TRUNCATED MOMENTS OF NONSINGLET PARTON DISTRIBUTIONS IN THE DOUBLE LOGARITHMIC $\ln^2 x$ APPROXIMATION

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The method of truncated Mellin moments in a solving QCD evolution equations of the nonsinglet structure functions $F_2^{\mathrm{NS}}(x,Q^2)$ and $g_1^{\mathrm{NS}}(x,Q^2)$ is presented. All calculations are performed within double logarithmic $\ln^2 x$ approximation. An equation for truncated moments which incorporates $\ln^2 x$ effects is formulated and solved for the unintegrated structure function $f^{\mathrm{NS}}(x,Q^2)$. The contribution to the Bjorken sum rule coming from the region of very small x is quantified. Further possible improvement of this approach is also discussed.

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1. Introduction

Structure functions play a central role in the perturbative QCD. Experimental measurements of the spin dependent and unpolarised structure functions of the nucleon allow verification of sum rules and determination of free parameters of the input parton distributions. From the other side, theoretical analysis within perturbative methods investigates the available experimentally region of the variables x and Q^2 and the interesting very small x region (still unmeasurable) as well. In this low x region QCD predicts a strong growth of structure functions with decreasing x — the longitudinal momentum fraction of a hadron carried by a parton. Small x behaviour of the singlet unpolarised structure functions, driven by gluons, is governed by

BFKL [1] or CCFM [2,3] equations which generate the steep $x^{-\lambda}$ ($\lambda \sim 0.3$) shape of $F_2^{\rm S}(x,Q^2)$ [4]. For the nonsinglet unpolarised $F_2^{\rm NS}(x,Q^2)$ the driving term at small x is a nonperturbative contribution of the A_2 Regge pole $F_2^{\rm NS} \sim x^{0.5}$ [4] which dominates even over the $\alpha_{\rm s}\ln^2 x$ effects. These double logarithmic terms control however the small x behaviour of both nonsinglet and singlet spin dependent structure function $g_1(x,Q^2)$ [4, 5]. The knowledge of structure functions at very low x is very important. The sum rules, which can be verified by experiments, concern moments of structure functions $\int_0^1 dx x^{n-1} g_1(x,Q^2)$ and $\int_0^1 dx x^{n-1} F_2(x,Q^2)$ and hence require the knowledge of g_1 and F_2 over the entire region of $x \in (0;1)$. The lowest limit of x in present experiments is about $x \sim 10^{-5}$ so in theoretical analysis one should extrapolate results to $x = 0$ and $x = 1$. More important is, however, the extrapolation to $x \to 0$, where structure functions grow stronger than in the extrapolation to $x = 1$, where structure functions are equal to 0. The limit $x \to 0$ which implies that the invariant energy W^2 of the inelastic lepton-hadron scattering becomes infinite ($W^2 = Q^2(\frac{1}{x} - 1)$) will never be attained experimentally. So we will really never know "what happens" with the structure functions at $x \to 0$. This situation is however not quite hopeless. One can combine the QCD perturbative analysis in the very interesting small x region with experimental data without uncertainty from the region where $x \to 0$. It could be achieved through dealing with truncated moments of the structure functions, where one takes an integral over $x_0 \leq x \leq 1$ instead of over the whole region $0 \leq x \leq 1$. In usually used method of solving evolution equations in QCD one takes Mellin (full) transforms of these equations what gives possibility of analytical solutions. Then after inverse Mellin transform (performed numerically) one can obtain suitable solutions of original equations in x space. In this way *e.g.* in a case of DGLAP approximation, the differentio-integral equations for parton distributions $q(x,Q^2)$ after Mellin transform change into simple differential and diagonalized ones in moment space n. The only problem is the knowledge of input parametrisations for the whole region $0 \leq x \leq 1$ what is necessary in the determination of moments of distribution functions. Using truncated moments approach one can avoid uncertainty from the unmeasurable $x \to 0$ region and also obtain important theoretical results incorporating perturbative QCD effects at small x, which could be verified experimentally. Truncated moments of parton distributions in solving DGLAP equations have been presented in [6]. Authors have shown that the evolution equations for truncated moments though not diagonal can be solved with good precision. This is because each n-th truncated moment couples only with $n+j$-th ($j \geq 0$) truncated moments. In our paper we adopt the truncated moments method to double logarithmic $\ln^2 x$ resummation. However, a technique we use in this approach is quite different because suitable integral equations which correspond to $\ln^2 x$ terms are, of course, different from the DGLAP ones. As a result we obtain the equations for truncated moments of the unintegrated structure function $f^{\rm NS}(x,Q^2)$, where each n-th moment ($n \neq 0$) couples only with itself and with 0-th moment. For fixed coupling constant $\alpha_{\rm s}$ the result for n-th truncated moment can be found analytically. The purpose of this paper is to start the truncated moments method in the case of perturbative QCD formalism, describing double logarithmic $\ln^2 x$ terms in the nonsinglet structure functions $F_2^{\rm NS}$ and $g_1^{\rm NS}$. In the next section we recall the approach which resums the double logarithmic terms. The integral equation for the nonsinglet unintegrated quark distributions $f^{\rm NS}(x,Q^2)$ is presented. In Section 3 the equation for truncated moments of $f^{\rm NS}(x,Q^2)$ within $\ln^2 x$ approach is derived. This equation is solved analytically for fixed $\alpha_{\rm s}$. Agreement for the limit case $x_0 \to 0$ (full moments) of our results is shown. Results for truncated moments of $F_2^{\rm NS}$ and $g_1^{\rm NS}$ for simple Regge-type input parametrisations and different x_0 are presented in Section 4. We calculate also the contribution to the Bjorken sum rule coming from the region of very small x. Finally in Section 5 we summarise our results and discuss further possible improvement of our treatment.

