

LÉVY STABLE PROCESSES. FROM STATIONARY TO SELF-SIMILAR DYNAMICS AND BACK. AN APPLICATION TO FINANCE*

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We employ an ergodic theory argument to demonstrate the foundations of ubiquity of Lévy stable self-similar processes in physics and present a class of models for anomalous and nonextensive diffusion. A relationship between stationary and self-similar models is clarified. The presented stochastic integral description of all Lévy stable processes could provide new insights into the mechanism underlying a range of self-similar natural phenomena. Finally, this effect is illustrated by self-similar approach to financial modelling.

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1. Introduction

Over the past decade there has been much interest in the asymptotic behaviour of dynamical systems, in particular in detecting self-similar character of these systems and testing for the existence of so called “long memory” or “long-range dependence”. It turns out that the self-similar processes are very important mathematical objects which can be used to model many physical phenomena (see [1–10] and references therein). After the first step made by Einstein and Smoluchowski who explained why the range reached by a Brownian particle is proportional to the square root of the movement duration, there were constructed many other self-similar processes including

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the most prominent examples: fractional Brownian, Lévy stable, and fractional Lévy stable motions [10–12]. However, as we will show here they are completely described by a deterministic kernel and the stochastic integral with respect to the Lévy symmetric α -stable process.

In Section 2 we discuss the canonical decomposition of H -self-similar Lévy symmetric α -stable processes. The aim is to show the structure of this class of self-similar processes. A variety of mathematical models for anomalous [2, 6] or nonextensive [13] diffusion and other physical processes [6, 7] is provided. To be more precise we develop tools for study of diffusion processes described by the following stochastic differential equation $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dZ_t^\alpha$, where dZ_t^α stands for the increments of Lévy α -stable motion Z_t^α , see [11]. Two basic examples: fractional Brownian motion and fractional Lévy stable motion are described first in the form of stochastic integrals. Next we discuss the integral representation of all Lévy stable self-similar processes in the language of nonsingular flows and exploit the connection with the Hopf decomposition. We identify the three components of the decomposition with mixed fractional motion, harmonizable and evanescent processes, respectively. The first process corresponds to a dissipative part and two other to a conservative part of the dynamics given by the nonsingular flow representing a Lévy stable and self-similar process. A number of special examples discussed in the paper demonstrate that the proposed integral representation is user-friendly.

In his pioneering paper [14] Lamperti defined a transformation $X(t) = t^H Y(a \log t)$ which changes stationary processes $Y(t)$ to the corresponding self-similar ones $X(t)$. In this context a question arises whether the transformations proposed by Lamperti are unique. In Section 3 we search for functions ϕ , ψ , ζ and η such that $X(t) = \phi(t)Y(\psi(t))$ is H -ss for a non-trivial stationary process Y , and $Y(t) = \zeta(t)X(\eta(t))$ is stationary for a non-trivial H -ss process X . We present two constructions in Section 3 which lead to the conclusion that essentially $\phi(t) = t^H$, $\psi(t) = a \log t$, $\zeta(t) = e^{-bHt}$ and $\eta(t) = e^{bt}$ for some $a, b \in \mathbf{R}$. In Section 3 also a visualization of the Lamperti transformation is provided. Next we study the influence of various a 's and b 's on distributions of corresponding processes. This is illustrated in Section 4 by four processes chosen to express a difference between the Gaussian and non-Gaussian case. As a result of this investigation, we construct, in a natural way, a pair of distinct S α S Ornstein–Uhlenbeck processes for $\alpha < 2$, already known in the literature [15].

In Section 5 we present a test on a DJIA index financial data which justifies using self-similar models as asset price processes. A modification of the Black–Scholes model is presented. The idea is to change, in the stochastic differential equation describing discounted stock prices process Z_t with respect to the reference measure Q , the differential dB_t to dM_t , where

M_t is a martingale generating the same filtration (history) as B_t^H and is well defined for $\frac{1}{2} < H < 1$. As a result of the investigation we obtain an option pricing formula which appears to be an extension of the Black–Scholes one for dependent stock returns. The differences are illustrated graphically. We hope that this type of modelling can be used not only in econophysics.

2. Stochastic representation

A process $\mathbf{X} = \{X(t)\}_{t \geq 0}$ is called self-similar [14] if for some $H > 0$,

$$X(at) \stackrel{d}{=} a^H X(t) \quad \text{for every } a > 0, \quad (1)$$

where $\stackrel{d}{=}$ denotes equality of all finite-dimensional distributions of the processes on the left and right. $\mathbf{X}$ is also called a H -self-similar process and the parameter H is called the self-similarity index or exponent. If we interpret t as “time” and X_t as “space” then (1) tells us that every change of time scale $a > 0$ corresponds to a change of space scale a^H . The bigger H , the more dramatic is the change of the space co-ordinate. Notice that (1), indeed, means a “scale-invariance” of all finite-dimensional distributions of $\mathbf{X}$. This property of a self-similar process does not imply the same for the sample paths. A convenient mathematical tool to observe self-similarity is provided by so-called quantile lines [11] which will be exploited in Section 3.

