GENERALIZED FACTORIAL MOMENTS

A. Bialas

M. Smoluchowski Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Kraków, Poland
e-mail: bialas@th.if.uj.edu.pl

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It is shown that the method of eliminating the statistical fluctuations from event-by-event analysis proposed recently by Fu and Liu can be rewritten in a compact form involving the generalized factorial moments.

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1. The factorial moments of the multiplicity distribution

$$F_k \equiv \langle n(n-1)\dots(n-k+1)\rangle \,, \tag{1}$$

where $\langle\dots\rangle$ denotes an average over events, proved to be a very useful tool in investigations of the multiplicity fluctuations. The main reason of this success was the observation [1] that the measurement of F_k gives a direct access to the "dynamical" fluctuations of the multiplicity.

The problem of separation of the "dynamical" and "statistical" multiplicity fluctuations can be formulated as follows [1]. Assume that the average multiplicity in a certain phase space region undergoes "dynamical" fluctuations with the probability distribution $W(\bar{n})d\bar{n}$ where $\bar{n}$ is the average multiplicity in this region. At fixed $\bar{n}$ there are additional "statistical" fluctuations because the actually observed number of particles obviously cannot be identical with $\bar{n}$. Assuming that these statistical fluctuations do not introduce new correlations in the system, we conclude that they must be in the form of the Poisson distribution. Consequently, the actually observed multiplicities are distributed according to

$$P(n) = \int d\bar{n} W(\bar{n}) \mathrm{e}^{-\bar{n}} \frac{\bar{n}^n}{n!} \,. \tag{2}$$

A simple calculation shows that the *factorial* moments of this distribution $P(n)$ are equal to the *normal* moments of $W(\bar{n})$:

$$F_k = \sum_n n(n-1)\dots(n-k+1)P(n) = \int d\bar{n}\bar{n}^k W(\bar{n}) \,, \tag{3}$$

and thus indeed, F_k gives a direct access to the *dynamical* distribution $W(\bar{n})$.

At this point it is also useful to recall that the factorial moments are simply related to the integrals of the inclusive particle densities:

$$\langle n(n-1)\dots(n-k+1)\rangle = \int dp_1 dp_2 \dots dp_k \rho(p_1, p_2, \dots, p_k)\,. \tag{4}$$

This formula relates multiplicity fluctuations (as expressed by the factorial moment on the l.h.s.) and multiparticle correlations (as expressed by the integral on the r.h.s.). Note also that in absence of correlations between particles we obtain $F_k = \bar{n}^k$.

2. Recently, Fu and Liu [2] proposed an extension of this method which can be used to eliminate statistical fluctuations from distributions of other dynamical quantities. In the present note we show that their result can be elegantly formulated in terms of the *generalized factorial moments.*

Following [3], we are going to study fluctuations (in a given phase-space region) of an *extensive* quantity X defined for each event as

$$X = \sum_{i=1}^{n} x_i\,, \tag{5}$$

where $i = 1, \dots n$ labels the particles in the event and x_i is a dynamical variable which may depend on momentum and other quantum numbers of particle i. Some examples of x_i are charge, momentum, or energy of the particle. Note that taking $x_i \equiv 1$ we obtain $X = n$, *i.e.*, the problem is reduced to the previous one.

Let us now define the generalized factorial moments as

$$F_k[X] \equiv \langle [X-(k-1)\hat{x}]\dots[X-\hat{x}]X\rangle\,, \tag{6}$$

where $\hat{x}$ is the operator acting on the variable X in the following way:

$$[\hat{x}]^l X = X_{l+1} \tag{7}$$

with

$$X_l \equiv \sum_{i=1}^{n} [x_i]^l, \qquad X_1 \equiv X\,. \tag{8}$$

To see the physical interpretation of $F_k[X]$, let us assume — in the spirit of the argument which led to the Eq. (2) — that the "dynamical" fluctuations of the average values $\bar{n}$ and $\bar{x}$ in a given phase-space region are described by the distribution $W(\bar{n},\bar{x})d\bar{n}d\bar{x}$. To obtain the actual distribution of n and x, it is necessary to add "statistical" fluctuations. Demanding again that they do not introduce additional correlations in the system, we arrive at the formula

$$P(n,x_1,\ldots,x_n)dx_1\ldots dx_n = \int d\bar{n}d\bar{x}W(\bar{n},\bar{x})\mathrm{e}^{-\bar{n}}\frac{\bar{n}^n}{n!} \times p(x_1,\bar{x})dx_1\ldots p(x_n,\bar{x})dx_n\,, \tag{9}$$

where $p(x,\bar{x})dx$ is the distribution of x at a fixed $\bar{x}$.