2. Idea of double logarithmic $\ln^2 x$ resummation for the nonsinglet unpolarised and polarised structure functions of the nucleon

It has been noticed [4, 5] that the nonsinglet structure functions of the nucleon in the small x region are governed by double logarithmic terms *i.e.* powers of $\alpha_{\rm s}\ln^2 x$ at each order of the perturbative expansion. This contribution to the $\ln^2 x$ resummation comes from the ladder diagram with quark and gluon exchanges along the chain — *cf.* Fig.1. In contrast to the singlet spin dependent structure function, for the nonsinglet one the "bremsstrahlung" nonladder corrections vanish for the unpolarised structure function and are negligible for the spin dependent one [5, 7]. In this way we do not need to take into account the nonladder diagrams in the case of nonsinglet (both polarised and unpolarised) structure functions. The Regge theory [8], which concerns the Regge limit: $x \to 0$ predicts singular behaviour of nonsinglet, unpolarised distributions and nonsingular (flat) shape of nonsinglet polarised ones

$$q^{\rm NS} \sim x^{-0.5}\,, \tag{2.1}$$

$$\Delta q^{\rm NS} \sim x^{0} \div x^{-0.5}\,, \tag{2.2}$$

where $q^{\rm NS}$ denotes nonsinglet unpolarised and $\Delta q^{\rm NS}$ nonsinglet polarised quark distributions. The nonsinglet part of the unpolarised structure func-

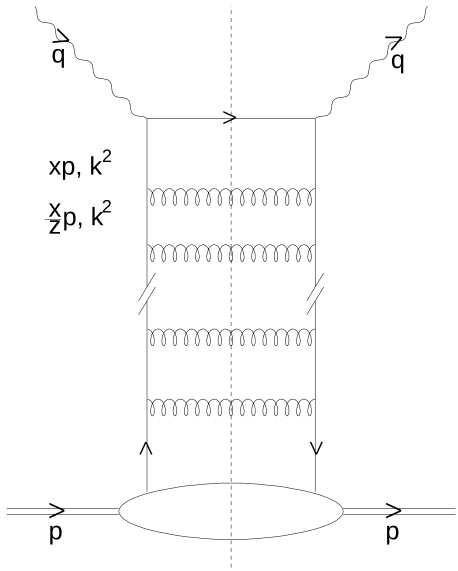

Fig. 1. A ladder diagram generating double logarithmic $\ln^2(1/x)$ terms in the non-singlet spin structure functions.

tion has the form

$$F_2^{\rm NS}(x,Q^2) = F_2^p(x,Q^2) - F_2^n(x,Q^2)\,, \tag{2.3}$$

where

$$F_2(x,Q^2) = \sum_{i=u,d,s,\dots} e_i^2[xq_i(x,Q^2) + x\bar{q}_i(x,Q^2)]\,. \tag{2.4}$$

The spin dependent structure function is

$$g_1^{\rm NS}(x,Q^2) = g_1^p(x,Q^2) - g_1^n(x,Q^2) \tag{2.5}$$

and

$$g_1(x,Q^2) = \frac{1}{2}\sum_{i=u,d,s,\dots} e_i^2 \Delta q_i(x,Q^2)\,. \tag{2.6}$$

p and n in above formulae denote proton and neutron respectively, e_i is a charge of the i-flavour quark. Hence finally we get

$$xq^{\rm NS} = F_2^{\rm NS}(x,Q^2) = \frac{x}{3}(u_{\rm val} - d_{\rm val})(x,Q^2) \tag{2.7}$$

and

$$\Delta q^{\rm NS} = g_1^{\rm NS}(x,Q^2) = \frac{1}{6}(\Delta u_{\rm val} - \Delta d_{\rm val})(x,Q^2)\,, \tag{2.8}$$

where $u_{\rm val}$, $d_{\rm val}$, $\Delta u_{\rm val}$, $\Delta d_{\rm val}$ are respectively spin and nonspin valence quark distributions in the proton. In the double logarithmic approximation the unintegrated nonsinglet quark distribution function $f^{\rm NS}(x,k^2)$ satisfies the following integral equation [4]

$$f^{\rm NS}(x,k^2) = f_0^{\rm NS}(x) + \bar{\alpha_{\rm s}}\int\limits_x^1 \frac{dz}{z}\int\limits_{k_0^2}^{k^2/z}\frac{dk'^2}{k'^2}f^{\rm NS}(\frac{x}{z},k'^2)\,, \tag{2.9}$$

where

$$\bar{\alpha_{\rm s}} = \frac{2\alpha_{\rm s}}{3\pi} \tag{2.10}$$

and $f_0^{\rm NS}(x)$ is a nonperturbative contribution which has a form

$$f_0^{\rm NS}(x) = \bar{\alpha_{\rm s}}\int\limits_x^1 \frac{dz}{z}q^{\rm NS}(z) \sim \bar{\alpha_{\rm s}}x^{-0.5} \tag{2.11}$$

for nonspin distributions or

$$f_0^{\rm NS}(x) = \bar{\alpha_{\rm s}}\int\limits_x^1 \frac{dz}{z}\Delta q^{\rm NS}(z) \sim \bar{\alpha_{\rm s}}\ln\frac{1}{x} \tag{2.12}$$