The fractional Brownian motion (fBm) has the integral representation

$$B_H(t) = \int_{-\infty}^{\infty} \left((t-u)_+^{H-1/2} - (-u)_+^{H-1/2} \right) B(du), \quad (2)$$

where $x_+ = \max(x, 0)$ and $B(du)$ is a symmetric Gaussian independently scattered random measure [12]. The classic Brownian motion $B(t)$, is simply a special case of fBm when $H = 1/2$. In this case $B((s, t]) = B(t) - B(s)$ and the above integral with the deterministic kernel has to be understood in the Itô sense.

The most commonly used extension of fBm to the α -stable case is the fractional Lévy stable motion (fLsm). The process $\mathbf{Z}_\alpha^H = \{Z_\alpha^H(t)\}_{t \in \mathbf{R}}$ is defined by the following integral representation

$$Z_\alpha^H(t) = \int_{-\infty}^{\infty} \left((t-u)_+^{H-\frac{1}{\alpha}} - (-u)_+^{H-\frac{1}{\alpha}} \right) Z_\alpha(du), \quad (3)$$

where Z_α is a symmetric Lévy α -stable independently scattered random measure [11, 12]. The integral is well defined for $0 < H < 1$ and $0 < \alpha \leq 2$ as a weighted average of the Lévy stable motion $Z_\alpha(u)$ over the infinite past.

This process is H -self-similar and has stationary increments. Let us observe that H -self-similarity follows from the above integral representation and the fact that the kernel is d -self-similar with $d = H - 1/\alpha$, when the integrator $Z_\alpha(du)$ is $1/\alpha$ -self-similar. This implies the following important relation $H = d + 1/\alpha$ [10].

The representation (3) of fLsm is similar to the representation (2) of fBm. Therefore fLsm reduces to fBm if one sets $\alpha = 2$. When we put $H = 1/\alpha$ we obtain the Lévy α -stable motion which is an extension of the Brownian motion to the α -stable case.

Now we exploit the connection between theory of self-similar Lévy stable processes and ergodic theory of nonsingular flows. We use the minimal integral representation of any H -self-similar Lévy symmetric α -stable (SaS) process $\{X_t\}_{t \in \mathbf{R}_+}$ of the form

$$X_t = \int_S t^H [a_t f \circ \phi_t] m_t^{1/\alpha} dM, \quad t \in \mathbf{R}_+. \quad (4)$$

Here SaS stands for the Lévy symmetric α -stable distribution of the process $\{X_t\}_{t \in \mathbf{R}}$, $\{\phi_t\}_{t \in \mathbf{R}_+}$ is a nonsingular multiplicative flow on (S, μ) , $\{a_t\}_{t \in \mathbf{R}_+}$ is a cocycle for this flow taking values in $\{-1, 1\}$, $m_t = d(\mu \circ \phi_t)/d\mu$, $f \in L^\alpha(S, \mu)$ and M is a SaS random measure [16].

The stochastic process X_t defined in (4) can be interpreted as a weighted average of the SaS random measure dM over the infinite past with the weight given by the kernel $f_t(u) = t^H [a_t f \circ \phi_t] m_t^{1/\alpha}$. Let us point out that formula (4) can be given in an equivalent form as the following stochastic differential $dX_t = f_t(u) dM(t)$, where $dM(t)$ corresponds to the increment of SaS-stable motion Z_t^α . Thus the stochastic integral (4) is equivalent to the diffusion without drift (*i.e.* $\mu = 0$) and with diffusion coefficient ($\sigma = f_t(\cdot)$). The self-similarity property of the above integral with parameter H , follows directly from $\frac{1}{\alpha}$ -self-similarity of the random measure M and the following property of the kernel $f_{ct}(u) = c^{H-1/\alpha} f_t(uc^{-1})$.

It follows from (4) that every measurable Lévy SaS self-similar process is generated by a nonsingular flow. The standard Hopf decomposition [17] of flows onto conservative and dissipative parts in ergodic theory induce natural decomposition of the Lévy stable self-similar processes. Consequently, every Lévy SaS self-similar process $\{X_t\}_{t>0}$ admits a unique decomposition into three independent parts

$$\{X_t\}_{t>0} \stackrel{d}{=} \{X_t^{(1)}\}_{t>0} + \{X_t^{(2)}\}_{t>0} + \{X_t^{(3)}\}_{t>0}, \quad (5)$$

where the first process on the right-hand side is a Mixed Fractional Motion (MFM), the second is harmonizable, and the third one is an H -ss evanescent process, see [16] for details.