Using this we obtain

$$\begin{aligned}
\langle X\rangle &= \int d\bar{n}d\bar{x}W(\bar{n},\bar{x})\bar{n}\bar{x};\\
\langle X^2\rangle &= \int d\bar{n}d\bar{x}W(\bar{n},\bar{x})[(\bar{n}\bar{x})^2+\bar{n}\overline{x^2}] = \int d\bar{n}d\bar{x}W(\bar{n},\bar{x})(\bar{n}\bar{x})^2+\langle X_2\rangle;\\
\langle X^3\rangle &= \int d\bar{n}d\bar{x}W(\bar{n},\bar{x})[(\bar{n}\bar{x})^3+3\bar{n}\bar{x}\bar{n}\overline{x^2}+\bar{n}\overline{x^3}]\\
&= \int d\bar{n}d\bar{x}W(\bar{n},\bar{x})(\bar{n}\bar{x})^3+3[\langle XX_2\rangle-\langle X_3\rangle]+\langle X_3\rangle;\\
\langle X^4\rangle &= \int d\bar{n}d\bar{x}W(\bar{n},\bar{x})[(\bar{n}\bar{x})^4+6(\bar{n}\bar{x})^2\bar{n}\overline{x^2}+3(\bar{n}\overline{x^2})^2+4\bar{n}\bar{x}\bar{n}\overline{x^3}+\bar{n}\overline{x^4}]\\
&= \int d\bar{n}d\bar{x}W(\bar{n},\bar{x})(\bar{n}\bar{x})^4+6[\langle X^2X_2\rangle-2(\langle XX_3\rangle-\langle X_4\rangle)-(\langle (X_2)^2\rangle\\
&\quad -\langle X_4\rangle)-\langle X_4\rangle]+3(\langle (X_2)^2\rangle-\langle X_4\rangle)+4(\langle XX_3\rangle-\langle X_4\rangle)+\langle X_4\rangle
\end{aligned} \tag{10}$$

and similarly for higher moments. Here $\overline{x^k}=\int dxx^kp(x,\bar{x})$.

Comparing (10) with (6) we conclude that

$$F_k[X]=\int d\bar{n}d\bar{x}W(\bar{n},\bar{x})(\bar{n}\bar{x})^k \tag{11}$$

which shows that, indeed, the generalized factorial moment $F_k[X]$ measures directly the k-th moment of the *dynamical* distribution

$$W(\bar{X})\equiv\int d\bar{n}d\bar{x}W(\bar{n},\bar{x})\delta(\bar{X}-\bar{n}\bar{x})\,. \tag{12}$$

3. It was shown in [3] that the moments of X can be expressed in terms of finite number of integrals of the products of *inclusive* particle densities and powers of x, *e.g.*,

$$\begin{aligned}
\langle X\rangle &= \int dp\rho(p)x(p);\\
\langle X^2\rangle &= \int dp_1dp_2\rho(p_1,p_2)x(p_1)x(p_2)+\int dp\rho(p)[x(p)]^2;
\end{aligned}$$

$$\begin{aligned}\langle X^3\rangle &= \int dp_1dp_2dp_3\rho(p_1,p_2,p_3)x(p_1)x(p_2)x(p_3)\\ &+3\int dp_1dp_2\rho(p_1,p_2)x(p_1)[x(p_2)]^2+\int dp\rho(p)[x(p)]^3\,, \quad (13)\end{aligned}$$

and similar but more complicated relations for higher moments.

Observing that

$$\int dp_1dp_2\rho(p_1,p_2)[x(p_1)]^{s_1}[x(p_2)]^{s_2}=\langle X_{s_1}X_{s_2}\rangle-\langle X_{s_1+s_2}\rangle \qquad (14)$$

one sees that (13) can be summarized in the single formula

$$F_k[X]=\int dp_1\ldots dp_k\rho(p_1,\ldots,p_k)x(p_1)\ldots x(p_k)\,, \qquad (15)$$

which is a generalization of (4) and relates the fluctuations of the variable X with the correlations in the system. Although we have only shown that (15) is valid for $k=1,2,3$, it is not difficult to see that it is actually valid for any k.

4. In summary, it was shown that the recently proposed method of eliminating the statistical fluctuations from event-by-event analysis [2] can be rewritten in a compact form in terms of the generalized factorial moments. It was also shown that the well-known relation between factorial moments and integrals of inclusive multiparticle densities can be extended to the generalized factorial moments.

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