for spin dependent distributions. The driving terms of these nonperturbative contributions $q^{\rm NS}$ and $\Delta q^{\rm NS}$ are shown in (2.1) and (2.2). The unintegrated distribution $f^{\rm NS}(x,k^2)$ is related to the quark distributions $q^{\rm NS}$ ($\Delta q^{\rm NS}$) via

$$f^{\rm NS}(x,k^2) = \frac{\partial\frac{1}{x}F_2^{\rm NS}(x,k^2)}{\partial\ln k^2} \tag{2.13}$$

in the unpolarised case and

$$f^{\rm NS}(x,k^2) = \frac{\partial g_1^{\rm NS}(x,k^2)}{\partial\ln k^2} \tag{2.14}$$

in the polarised one. Using the method of the Mellin moment functions one can obtain from (2.9) the following equation

$$\bar{f}^{\rm NS}(n,k^2) = \bar{f_0}^{\rm NS}(n) + \frac{\bar{\alpha_{\rm s}}}{n}\left[\int\limits_{k_0^2}^{k^2}\frac{dk'^2}{k'^2}\bar{f}^{\rm NS}(n,k'^2) + \int\limits_{k^2}^{\infty}\frac{dk'^2}{k'^2}\left(\frac{k^2}{k'^2}\right)^n\bar{f}^{\rm NS}(n,k'^2)\right], \tag{2.15}$$

where the full Mellin moment of the function $f(x)$ is defined as:

$$M(n) \equiv \int\limits_0^1 dx x^{n-1} f(x) \tag{2.16}$$

and in our case

$$\bar{f}^{\mathrm{NS}}(n,k^2) = \int\limits_0^1 dx x^{n-1} f^{\mathrm{NS}}(x,k^2)\,. \tag{2.17}$$

As it was shown in [4], equation (2.15) for fixed coupling $\bar{\alpha_{\mathrm{s}}}$ can be solved analytically. Because later we would like to compare full-moments approach with the truncated moments one, let us recall the way to obtain the analytical solution of (2.15). A short explanation of this is given in Appendix A. Thus we get the solution of (2.15) in the form

$$\bar{f}^{\mathrm{NS}}(n,k^2) = \bar{f_0}^{\mathrm{NS}}(n) \frac{n\gamma}{\bar{\alpha_{\mathrm{s}}}} \left(\frac{k^2}{k_0^2}\right)^{\gamma}, \tag{2.18}$$

where

$$\gamma = \frac{n}{2}\left[1 - \sqrt{1-(\frac{n_0}{n})^2}\right] \tag{2.19}$$

and

$$n_0 = 2\sqrt{\bar{\alpha_{\mathrm{s}}}}\,. \tag{2.20}$$

The inhomogeneous term $\bar{f_0}^{\mathrm{NS}}(n)$ in (2.15) and (2.18) according to (2.16), (2.11)–(2.12) and (2.1)–(2.2) behaves as

$$\bar{f_0}^{\mathrm{NS}}(n) \sim \frac{\bar{\alpha_{\mathrm{s}}}}{n(n-0.5)} \tag{2.21}$$

for unpolarised structure functions and

$$\bar{f_0}^{\mathrm{NS}}(n) \sim \frac{\bar{\alpha_{\mathrm{s}}}}{n^2} \tag{2.22}$$

for the spin dependent ones. The anomalous dimension of the moment of the nonsinglet structure function γ from (2.19) has a (square root) branch point singularity at $n = n_0$. This gives the following behaviour of the nonsinglet structure functions at small x:

$$f^{\mathrm{NS}}(x,k^2) \sim x^{-n_0} \left(\frac{k^2}{k_0^2}\right)^{n_0/2} \tag{2.23}$$

and hence also

$$\frac{1}{x}F_2^{\rm NS}(x,k^2) \sim g_1^{\rm NS}(x,k^2) \sim x^{-n_0}\left(\frac{k^2}{k_0^2}\right)^{n_0/2} . \tag{2.24}$$

This low x shape $\sim x^{-n_0}$, where n_0 given in (2.20) is equal to 0.39 for $\bar{\alpha_{\rm s}} = 0.038$ ($\alpha_{\rm s} = 0.18$), remains nonleading in the case of the nonsinglet unpolarised structure function in comparison to the contribution of the nonperturbative A_2 Regge pole (2.1). In this way QCD perturbative singularity at small x generated by the double logarithmic $\ln^2 x$ resummation for nonsinglet unpolarised quark distributions is hidden behind the leading Regge contribution:

$$q^{\rm NS}(x,Q^2) \sim x^{-n_0} < x^{-\alpha_{A_2}(0)} \quad \alpha_{A_2}(0) = 0.5 . \tag{2.25}$$

Quite different situation occurs for the nonsinglet spin dependent functions, where the double logarithmic contribution becomes important:

$$\Delta q^{\rm NS}(x,Q^2) \sim x^{-n_0} > x^{-\alpha_{A_1}(0)} \quad \alpha_{A_1}(0) \leq 0 . \tag{2.26}$$

This takes place because the nonperturbative Regge part for spin dependent quark distributions involves a very low intercept $\alpha_{A_1}(0) \leq 0$. Small x behaviour $\sim x^{-2\sqrt{\bar{\alpha_{\rm s}}}}$ of the nonsinglet structure functions originating from double logarithmic $\ln^2 x$ resummation is a very interesting feature. Particularly for the polarised structure functions, where the double logarithmic analysis enables one to estimate of parton parametrisations at low x. In the next section we introduce the truncated moments method and combine it with the $\ln^2 x$ approach. This technique will give a novel advantage in the QCD perturbative analysis: enable one to avoid dealing with the unmeasurable $x \to 0$ region.

