;=\; 0\,, \tag{73}$$

$$(-E_p - i\partial_t - g\Phi(t))\tilde{F}(t)\eta - F(t)p_k\sigma^k\zeta \;=\; 0 \tag{74}$$

or, using (72),

$$(E_p + i\partial_t - g\Phi(t))F(t) - (E - M)\tilde{F}(t) \;=\; 0\,, \tag{75}$$

$$(E_p + i\partial_t + g\Phi(t))\tilde{F}(t) - (E + M)F(t) \;=\; 0\,; \tag{76}$$

the boundary condition (70) is obeyed provided

$$F\mid_{\substack{g=0\\ gv=M}} = 1, \quad \tilde{F}\mid_{\substack{g=0\\ gv=M}} = 1\,. \tag{77}$$

It is not difficult to find the solution to (75), (76) obeying (77). We apply $(E_p + i\partial_t + g\Phi(t))$ to (75) and use (76); then

$$\left(\partial_t^2 - 2iE_p\partial_t + g^2\Phi^2(t) + ig\partial_t\Phi(t) + M^2\right)F = 0\,. \tag{78}$$

Changing the variable,

$$y = \frac{-\frac{z(t)}{2v}}{1-\frac{z(t)}{2v}}\,, \tag{79}$$

we arrive at the hipergeometric equation. Its solution reads

$$F(t) = F(\alpha,\ \beta,\ \gamma;\ y)\,, \tag{80}$$

$$\begin{aligned} \alpha &\equiv \frac{2M}{m_\rho}\,, \\ \beta &\equiv 1-\frac{2M}{m_\rho}\,, \\ \gamma &\equiv 1-\frac{2E_p}{m_\rho}\,. \end{aligned} \tag{81}$$

For $\tilde{F}(t)$ we obtain, respectively,

$$\tilde{F}(t) = F(\alpha+1,\ \beta-1,\ \gamma;\ y)\,; \tag{82}$$

(80) and (82) satisfy (75), (76).

We are now ready to calculate the exact expression for the tree-graph threshold amplitude $A(f\bar{f} \to n\rho)$. To this end we reduce the fermionic antiparticle by LSZ formula. Then

$$-i\int d^4x \bar{v}_{p's'}(x)(i\gamma^\mu\partial_\mu - M)\psi_{ps}(x) \tag{83}$$

becomes the generating functional for all such amplitudes; here

$$\bar{v}_{p's'}(x) = \sqrt{\frac{M}{(2\pi)^3 E_{p'}}}\bar{v}(p',s')\mathrm{e}^{-ip'x}\,. \tag{84}$$

In order to calculate (83) let us write

$$\psi_{ps}(x) = \sqrt{\frac{M}{(2\pi)^3 E_p}}\tilde{\psi}(x^0)\mathrm{e}^{-ipx}\,; \tag{85}$$

then (83) can be rewritten as follows

$$\frac{-iM}{(2\pi)^3\sqrt{E_pE_{p'}}}\int d^4x\bar{v}(p',s')\mathrm{e}^{-ip'x}(i\gamma^\mu\partial_\mu - M)\tilde{\psi}(x^0)\mathrm{e}^{-ipx}$$
$$= \frac{-iM}{(2\pi)^3\sqrt{E_{p'}E_{p'}}}\int d^4x\bar{v}(p',s')\mathrm{e}^{-i(p+p')x}(i\gamma^\mu p_\mu - M + i\gamma^0\partial_0)\tilde{\psi}(x^0)$$
$$= \frac{-iM}{E_p}\delta^{(3)}(\vec{p}+\vec{p}')\int\limits_{-\infty}^{\infty} dx^0\mathrm{e}^{-2iE_px^0}\bar{v}(p',s')(\gamma^\mu p_\mu - M + i\gamma^0\partial_0)\tilde{\psi}(x^0)\,. \quad (86)$$

Let us note that, due to $\vec{p}\,' = -\vec{p}\,'$,

$$\bar{v}(p',s')(\gamma^\mu p_\mu - M) = 2E_p\bar{v}(p',s')\gamma^0 \quad (87)$$

and our functional takes the form

$$\frac{-iM}{E_p}\delta^{(3)}(\vec{p}+\vec{p}')\int\limits_{-\infty}^{\infty} dx^0\mathrm{e}^{-2iE_px^0}\bar{v}(p',s')\gamma^0(2E_p + i\partial_0)\tilde{\psi}(x^0)\,. \quad (88)$$