3. The Lamperti transformation

We start from a illustration of the Lamperti transformation by demonstrating graphically the statistical behaviour of self-similar processes and corresponding stationary ones.

We generate the fractional S α S motion with parameters H and α , applying the algorithm presented in Janicki and Weron [11]. It is based on Maejima [18] who studied the domains of attraction of the fractional and log-fractional stable motions in terms of moving averages. They are examples of the first (conservative) part of the canonical decomposition given in formula (5). In Fig. 1 we can see four trajectories of the process (thin lines) for $\alpha = 1.6$ and $H = 0.8$. To give the insight view on the nature of the process, we follow [11]. We compute quantiles in the points of discretization for some fixed p ($0 < p < 0.5$), *i.e.* we compute $F^{-1}(p)$ and $F^{-1}(1-p)$, where F is the distribution function. Fig. 1 and Fig. 2 have the same graphical form of output. The thin lines represent four sample trajectories of the process. The thick lines stand for quantile lines, the bottom one for $p = 0.05$ and the top one for $1 - p = 0.95$. The lines determine the subdomain of $\mathbf{R}^2$ to which the trajectories of the approximated process should belong with probabilities 0.9 at any fixed moment of time. In Fig. 1 they are of the form $y = ct^H$, where $c = F^{-1}(p)$ is evaluated at $t = 1$, see formula (1). In Fig. 2 we can see the corresponding process obtained by the Lamperti transformation for the parameter $H = 0.8$. We can observe that now the quantile lines are “parallel”. This means they are time invariant, demonstrating the stationarity of the process.

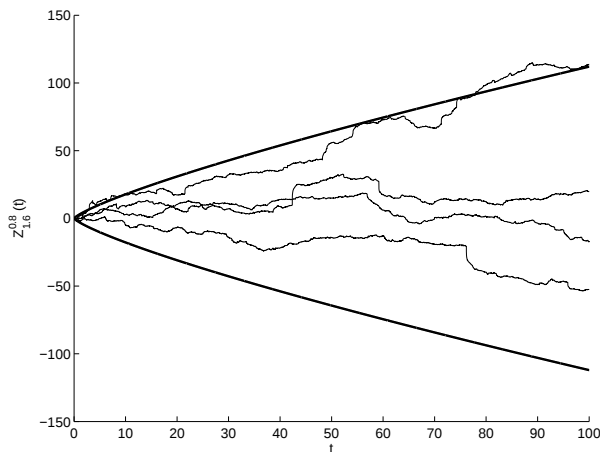


Fig. 1. Visualization of the fractional stable motion for $H = 0.8$ and $\alpha = 1.6$.

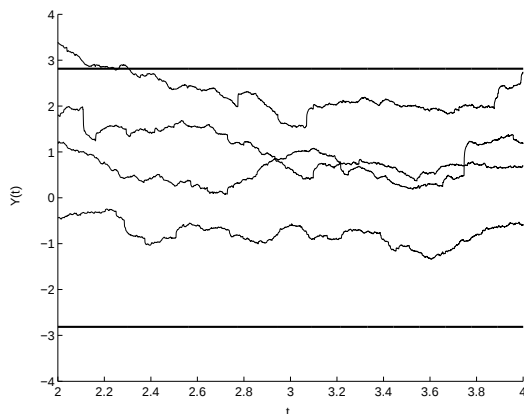


Fig. 2. Visualization of the stationary process obtained from the fractional stable motion (from Fig. 1) by the Lamperti transformation for $H = 0.8$.

When for two stochastic processes $X = (X(t))$ and $Y = (Y(t))$, $X(t) \stackrel{d}{=} aY(t)$ for some $a \in \mathbf{R} \setminus \{0\}$, we say that X and Y are *essentially equivalent*. Henceforth we will not distinguish between such processes. Furthermore, we will assume that all considered processes throughout this section are stochastically continuous.

Next we discuss the uniqueness of the generalized Lamperti transformations [19] leading from stationary to self-similar processes, and conversely.

Construction 1. (From stationary to self-similar dynamics)

Let $0 < H < \infty$.

(i) If $(Y(t))_{t \in \mathbf{R}}$ is a stationary process and $a \in \mathbf{R}$, then

$$X(t) = \begin{cases} t^H Y(a \log t), & \text{for } t > 0, \\ 0, & \text{for } t = 0 \end{cases} \quad (6)$$

is H -ss.

(ii) Conversely, if for some continuous functions ϕ, ψ on $(0, \infty)$ and for a non-trivial stationary process $Y = (Y(t))_{t \in \mathbf{R}}$,

$$X(t) = \begin{cases} \phi(t) Y(\psi(t)), & \text{for } t > 0, \\ 0, & \text{for } t = 0 \end{cases} \quad (7)$$

is H -ss, then $\phi(t) = t^H$ and $\psi(t) = a \log t$ for some $a \in \mathbf{R}$.