3. Truncated moments method within double logarithmic $\ln^2 x$ resummation for the nonsinglet structure functions

Truncated moments of parton distributions have been lately used in the LO and NLO DGLAP analysis [6]. Authors avoid in this way an extrapolation of well known quark distributions behaviour to the unmeasurable and unknown $x \to 0$ region. Apart from that they receive evolution equations for the truncated moments, in which n-th truncated moment couples only with $n + j$-th ($j \geq 0$) moments and the series of couplings is convergent, which ensures good accuracy. In our double logarithmic analysis we use truncated moments of the unintegrated function $f^{\rm NS}(x,k^2)$ and for fixed coupling $\bar{\alpha_{\rm s}}$ we can solve the suitable equation analytically. Let us shortly explain this treatment. The truncated n-th Mellin moment of the function $f^{\rm NS}(x,k^2)$ from (2.13)–(2.14) is defined as

$$\bar{f}^{\rm NS}(x_0,n,k^2) \equiv \int\limits_{x_0}^{1} dx x^{n-1} f^{\rm NS}(x,k^2)\,. \tag{3.1}$$

The evolution equation (2.9), generating double logarithmic terms $\ln^2 x$ in the truncated Mellin moment space takes a form

$$\bar{f}^{\rm NS}(x_0,n,k^2) = \bar{f_0}^{\rm NS}(x_0,n) + \bar{\alpha}_{\rm s} \int\limits_{k_0^2}^{k^2/x_0} \frac{dk'^2}{k'^2}$$
$$\times \int\limits_{x_0}^{1} dy y^{n-1} f^{\rm NS}(y,k'^2) \int\limits_{x_0/y}^{1} dz z^{n-1} \Theta\left(\frac{k^2}{k'^2} - z\right) \tag{3.2}$$

where $\Theta(t)$ is Heaviside's function

$$\Theta(t) = \begin{cases} 1 & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t \leq 0 \end{cases} \tag{3.3}$$

and we deal with the x-Bjorken region, where

$$x \geq x_0\,. \tag{3.4}$$

After taking into account the relations

$$\int\limits_{x_0/y}^{1} dz z^{n-1} \Theta\left(\frac{k^2}{k'^2} - z\right) = \frac{1}{n}\left[\Theta(k^2-k'^2) + \Theta(k'^2-k^2)\left(\frac{k^2}{k'^2}\right)^n - \frac{x_0^n}{y^n}\right];\, n \neq 0 \tag{3.5}$$

$$\int\limits_{x_0/y}^{1} dz z^{n-1} \Theta\left(\frac{k^2}{k'^2} - z\right) = \ln\frac{y}{x_0} + \Theta(k'^2-k^2)\ln\frac{k^2}{k'^2}; \qquad n = 0 \tag{3.6}$$

one can obtain from (3.2)

$$\bar{f}^{\rm NS}(x_0,0,k^2) = \bar{f_0}^{\rm NS}(x_0,0) + \bar{\alpha_{\rm s}}\left[\int\limits_{k^2}^{k^2/x_0} \frac{dk'^2}{k'^2}\ln\frac{k^2}{k'^2}\bar{f}^{\rm NS}(x_0,0,k'^2)\right.$$
$$\left. + \int\limits_{k_0^2}^{k^2/x_0} \frac{dk'^2}{k'^2}\int\limits_{x_0}^{1}\frac{dy}{y}\ln y f^{\rm NS}(y,k'^2) - \ln x_0 \int\limits_{k_0^2}^{k^2/x_0}\frac{dk'^2}{k'^2}\bar{f}^{\rm NS}(x_0,0,k'^2)\right] \tag{3.7}$$

and

$$\bar{f}^{\rm NS}(x_0,n,k^2) = \bar{f_0}^{\rm NS}(x_0,n) + \frac{\bar{\alpha_{\rm s}}}{n}\left[\int\limits_{k_0^2}^{k^2} \frac{dk'^2}{k'^2}\bar{f}^{\rm NS}(x_0,n,k'^2)\right.$$
$$\left. + \int\limits_{k^2}^{k^2/x_0} \frac{dk'^2}{k'^2}\left(\frac{k^2}{k'^2}\right)^n \bar{f}^{\rm NS}(x_0,n,k'^2) - x_0^n \int\limits_{k_0^2}^{k^2/x_0}\frac{dk'^2}{k'^2}\bar{f}^{\rm NS}(x_0,0,k'^2)\right];\, n\neq 0\,. \tag{3.8}$$

For $n \neq 0$ we get the following solution (for details see Appendix B)

$$\bar{f}^{\rm NS}(x_0,n,k^2) = \bar{f_0}^{\rm NS}(x_0,n)\left(\frac{k^2}{k_0^2}\right)^{\gamma}\frac{R}{1+(R-1)x_0^n}\,, \tag{3.9}$$

where

$$R \equiv R(n,\bar{\alpha_{\rm s}}) = \frac{n\gamma}{\bar{\alpha_{\rm s}}} \tag{3.10}$$

γ is given in (2.19) and $\bar{f_0}^{\rm NS}(x_0,n)$ is the inhomogeneous term, independent on k^2

$$\bar{f_0}^{\rm NS}(x_0,n) = \int\limits_{x_0}^{1} dx x^{n-1} f_0^{\rm NS}(x) = \frac{\bar{\alpha_{\rm s}}}{n}\int\limits_{x_0}^{1}\frac{dx}{x}(x^n - x_0^n)p_0(x)\,. \tag{3.11}$$

The input parton distribution $p_0(x)$ in the above formula denotes $q_0^{\rm NS}(x)$ for the unpolarised case or $\Delta q_0^{\rm NS}(x)$ for the polarised one, respectively. From (3.9) one can read that our solution for the truncated moment $\bar{f}^{\rm NS}(x_0,n,k^2)$ reduces to (2.18) when $x_0 = 0$ what must be, of course, fulfilled.