Consider the last expression. It can be analysed in a similar way as in purely bosonic case. In order to isolate the amplitude for the creation of n bosons one has to single out the term proportional to z_0^n. Due to the structure of the solution $\psi_{ps}(x)$ it can be expanded in Fourier series in $\mathrm{e}^{im_\rho t}$. The term $\mathrm{e}^{inm_\rho t}$ produces the energy delta function $\delta(2E_p - nm_\rho)$; however, because of time derivative in (88) this delta function is accompanied by $2E_p - nm_\rho$, so the result is zero unless there is an additional pole coming from the gamma functions entering hipergeometric series. It is straightforward to isolate this term arriving at the following final expression for the amplitude

$$A(f\bar{f}\to n\rho)$$
$$= \frac{-4\pi iM\delta(2E_p - nm_\rho)\delta^{(3)}(\vec{p}+\vec{p}')}{\left(\sqrt{(2\pi)^3 2m_\rho}\right)^n(n-1)!(2v)^n}\frac{\Gamma(n+\frac{2M}{m_\rho})\Gamma(n-\frac{2M}{m_\rho})}{\Gamma(\frac{2M}{m_\rho})\Gamma(1-\frac{2M}{m_\rho})}\bar{v}(p',s')u(p,s)\,. \quad (89)$$

The spin structure of this formula is dictated by invariance properties, angular momentum and parity conservation. In order to check the formula (89) we compare it with Feynman-graph computation. For $n = 2$ Eq. (89) takes the form

$$A(f\bar{f}\to 2\rho)$$
$$= \frac{-i}{4\pi^2}\delta(2E_p - 2m_\rho)\delta^{(3)}(\vec{p}+\vec{p}')\left(\frac{M^2}{2m_\rho^2v^2} - \frac{2M^4}{m_\rho^4v^2}\right)\bar{v}(p',s')u(p,s)\,. \quad (90)$$

The relevant Feynman rules are

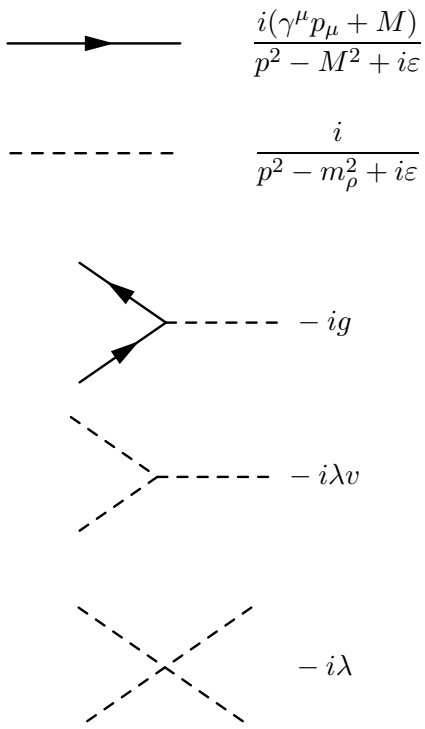


Fig. 2

The graphs contributing to $A(f\bar{f} \to 2\rho)$ are depicted in Fig. 3

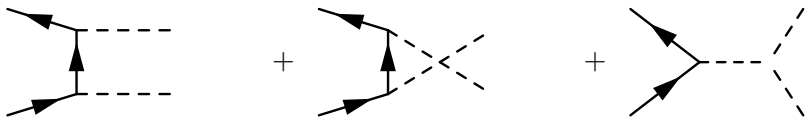


Fig. 3

Again, the amplitude exhibits nullification phenomenon [4, 5, 7, 9]. Due to

$$\frac{\Gamma(n-\alpha)}{\Gamma(1-\alpha)} = (1-\alpha)(2-\alpha)\dots(n-1-\alpha) \tag{91}$$

the amplitude vanishes for $n > N$ provided $\alpha \equiv \frac{2M}{m_\rho} = N$ is an integer (actually, $n \leq N$ amplitudes also vanish due to energy conservation and the properties of spinor wave functions). In particular, for $N = 1$ all amplitudes, vanish.

5. Final remarks

We presented a simple derivation of the Brown–Zhai Feynman wave function algorithm for general theories. It allows to compare our process with a purely soft one. The important point is that, for the latter, we can use Ward identities following from the integrability of reduced dynamics [2] to prove their vanishing. This argument works in purely scalar case (*cf.* also [2]) as well as fermionic one [14]. In this way we identify, in some cases, the symmetry underlying nullification of $2 \to n$ amplitudes.

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