Construction 2. (From self-similar to stationary dynamics)

Let $0 < H < \infty$.

- (i) If $(X(t))_{t \geq 0}$ is an H -ss process and $b \in \mathbf{R}$, then $Y(t) = e^{-bHt}X(e^{bt})$, $t \in \mathbf{R}$, is stationary.
- (ii) Conversely, if for some continuous functions ζ, η , where η is invertible, and for a non-trivial H -ss process $(X(t))$, $Y(t) = \zeta(t)X(\eta(t))$, $t \in \mathbf{R}$, is stationary, then $\zeta(t) = e^{-bHt}$, and $\eta(t) = e^{bt}$ for some $b \in \mathbf{R}$.

We note here that the above construction can be repeated in more general setup giving the relationship between the classes of semi-self-similar processes and periodically distributed processes, see [1, 20, 21].

Let us observe that marginal distributions do not depend on the choice of a and b , that is, $X(t) = t^HY(a \log t) \stackrel{d}{\sim} t^HY(1)$ since Y is stationary, and $Y(t) = e^{-bHt}X(e^{bt}) \stackrel{d}{\sim} X(1)$ since X is H -ss. The parameters a and b are meaningful when considering finite-dimensional distributions, hence the influence of a and b will be discussed in the sequel. We want to establish the influence of a 's and b 's on distributions of the corresponding processes.

Proposition. (A correspondence principle)

Let $0 < H < \infty$.

- (i) If $Y = (Y(t))_{t \in \mathbf{R}}$ is a non-trivial stationary process and if for some $a, a' \in \mathbf{R} \setminus \{0\}$ $t^HY(a \log t) \stackrel{d}{=} t^HY(a' \log t)$, then either $a = a'$ or $a = -a'$.
- (ii) If $X = (X(t))_{t \geq 0}$ is a non-trivial H -ss process and if for some $b, b' \in \mathbf{R} \setminus \{0\}$ $e^{-bHt}X(e^{bt}) \stackrel{d}{=} e^{-b'Ht}X(e^{b't})$, then either $b = b'$ or $b = -b'$.

Part (i) follows directly from the fact that if $Y = (Y(t))_{t \in \mathbf{R}}$ is a non-trivial stationary stochastic process and if $Y(ct) \stackrel{d}{=} Y(t)$, for some $c \in \mathbf{R} \setminus \{0\}$, then either $c = -1$ or $c = 1$. In order to check (ii) it is enough to apply the same fact to $Y(t) = e^{-Ht}X(e^t)$.

Up to now we have considered processes merely assuming that they are stochastically continuous. In order to gain insight into the influence of different a 's and b 's on finite-dimensional distributions of corresponding processes we are to concentrate on α -stable processes. We will study Gaussian and non-Gaussian examples to take a different view of the foregoing results.

4. Ornstein–Uhlenbeck processes

Note that for Gaussian stationary processes $Y(t) \stackrel{d}{=} Y(-t)$. Hence if Y is Gaussian, then the statement (i) in the above proposition can be replaced by that $t^H Y(a \log t) \stackrel{d}{=} t^H Y(a' \log t)$ if and only if $a = \pm a'$, and if X is Gaussian, then (ii) can be replaced by that $e^{-bHt} X(e^{bt}) \stackrel{d}{=} e^{-b'Ht} X(e^{b't})$ if and only if $b = \pm b'$. Therefore we have the following.

Example 1 Let $0 < H < \infty$ and $(Y_\lambda(t))_{t \in \mathbf{R}}$ be a Gaussian Ornstein–Uhlenbeck process, namely

$$Y_\lambda(t) = \int_{-\infty}^t e^{-\lambda(t-x)} B(dx), \quad t \in \mathbf{R}, \quad (8)$$

where $B(t)$ is a standard Brownian motion. Then

$$t^H Y_\lambda(a \log t) \stackrel{d}{=} t^H Y_\lambda(a' \log t), \quad \text{for } t > 0 \text{ if and only if } a = \pm a'.$$

Example 2 Let $(X(t))_{t \geq 0}$ be a Gaussian H -ss process and $0 < H < 1$. (If, in addition, it has stationary increments, it is the fractional Brownian motion defined by the stochastic integral (2)). Then $e^{-bHt} X(e^{bt}) \stackrel{d}{=} e^{-b'Ht} X(e^{b't})$, for $t \in \mathbf{R}$, if and only if $b = \pm b'$.

Let us recall that the Gaussian Ornstein–Uhlenbeck process can be obtained by transforming the Brownian motion by the Lamperti transformation and there exists only one such process (this was observed by Doob and Itô [11]). How does this fact match the above results? Comparing the covariance functions we obtain that generalized Lamperti transformation with parameter a maps the Brownian motion $B(t)$ to the Gaussian Ornstein–Uhlenbeck process $Y_\lambda(at)$ (characterized by parameter λ , where $\lambda = \frac{1}{2}$).