4. Some results for truncated moments $\bar{f}^{\rm NS}(x_0,n,k^2)$, $\bar{F_2}^{\rm NS}(x_0,n,k^2)$, $\bar{g_1}^{\rm NS}(x_0,n,k^2)$ in the double logarithmic $\ln^2 x$ approximation

The truncated n-th moment of the unintegrated nonsinglet function $f^{\rm NS}(x,k^2)$ is given by Eqs. (3.9)–(3.11). In Figs. 2–3 we plot the moments of $f^{\rm NS}(x,k^2)$ for different n as a function of x_0 at $k^2 = 10{\rm GeV}^2$. One can see that the ratio $p_f(x_0,n)$ defined as

$$p_f(x_0,n) \equiv \frac{\bar{f}^{\rm NS}(x_0,n,k^2)}{\bar{f}^{\rm NS}(0,n,k^2)} \tag{4.1}$$

becomes very large ($\simeq 1$) at $x_0 \approx 10^{-4}$ ($p_f(x_0 = 10^{-4}, n=1) = 0.997$).

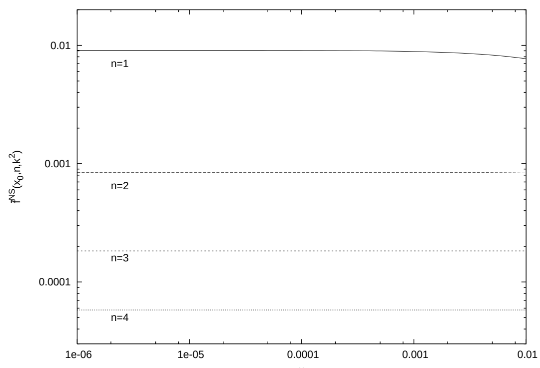

Fig. 2. Truncated Mellin moments $\bar{f}^{\rm NS}(x_0,n,k^2) = \int_{x_0}^{1} dx x^{n-1} f^{\rm NS}(x,k^2)$ as a function of x_0 for different n at $k^2 = 10{\rm GeV}^2$ in $\ln^2 x$ approach.

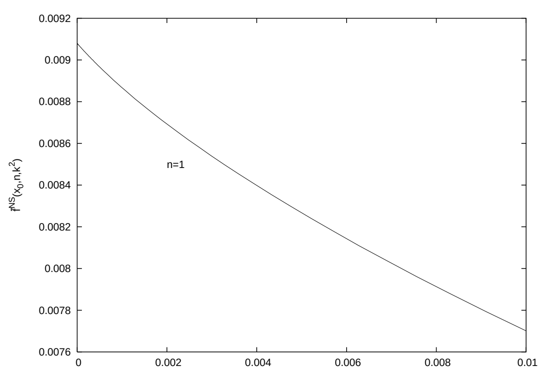

Fig. 3. The truncated Mellin moment $\bar{f}^{\rm NS}(x_0, n=1, k^2 = 10{\rm GeV}^2)$, similarly as in Fig.2 but in a linear scale.

This could be an advice that the truncated moments method is useful in the region $x \geq x_0 \sim 10^{-4}$ because at lower x the results for the full and the truncated moments are practically the same (at least for the Regge input). From the other side double logarithmic $\ln^2 x$ terms from our approach become important in the small x region $x \leq 10^{-2}$ [7], [9]. So taking into account both above facts one should choose the limit point x_0 in the truncated moments of $f^{\rm NS}$, $g_1^{\rm NS}$ or $F_2^{\rm NS}$ as

$$10^{-2} \geq x_0 \geq 10^{-4}\,. \tag{4.2}$$

Truncated moments of the nonsinglet quark distributions are related to the moments of $f^{\rm NS}$ via

$$\int\limits_{x_0}^{1} dx x^{n-1} g_1^{\rm NS}(x,k^2) = \int\limits_{x_0}^{1} dx x^{n-1} g_1^{0\rm NS}(x) + \int\limits_{k_0^2}^{k^2/x_0} \frac{dk'^2}{k'^2(1+\frac{k'^2}{k^2})} \bar{f}^{\rm NS}(x_0,n,k'^2) \tag{4.3}$$

for the spin dependent structure function and via

$$\int\limits_{x_0}^{1} dx x^{n-2} F_2^{\rm NS}(x,k^2) = \int\limits_{x_0}^{1} dx x^{n-2} F_2^{0\rm NS}(x) + \int\limits_{k_0^2}^{k^2/x_0} \frac{dk'^2}{k'^2(1+\frac{k'^2}{k^2})} \bar{f}^{\rm NS}(x_0,n,k'^2) \tag{4.4}$$

for the unpolarised case. This together with (3.9)–(3.11) gives the following formulae

$$\begin{aligned} I_1(x_0,n,k^2) \;\equiv\; & \int\limits_{x_0}^{1} dx x^{n-1} g_1^{\rm NS}(x,k^2) = \bar{g_1}^{0\rm NS}(x_0,n) \\ & + B(x_0,n,k^2)[\bar{g_1}^{0\rm NS}(x_0,n) - x_0^n \bar{g_1}^{0\rm NS}(x_0,0)]\,, \end{aligned} \tag{4.5}$$

$$\begin{aligned} I_2(x_0,n,k^2) \;\equiv\; & \int\limits_{x_0}^{1} dx x^{n-1} F_2^{\rm NS}(x,k^2) = \bar{F_2}^{0\rm NS}(x_0,n) \\ & + B(x_0,n+1,k^2)[\bar{F_2}^{0\rm NS}(x_0,n) - x_0^{n+1} \bar{F_2}^{0\rm NS}(x_0,0)]\,, \end{aligned} \tag{4.6}$$

where

$$B(x_0,n,k^2) = \frac{\gamma(\frac{k^2}{k_0^2})^{\gamma}}{1+(R-1)x_0^n} \int\limits_{\ln\frac{k_0^2}{k^2}}^{\ln\frac{1}{x_0}} dt \frac{{\rm e}^{\gamma t}}{1+{\rm e}^t}\,. \tag{4.7}$$

In Table I we collect results for different n-th moments of $F_2^{\rm NS}(x,k^2)$ and $g_1^{\rm NS}(x,k^2)$ functions obtained from (4.5)–(4.6). The moments are truncated at $x_0 = 10^{-1}, 10^{-2}, 10^{-3}$ and 10^{-4}. For all moments in Table I $k^2 = 10{\rm GeV}^2$, $\alpha_{\rm s} = 0.18$ and

$$\bar{F_2}^{0{\rm NS}}(x_0,n) \equiv \int\limits_{x_0}^{1} dx x^{n-1} F_2^{0{\rm NS}}(x)\,, \tag{4.8}$$

$$\bar{g_1}^{0{\rm NS}}(x_0,n) \equiv \int\limits_{x_0}^{1} dx x^{n-1} g_1^{0{\rm NS}}(x)\,. \tag{4.9}$$

TABLE I

Truncated moments $I_1(x_0,n,k^2) \equiv \int_{x_0}^1 dx x^{n-1} g_1^{\rm NS}(x,k^2)$ and $I_2(x_0,n,k^2) \equiv \int_{x_0}^1 dx x^{n-1} F_2^{\rm NS}(x,k^2)$ for different x_0 and n in the $\ln^2 x$ approach.

x_0	n	$I_1(x_0,n,k^2)$	$I_2(x_0,n,k^2)$
10^{-1}	1	0.144631	0.031977
	2	0.039952	0.010018
	3	0.014137	0.003945
	4	0.006101	0.001843
10^{-2}	1	0.219244	0.038540
	2	0.043831	0.010413
	3	0.014400	0.003975
	4	0.006124	0.001846
10^{-3}	1	0.229486	0.038791
	2	0.043892	0.010416
	3	0.014402	0.003975
	4	0.006125	0.001847
10^{-4}	1	0.230692	0.038801
	2	0.043894	0.010416
	3	0.014402	0.003975
	4	0.006125	0.001847

Input parametrisations $g_1^{0{\rm NS}}(x)$ and $F_2^{0{\rm NS}}(x)$ are chosen at $k_0^2 = 1{\rm GeV}^2$ in the simple Regge form

$$F_2^{0{\rm NS}}(x) = \frac{35}{96}\sqrt{x}(1-x)^3\,, \tag{4.10}$$

$$g_1^{0{\rm NS}}(x) = 0.838(1-x)^3 \tag{4.11}$$

which satisfy the sea flavour symmetric Gottfried [10] and Bjorken [11] sum rules

$$I_{\rm GSR} \;=\; \int\limits_0^1 \frac{dx}{x} F_2^{\rm NS}(x,k^2) = \frac{1}{3}\,, \tag{4.12}$$

$$I_{\rm BSR} \;=\; \int\limits_0^1 dx g_1^{\rm NS}(x,k^2) = \frac{1}{6} g_{\rm A} = 0.21\,, \tag{4.13}$$

where $g_{\rm A} = 1.257$ is the axial vector coupling. In order to determine the moment integrals we need knowledge of structure functions over the entire region of x. The small x behaviour of structure functions is driven by the double logarithmic $\ln^2 x$ terms. This $\ln^2 x$ approximation is, however, inaccurate in describing the Q^2 evolution for large values of x. Therefore, double logarithmic $\ln^2 x$ approach should be completed by LO DGLAP Q^2 evolution. Dealing with truncated moments of $f^{\rm NS}$ within unified $\ln^2 x$+LO resummation one encounters difficulties because suitable evolution equations are, of course, more complicated than in the pure double logarithmic approach presented here. Probably the only successful method to solve the evolution equations for truncated moments in $\ln^2 x$+LO treatment is the numerical calculus. We will discuss this in detail in the forthcoming paper. Now we are able to employ the truncated moment method in calculation of the contribution to moment integrals coming from the region of very small x. Using the definition of the truncated moment (3.1) we can in a very easy way find the double truncated moments of $f^{\rm NS}$, $g_1^{\rm NS}$ or $F_2^{\rm NS}$:

$$\int\limits_{x_1}^{x_2} dx x^{n-1} f^{\rm NS}(x,k^2) = \int\limits_{x_1}^{1} dx x^{n-1} f^{\rm NS}(x,k^2) - \int\limits_{x_2}^{1} dx x^{n-1} f^{\rm NS}(x,k^2) \tag{4.14}$$

etc. In this way for the partial Bjorken sum rule, according to (4.5), we have

$$\int\limits_{x_1}^{x_2} dx g_1^{\rm NS}(x,k^2=10) = I_1(x_1,1,10) - I_1(x_2,1,10)\,, \tag{4.15}$$

where x_1 and x_2 are both very small

$$x_1 \leq x_2 \leq 10^{-2}\,. \tag{4.16}$$

We have thus estimated a contribution

$$\Delta I_{\rm BSR}(x_1,x_2,k^2) = \int\limits_{x_1}^{x_2} dx g_1^{\rm NS}(x,k^2) \tag{4.17}$$

and found that $\Delta I_{\mathrm{BSR}}(10^{-4}, 10^{-2}, 10) = 0.011$, $\Delta I_{\mathrm{BSR}}(10^{-5}, 10^{-3}, 10) = 0.0013$ what is equal to around 5% and 0.6%, respectively, of the value of the full sum (4.13). In our above estimation we have assumed the simple Regge input parametrisation of $g_1^{0\mathrm{NS}}$ (4.11).