Observe that $Y_\lambda(at)$ and $Y_\lambda(a't)$ are different processes when $a \neq \pm a'$ (with respect to finite-dimensional distributions) but nevertheless they are still in the same class of processes because $Y_\lambda(at) \stackrel{d}{=} \sqrt{a} Y_{a\lambda}(t)$, (see Example 1). Due to the above generalization of the Lamperti theorem we are able to obtain the complete class of Ornstein–Uhlenbeck processes from the standard Brownian motion.

Using the generalized Lamperti transformation with different a 's, one can generate the entire class of H -ss Gaussian Markov processes starting from the standard Ornstein–Uhlenbeck process with $\lambda = 1$, (see Example 1). They are given by the covariance function in the following way: $E[X(t)X(s)] = t^H s^H E[Y_1(a \log t)Y_1(a \log s)] = t^H s^H e^{-a(\log t - \log s)} = t^{H-a} s^{H+a}$, where $a > 0$ and $s < t$.

We proceed to non-Gaussian stable cases.

Example 3 Let $0 < H < \infty$ and $(Y_\lambda(t))_{t \in \mathbf{R}}$ be a S α S Ornstein–Uhlenbeck process, namely $Y_\lambda(t) = \int_{-\infty}^t e^{-\lambda(t-x)} Z_\alpha(dx)$, $t \in \mathbf{R}$ where $0 < \alpha < 2$. Then $t^H Y_\lambda(a \log t) \stackrel{d}{=} t^H Y_\lambda(a' \log t)$, for $t > 0$, if and only if $a = a'$.

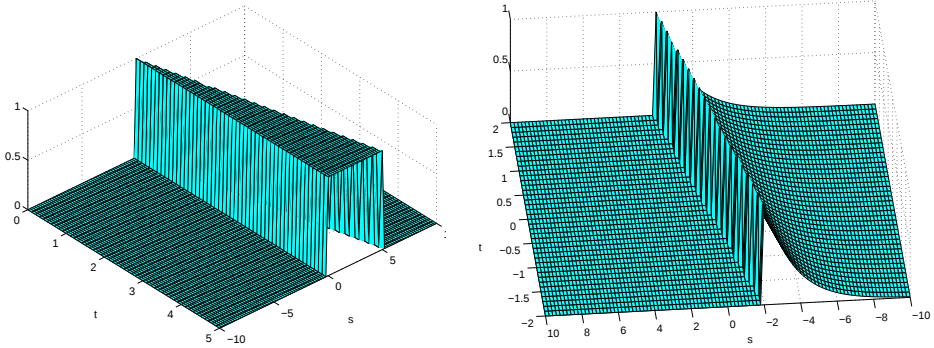


Fig.3. The kernel of the integral representation of the S α S Lévy motion with $\alpha = 1.6$ (left panel), the kernel of the corresponding stationary process through the Lamperti transformation for $H = 1/1.6$, i.e. the S α S Ornstein–Uhlenbeck process (right panel).

Example 4 Let $0 < \alpha < 2$, $H = \frac{1}{\alpha}$ and $(Z_\alpha(t))_{t \geq 0}$ be a S α S Lévy motion. Then $e^{-bHt} Z_\alpha(e^{bt}) \stackrel{d}{=} e^{-b'Ht} Z_\alpha(e^{b't})$, for $t \in \mathbf{R}$ if and only if $b = b'$.

By the proposition it is enough to show that $e^{-Ht} Z_\alpha(e^t) \stackrel{d}{\neq} e^{Ht} Z_\alpha(e^{-t})$, which is equivalent to $Z_\alpha(t) \stackrel{d}{\neq} t^{2H} Z_\alpha(t^{-1})$. For that, we show that the process on the right hand side does not have independent increments. To this end, it suffices to represent the process by a stable integral $t^{2H} \int_0^{t^{-1}} dZ_\alpha(u)$ and to check its increments. Use the fact that two non-Gaussian stable random variables $\int f dZ_\alpha$ and $\int g dZ_\alpha$ are independent if and only if $f \cdot g = 0$ a.e. (see [11, 12]).

As in the Gaussian case there is a correspondence between the S α S Lévy motion $Z_\alpha(t)$ (characterized by the parameter α) and the S α S Ornstein–Uhlenbeck process $Y_\lambda(at)$ (determined by α and λ , where $\lambda = \frac{1}{\alpha}$) through the generalized Lamperti transformation with parameter a . It is enough to compute and compare the characteristic functions of processes $\{e^{-at/\alpha} Z_\alpha(e^{at})\}$ and $\{Y_{1/\alpha}(at)\}$.

Contrary to the Gaussian case, $Y_\lambda(at)$ defines distinct processes for a and for $-a$ (see Example 3). This is a kind of symmetry breaking effect. For example, $a = 1$ and $a = -1$ produce the SaS Ornstein–Uhlenbeck and the reverse SaS Ornstein–Uhlenbeck process, respectively (which are different when $0 < \alpha < 2$), see [11, 12]. Since $Y_\lambda(at) \stackrel{d}{=} a^{1/\alpha} Y_{a\lambda}(t)$, so we can construct only two different Ornstein–Uhlenbeck processes.

5. Self-similar processes in financial modelling

A “self-similar” structure is one that looks the same on a small or a large scale. For example, share prices of stock when plotted against time have very much the same shape on a yearly, monthly, weekly and even on a daily basis. Brownian motion ($\frac{1}{2}$ -ss process) as a limit process is an unavoidable tool in finance. In his famous paper, Bachelier proposed Brownian motion as an appropriate model for pricing. More recently, in the traditional approach to contingent pricing, in the Black–Scholes model, the log-Brownian model for the movement of share prices was used. However it has been empirically demonstrated to be incorrect in a number of ways (stochastic volatility, volatility smile, *etc.*). Certain attempts have been made to replace Brownian motion by another self-similar process — α -stable Lévy motion; see [11] and [22]. Similarly, one can apply stationary Ornstein–Uhlenbeck processes since they correspond to the popular in finance Vasicek model of term structure. It is believed that, to some extent, such Lévy stable model would explain the large jumps which evidently occur in prices and which are caused by dramatic political or economic events [23]. Moreover, various alternatives have been suggested to account for empirically observed defiances, among them the fractional Brownian motion which displays dependence between returns on different days, in stark contrast to Brownian motion [24]. However, fBm is not a semimartingale (except in the Brownian case), and therefore there can be no equivalent martingale measure. Hence, by general results (*cf.* Rogers [25]) this leads to a conclusion that there must be arbitrage. This practically disqualifies the fBm model. Nonetheless, fBm has attracted some attention in mathematical finance [26].

Now we present a test on a DJIA index financial data which justifies using self-similar models as asset price processes. Some modification of the Black–Scholes model is presented. The new idea is to change, in the stochastic differential equation describing discounted stock prices process Z_t with respect to the reference measure Q , the differential $d\tilde{B}_t$ to dM_t , where M_t is a martingale generating the same filtration as B_t^H and is well defined for $\frac{1}{2} < H < 1$. As a result of the investigation we will obtain an option pricing formula which appears to be distinct from the Black–Scholes one. The differences are illustrated graphically.

We are going to apply a method from [27], which was called *variance-time plots*, for the DJIA index process. The method can be summarized as follows. Let $(X_t)_{t \geq 0}$ be an H -self-similar process with stationary increments. It is well known that if $EX_t^2 < \infty$ and $H \in (\frac{1}{2}, 1)$ then the increment process $(Y_k = (X_{k+1} - X_k) : k = 0, 1, \dots)$ exhibits long-range dependence. This means the time series Y_k has the autocovariance function of the form $r(k) = \text{Cov}(Y_0, Y_k) \sim_{k \rightarrow \infty} L_1(k)k^{2H-2}$, $H \in (\frac{1}{2}, 1)$, where $L_1(k)$ is a slowly varying function as $k \rightarrow \infty$. This property implies that the correlations are not summable and the spectral density has a pole at zero. More specifically, under suitable conditions on $L_1(\cdot)$, the spectral density has the property $f(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} r(k) \exp(-ikx) \sim_{|x| \rightarrow 0} L_2(x)|x|^{1-2H}$ for some $L_2(\cdot)$ that is slowly varying at the origin. The best known models of the above long-range dependence are the fractional Gaussian noise model and the fractional autoregressive integrated moving-average model (FARIMA). The parameter H describes the long-memory behaviour of the series. Now, for each $m = 1, 2, \dots$, let $(Y^{(m)} = (Y_k^{(m)}) : k = 1, 2, \dots)$ denote a new time series obtained by averaging the original series Y over nonoverlapping blocks of size m ; that is, for each $m = 1, 2, \dots$, $Y^{(m)}$ is given by $Y_k^{(m)} = 1/m(Y_{km-m+1} + \dots + Y_{km})$, $k = 1, 2, \dots$.