5. Summary and conclusions

In our paper we have derived the integral equation for truncated moments of the unintegrated nonsinglet structure function $f^{\mathrm{NS}}(x, Q^2)$ in the case of double logarithmic $\ln^2 x$ approximation. Analytical results at fixed α_{s} for truncated moments of F_2^{NS} and g_1^{NS} have been presented. We have received the clear, new solutions and what is important, in the limit $x_0 \to 0$ our calculations confirm the well known analytical result for full moments of $f^{\mathrm{NS}}(x, Q^2)$. The resummation of $\ln^2 x$ terms at small x goes beyond the standard LO (and even NLO) QCD evolution of spin dependent parton densities. In the case of unpolarised structure functions, the Regge behaviour, originating from the nonperturbative contribution is the leading one at small x. Therefore, the double logarithmic approximation is very important particularly for the polarised structure function g_1, which at low x is dominated just by $\ln^2 x$ terms. In our paper we have obtained analytical solutions for truncated Mellin moments of g_1^{NS} (and F_2^{NS} too). Dealing with truncated moments at x_0: $\int_{x_0}^{1} dx x^{n-1} g_1(x, Q^2)$ one can avoid uncertainty from the unmeasurable very small $x \to 0$ region. In this way the theoretical predictions of QCD analysis for structure functions at small x can be compared with experimental data without necessity to extrapolate results into nonavailable $x < x_0$ range. In our analysis for nonsinglet structure functions g_1^{NS} and F_2^{NS} we have found their truncated at x_0 moments, what could be helpful in the estimation of Bjorken (g_1^{NS}) and Gottfried (F_2^{NS}) sum rules. We have estimated the contribution to the Bjorken sum rule from the very small x region ($10^{-4} < x < 10^{-2}$) and found it to be around 5% of the value of the sum. Evolution equations, we have derived for truncated Mellin moments of $f^{\mathrm{NS}}(x, Q^2)$ generate correctly the leading small x behaviour but do not describe Q^2 evolution at large values of x. In the integrals for moments of structure functions one needs to have values of g_1 or F_2 from the entire range of x: $x_0 \leq x \leq 1$ so for large x one should include in the formalism LO Altarelli–Parisi (DGLAP) evolution. The aim of our paper was to focus attention on the truncated moments technique within $\ln^2 x$ approach itself. We found an analytical solution for the truncated moments of structure functions within $\ln^2 x$ approximation. This is an important stage before further investigations. The next step in the improvement of our analysis will be including LO DGLAP evolution into equations generating double logarithmic terms $\ln^2 x$ for truncated moments of the nonsinglet structure functions. This should give a proper determination of truncated and thus experimentally verifiable sum rules.

The work presented here is the result of discussions of one of the authors (D.K.) with Jan Kwieciński. He was, as usual, very helpful, patient and stimulating. We are greatly indebted to him for many years of teaching us QCD. It is painful to us to express our appreciation of the help we received from Jan Kwieciński after he is gone.

Appendix A

Analytical solution of the evolution equation for the full moments of the function $f^{NS}(x,k^2)$ generating double logarithmic $\ln^2 x$ effects at small x

After double differentiation of (2.15) with respect to $\ln k^2$ one obtains

$$\frac{\partial^2 \bar{f}^{\rm NS}(n,k^2)}{\partial t^2} = n\frac{\partial \bar{f}^{\rm NS}(n,k^2)}{\partial t} - \bar{\alpha_{\rm s}}\bar{f}^{\rm NS}(n,k^2) \tag{A.1}$$

with

$$t = \ln k^2 \,. \tag{A.2}$$

Hence function $\bar{f}^{\rm NS}(n,k^2)$ has a form

$$\bar{f}^{\rm NS}(n,k^2) = \left(\frac{k^2}{k_0^2}\right)^{\gamma} H(n,\bar{\alpha_{\rm s}})\,, \tag{A.3}$$

where $H(n,\bar{\alpha_{\rm s}})$ from initial conditions should be

$$H(n,\bar{\alpha_{\rm s}}) = \bar{f_0}^{\rm NS}(n)R(n,\bar{\alpha_{\rm s}}) \tag{A.4}$$

and γ is given by (2.19)–(2.20). After inserting the solution (A.3)–(A.4) into (2.15) one finds that R is connected with $\bar{\alpha_{\rm s}}$, n and γ via

$$R(n,\bar{\alpha_{\rm s}}) = \frac{n\gamma}{\bar{\alpha_{\rm s}}}\,. \tag{A.5}$$

Finally, the solutions (A.3)–(A.5) for n-th moment $\bar{f}^{\rm NS}(n,k^2)$ giving (2.18), imply via inverse Mellin transform

$$f^{\rm NS}(x,k^2) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} dn x^{-n}\bar{f}^{\rm NS}(n,k^2) \tag{A.6}$$

the following small x behaviour for $f^{\rm NS}(x,k^2)$

$$f^{\rm NS}(x,\tau) \sim \frac{\sqrt{2n_0}}{4\bar{\alpha}_{\rm s}\sqrt{\pi}}(n_0 \ln\tau + 1)(\frac{\tau}{x})^{n_0}(\ln\frac{\tau}{x})^{-3/2}\,, \tag{A.7}$$

where

$$\tau = \left(\frac{k^2}{k_0^2}\right)^{1/2} \tag{A.8}$$

and n_0 defined in (2.20) denotes branch point singularity of the anomalous dimension γ.