From a statistical point of view, the most salient feature of the process Y_k is that the variance of the arithmetic mean decreases more slowly than the reciprocal of the sample size; that is it behaves like n^{2H-2} for some $H \in (\frac{1}{2}, 1)$ instead of like n^{-1} for the processes whose aggregated series converge to a second-order pure noise. A specification of the autocovariance function $r(k)$ (or equivalently of the spectral density function $f(x)$) is the same as a specification of the sequence $(\text{Var}(Y^{(m)} : m \geq 1))$ with the property $\text{Var}(Y^{(m)}) \sim_{m \rightarrow \infty} am^{2H-2}$, where a is a finite positive constant independent of m , and $H \in (\frac{1}{2}, 1)$. On the other hand, for covariance stationary processes whose aggregate series $Y^{(m)}$ tend to second-order pure noise it is easy to see that the sequence $(\text{Var}(Y^{(m)} : m \geq 1))$ satisfies $\text{Var}(Y^{(m)}) \sim_{m \rightarrow \infty} bm^{-1}$, where b is a finite positive constant independent of m . Thus, for self-similar processes with stationary increments the variances of the aggregated processes $Y^{(m)}$, $m = 1, 2, \dots$, decrease linearly (for large m) in log-log plots against m with slopes arbitrary flatter than -1 . The so-called *variance-time plots* are obtained by plotting $\log(\text{Var}(Y^{(m)}))$ against $\log(m)$ ("time") and by fitting a line through the resulting points in the plane, ignoring the small values for m . Values of the estimate $\hat{H}$ of the asymptotic slope between -1 and 0 suggest self-similarity.

Example 5 Let us consider the DJIA index analysed from January 2, 1901 to December 31, 2000. We define Y_k 's as log-returns of the index. We normalize the data in order to set the variance of the process Y_k to 10. Figure 4 shows an asymptotic slope that is clearly different from -1 and is estimated to be about -0.92 , resulting in an estimate of the parameter H of about 0.54.

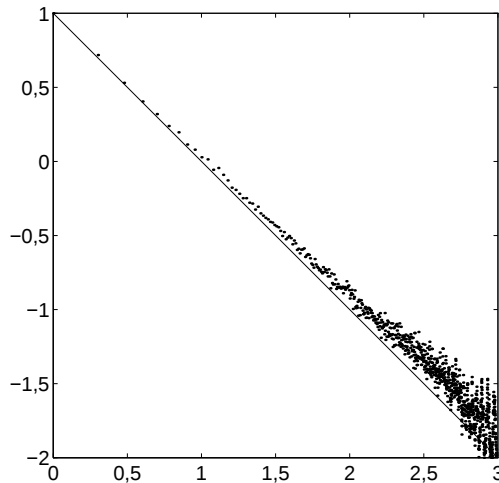


Fig. 4. Variance-time plot of the sequence of log-returns of DJIA index from January 2, 1901 to December 31, 2000.

Let B_t be, as usually, a standard (zero drift and unit variance) Brownian motion on some probability space (Ω, F, P) . Let r , μ and σ be real constants with $\sigma > 0$. A market in the classical Black–Scholes model is defined as a pair (A_t, S_t) , where $A_t = \exp(rt)$, $S_t = S_0 \exp(\sigma B_t + \mu t)$. Interpret A_t as the price at time t of a riskless bond and S_t as the price, in dollars per share, of a stock which pays no dividends. Furthermore r is called a fixed (riskless) interest rate, σ the volatility of the stock price process S_t and μ is his drift. Moreover, in the model, we assume a frictionless market with continuous trading, namely we demand that the two fundamental securities are traded continuously with no transaction costs with publically announced prices. Now we consider a ticket which entitles its bearer to buy one share of stock at the terminal date T , if he wishes, for a specified price of K dollars. This is a *European call option* on the stock, with *exercise price* K and *expiration date* T . It is easy to see that the call option is equivalent to a ticket which entitles a bearer to a payment of $X = (S_T - K)^+$ dollars at time T . Black and Scholes asserted that there exist a unique rational value

V for the option. Originally Black and Scholes obtained the price formula by solving a differential equation. Our approach to option pricing is based on a martingale method, which generalizes the ideas to arbitrary contingent pricing. The Black–Scholes formula in this approach is proved by considering so called *completeness* of the market, finding the *reference measure* Q via Girsanov theorem, asserting the measure is unique, due to the representation theorem for martingales, and computing $V = \exp(-rT)E^Q(X)$, where $X = (S_T - K)^+$.

In the Black–Scholes model stock price fluctuations are modelled by a stochastic differential equation with respect to the white noise dB_t , i.e. $dS_t = S_t(\sigma dB_t + (\mu + \frac{1}{2}\sigma^2)dt)$. If we introduce discounted stock price process $Z_t = \Lambda_t^{-1}S_t$, then applying Itô formula and Girsanov theorem for the Brownian motion we may claim the existence of a measure Q , such that $dZ_t = \sigma Z_t d\tilde{B}_t$ is a martingale under Q , where $\tilde{B}_t$ is a standard BM with respect to that measure.

Now the idea is to replace the process $\tilde{B}_t$ in the stochastic differential equation for discounted stock price Z_t by a process M_t , namely $M_t = \int_0^t c_1 s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H} dB_s^H$, where $c_1 = [H(2H-1)\mathcal{B}(\frac{3}{2}-H, H-\frac{1}{2})]^{-1}$, where $\mathcal{B}$ stands for the Beta function and $\frac{1}{2} < H < 1$.