Appendix B

Analytical solution of the evolution equation for truncated moments of the function $f^{\rm NS}(x,k^2)$ generating double logarithmic $\ln^2 x$ effects at small x

From the definition of the n-th ($n \neq 0$) truncated moment of $f(x,k^2)$ (3.1) one can read the following relation

$$\frac{\partial \bar{f}^{\rm NS}(x_0,n,k^2)}{\partial x_0} = x_0^n \frac{\partial \bar{f}^{\rm NS}(x_0,0,k^2)}{\partial x_0}\,. \tag{B.1}$$

On the strength of this above relation we can replace $\bar{f}^{\rm NS}(x_0,0,k'^2)$ in (3.8) via

$$\bar{f}^{\rm NS}(x_0,0,k'^2) = x_0^{-n}\bar{f}^{\rm NS}(x_0,n,k'^2) - n\int\limits_{x_0}^{1}\frac{dy}{y^{n+1}}\bar{f}^{\rm NS}(y,n,k'^2) \tag{B.2}$$

and hence we get for (3.8)

$$\begin{aligned}\bar{f}^{\rm NS}(x_0,n,k^2) \;&=\; \bar{f_0}^{\rm NS}(x_0,n) + \frac{\bar{\alpha}_{\rm s}}{n}\int\limits_{k^2}^{k^2/x_0}\frac{dk'^2}{k'^2}\left[\left(\frac{k^2}{k'^2}\right)^n - 1\right]\bar{f}^{\rm NS}(x_0,n,k'^2)\\ &+\bar{\alpha}_{\rm s}x_0^n\int\limits_{k^2}^{k^2/x_0}\frac{dk'^2}{k'^2}\int\limits_{x_0}^{1}\frac{dy}{y^{n+1}}\bar{f}^{\rm NS}(y,n,k'^2)\,.\end{aligned} \tag{B.3}$$

Now we "guess" the solution of (3.8) in the form

$$\bar{f}^{\rm NS}(x_0,n,k^2) = \bar{f_0}^{\rm NS}(x_0,n)\left(\frac{k^2}{k_0^2}\right)^{\gamma} g(x_0,n,\bar{\alpha}_{\rm s})\,, \tag{B.4}$$

where we postulate the same anomalous dimension γ to be in agreement with the result for the full moment

$$\bar{f}^{\rm NS}(x_0 = 0, n, k^2) = \bar{f}^{\rm NS}(n, k^2)\,. \tag{B.5}$$

After insertion (B.4) into (B.3) we can find a simple form of the auxiliary function $g(x_0, n, \bar{\alpha}_{\rm s})$

$$g(x_0, n, \bar{\alpha}_{\rm s}) = \frac{n}{n - \gamma + \gamma x_0^n} \tag{B.6}$$

or equivalently

$$g(x_0, n, \bar{\alpha}_{\rm s}) = \frac{R}{1 + (R-1) x_0^n} \tag{B.7}$$

with $R = R(n, \bar{\alpha}_{\rm s})$ given by (3.10).

REFERENCES

[1] E.A. Kuraev, L.N. Lipatov, V. Fadin, *Zh. Eksp. Teor. Fiz.* **72**, 373 (1977) [*Sov. Phys. JETP* **45**, 199 (1977))]; Ya.Ya. Balitzkij, L.N. Lipatov, *Yad. Fiz.* **28**, 1597 (1978) [*Sov. J. Nucl. Phys.* **28**, 822 (1978))]; L.N. Lipatov, in *Perturbative QCD*, edited by A.H. Mueller, World Scientific, Singapore 1989, p.441; J.B. Bronzan, R.L. Sugar, *Phys. Rev.* **D17**, 585 (1978); T. Jaroszewicz, *Acta Phys. Pol. B* **11**, 965 (1980).

[2] M. Ciafaloni, *Nucl. Phys.* **B296**, 49 (1988); S. Catani, F. Fiorani, G. Marchesini, *Phys. Lett.* **B234**, 339(1990); *Nucl. Phys.* **B336**, 18 (1990); G. Marchesini, *Nucl. Phys.* **B445**, 49 (1995).

[3] J. Kwieciński, A.D. Martin, P.J. Sutton, *Phys. Rev.* **D52**, 1455 (1995).

[4] J. Kwieciński, *Acta Phys. Pol. B* **27**, 893 (1996).

[5] J. Bartels, B.I. Ermolaev, M.G. Ryskin, *Z. Phys.* **C70**, 273 (1996); J. Bartels, B.I. Ermolaev, M.G. Ryskin, *Z. Phys.* **C72**, 627 (1996).

[6] S. Forte, L. Magnea, *Phys. Lett.* **B448**, 295 (1999).

[7] B. Badełek, J. Kwieciński, *Phys. Lett.* **B418**, 229 (1998).

[8] P.D.B. Collins, *An Introduction to Regge Theory and High Energy Physics*, Cambridge University Press, Cambridge 1977.

[9] D. Kotlorz, *Acta Phys. Pol.* **B31**, 1721 (2000).

[10] K. Gottfried, *Phys. Rev. Lett.* **18**, 1174 (1967).

[11] J.D. Bjorken, *Phys. Rev.* **148**, 1467 (1966); *Phys. Rev.* **D1**, 1376 (1970).