First let us take a closer look at the properties of such defined process M_t . It turns out (see [28]) that M_t is a Gaussian martingale which generates the same filtration as B_t^H , with $EM_t = 0$ and the second moment $EM_t^2 = c_2^2 t^{2-2H}$, where $c_2 = (H(2H-1)(2-2H)\mathcal{B}(H-\frac{1}{2}, 2-2H))^{-1}$. Moreover, M_t is a $(1-H)$ -self-similar process of independent but not stationary increments, with continuous paths.

Thus we obtain $dZ_t = \sigma Z_t dM_t$. Since M_t is a martingale, the equation has a unique solution given by a stochastic exponential $Z_t = Z_0 \exp\{\sigma M_t - \frac{1}{2}c_2^2 \sigma^2 t^{2-2H}\}$, which is a martingale with continuous paths.

Now let us define $E_t = E(\Lambda_T^{-1}X|F_t)$. It is a martingale with respect to F_t (the filtration generated by M_t , so B_t^H). Nonetheless, since the martingale Z_t does not satisfy the representation theorem for martingales we can not guarantee the existence of a unique predictable process H_t such that $E_t = E_0 + \int_0^t H_s dZ_s$ for an arbitrary contingent claim X (the model is not complete). Thereby we can not construct an appropriate self-financing strategy. Nevertheless we may compute a non-arbitrage price $E(\Lambda_T^{-1}X)$ for a specific contingent claim X . Let us take as an example $X = (S_T - K)^+$. We are to compute $V^M = e^{-rT}E(S_T - K)^+$. Since we have $Z_t = e^{-rt}S_t$, the process S_t can be expressed as $S_t = S_0 \exp\{\sigma M_t + rt - \frac{1}{2}c_2^2 \sigma^2 t^{2-2H}\}$. Hence, it is enough to calculate $V^M = e^{-rT}E(S_0 \exp(Z + rT) - K)^+$, where $Z \sim N(-\frac{1}{2}c_2^2 \sigma^2 T^{2-2H}, c_2^2 \sigma^2 T^{2-2H})$.

Thus an European call option value in the model, driven by the martingale of fBm M_t , is given by

$$V^M = S_0 \Phi \left(\frac{\log \frac{S_0}{K} + rT + \frac{1}{2} c_2^2 \sigma^2 T^{2-2H}}{c_2 \sigma T^{1-H}} \right) - K e^{-rT} \Phi \left(\frac{\log \frac{S_0}{K} + rT - \frac{1}{2} c_2^2 \sigma^2 T^{2-2H}}{c_2 \sigma T^{1-H}} \right). \quad (9)$$

The formula we obtained is different from the Black–Scholes one, however it reduces to it when $H = 1/2$. It is not surprising as we are aware that the model we use in modelling stock prices has changed, *i.e.* we incorporated an additional parameter H — index of self-similarity.

Example 6 *We will compare the two formulas using the data from Example 5 in order to compute DJIA index options. Analysing the data we obtain that the estimated standard deviation $\hat{\sigma} = 0.010644$. We assume that $S_0 = 8000$ and the striking price $K = 7500 \dots 8500$. We consider index options on the interval $[0, 20 \text{ days}]$ and set the interest rate r to $0.05/365$. Furthermore, Example 5 justifies the parameter H equal to 0.54. Figure 5 depicts the difference between M price (obtained by the martingale M_t) and BS price.*

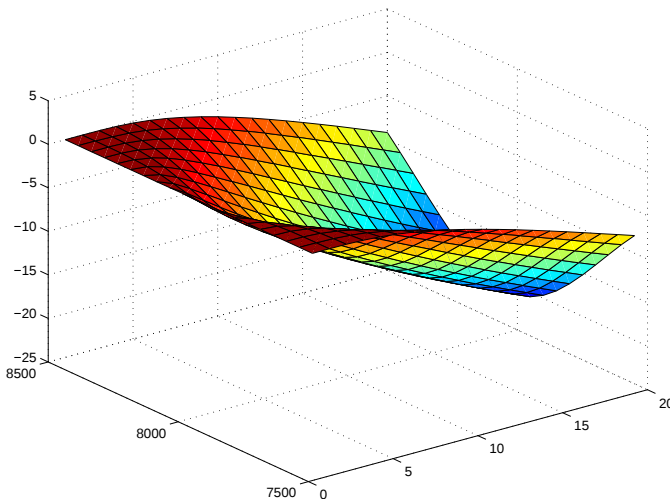


Fig. 5. M minus BS price for the DJIA index option.

In sum, we have just presented a martingale model based on a fractional Brownian motion, the model which stems from the classical Black–Scholes one. We may claim that despite of its disadvantages (further we can add an inevitable Gaussianity of the model) it possesses interesting features and the new parameter H provides additional information allowing to improve adjustment to the real-world phenomena.

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