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EXTREME-VALUE APPROACH TO THE TSALLIS' SUPERSTATISTICS*

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A novel approach to the Tsallis' superstatistics is discussed. On the basis of limit theorems of probability theory we have shown that the Tsallis generalization of the classical Boltzmann–Gibbs statistics can be represented by a distribution of an appropriately constructed scaled minima of a random variables sequence. This formalism provides a natural framework of construction of even more generalized statistics in which the Tsallis and the Boltzmann–Gibbs ones are special cases. It leads also to a new interpretation of the entropic index q.

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1. Introduction

Entropy is the fundamental concept of thermodynamics that connects the microscopic motion of particles to the macroscopic world [1]. Thus the Boltzmann–Gibbs–Shannon's expression

$$S = -k_{\mathrm{B}} \int_a^b \rho(x) \ln(\rho(x)) dx \qquad (1)$$

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is one of the most important formulas of the classical (extensive) statistical mechanics. According to the maximum entropy principle, given some partial information on a random variable, *i.e.*, information on related to it macroscopic observable, one finds the probability distribution which is consistent with that information but has otherwise a maximum uncertainty. As it is well-known, most of the probability distributions used in natural, biological, social, and economic sciences can be formally derived by maximizing entropy (1) with adequate constraints (see Ch.3 [2]). The maximum entropy probability density function (mepdf) depends on the choice of the limits of integration in (1) and on functions $g_i(x)$ whose expectation values are prescribed $\int_a^b g_i(x)\rho(x)dx < \infty$, $i = 1, 2....$ The pdf $\rho(x)$ that maximizes (1) subject to constraints: $\int_0^\infty \rho(x)dx = 1$ and $\int_0^\infty x\rho(x)dx = \frac{1}{B_0} < \infty$ has the corresponding distribution $F(x) = \int_0^x \rho(s)ds$ of an exponential form

$$1 - F(x) = G(x) = \mathrm{e}^{-B_0 x}, \tag{2}$$

which is known as the Boltzmann–Gibbs (BG) statistics, where the constant $\frac{1}{B_0}$ is an intensive physical quantity (*e.g.* temperature). However most distributions derived by maximizing (1) possess finite second moments, the heavy-tailed distributions like the Cauchy or Pareto ones are also available by imposing specific forms on the functions $g_i(x)$ [2].

The formalism of nonextensive statistical mechanics, as introduced by Tsallis in 1988 [3] and further developed by many others, has been proposed as a generalization of the classical approach with potential applications not only for equilibrium systems but also for nonequilibrium ones with a stationary state. It introduces a more general statistics which depends on an entropic index q and for $q = 1$ reduces to the ordinary BG form (2). The generalized statistics (or so-called superstatistics) has so far been observed to be relevant for three different classes of systems: for systems with long-range interactions [4–7], for multifractal systems [8, 9], and for systems with fluctuations of temperature or energy dissipation [10, 11].

The Tsallis' superstatistics follows from the extremization of the entropy

$$S_q = k_{\mathrm{B}} \frac{1 - \int_0^\infty [\rho_q(x)]^q dx}{q - 1}\,, \tag{3}$$

where q is the entropic index. In the limit of $q = 1$ the above formula leads to the classical entropy definition (1).

Subject to the normalization condition

$$\int_0^\infty \rho_q(x)dx = 1\,, \tag{4}$$

and to the finite q-expectation of the random variable X distributed according to $\rho_q(x)$ pdf

$$E_q\left[X\right] = \frac{\int\limits_0^\infty x\left[\rho_q(x)\right]^q dx}{\int\limits_0^\infty \left[\rho_q(x)\right]^q dx} < \infty \tag{5}$$

one obtains the mepdf $\rho_q(x)$ of the form known as the Tsallis' maximum-entropy density

$$\rho_q(x) = A\left(1 + Bx\right)^{-\frac{1}{q-1}} . \tag{6}$$

The positive constants are defined as $A = 1/E_q\left[X\right]$ and $B = \frac{q-1}{2-q}\left(1/E_q\left[X\right]\right)$.

The Tsallis' maximum-entropy density can be identified as the Pareto density supported on the positive half-line. It is defined in the range $1 < q < 2$. The properties of this density are determined by the asymptotic behavior of the tail of the corresponding distribution as $x \to \infty$. Namely, the n-th moment $E\left[X^n\right]$ of the random variable X, distributed according to (6), exists only if $\frac{2-q}{q-1} > n$ (and hence, for $1 < q < \frac{2+n}{n+1}$). The density $\rho_q(x)$ belongs to the domain of attraction of the completely asymmetric Lévy stable distributions if q exceeds the value $3/2$.

Although the applications of the Tsallis superstatistics are already numerous [4–18], the recent experimental data (see, *e.g.* [19]) provide evidence for a non-Tsallis superstatistics. Also, the theoretical studies [20, 21] clearly show a need for a new approach toward generalized statistics than that proposed by Tsallis.

In this paper, we propose a novel attempt to the Tsallis' superstatistics. By using of purely statistical arguments, we show that it can be derived in the framework of the extreme value theory [22]. This approach provides a natural way to construct a more generalized statistics in which the Tsallis and BG ones are special cases. It also allows us to find a new interpretation of the entropic index q.

2. The origins and consequences of the Tsallis statistics

The BG factor (2) can be generalized by taking average over the various β-parameter random fluctuations, *i.e.*,

$$G(x) = \left\langle \mathrm{e}^{-\beta x} \right\rangle_{f(b)} = \int\limits_0^\infty \mathrm{e}^{-bx} f(b) db \,. \tag{7}$$

As it has been already found, randomness of β may result either from local temperature fluctuations or fluctuations of an effective friction coefficient [10, 12] and the Tsallis statistics (6) is recovered as soon as the gamma distribution of the variable β is assumed. If the gamma pdf has the form

$$f(b) = \frac{1}{B\Gamma(c)} \left(\frac{b}{B}\right)^{c-1} \mathrm{e}^{-\frac{b}{B}} \tag{8}$$

we obtain [23]

$$G(x) = \int_0^\infty \mathrm{e}^{-bx} f(b)\, db = (1+Bx)^{-c} \,, \tag{9}$$

where $c = \frac{2-q}{q-1} > 0$ and $B = \frac{1}{c}$.

Such an approach to the Tsallis' statistics has been justified by considering the Langevin model (Brownian motion) with the parameters that are neither temporally nor spatially constant but have some random distributions. As a result one obtains $G(x)$ significantly different from the ordinary Boltzmann factor which is recovered itself for pdf of the random variable β being a Dirac delta function $f(b) = \delta(b - B_0)$. Unfortunately, the proposed models do not present satisfactory statistical reasons for appearing of the gamma distribution of the fluctuating parameter β [10, 12].

Let us introduce another derivation of the Tsallis statistics. It follows from the fact that the classical Boltzmann factor (2), as well as, the corresponding Tsallis statistics (9) has the meaning of the tail of the probability distribution of the random variable X [24]. More precisely,

$$\begin{aligned} 1 - F(x) = G(x) = \Pr\left(\frac{\Gamma_1}{\Gamma_c} > x\right) &= \int_0^\infty \Pr\left(\Gamma_1 > bx\right) f(b) db \\ &= \int_0^\infty \mathrm{e}^{-bx} f(b) db \,, \end{aligned} \tag{10}$$

where Γ_1 and Γ_c denote the exponential and gamma random variables, respectively.

On the basis of limit theorems [25] of probability theory we have that all the distributions of the ratio Γ_1/Y of independent, positive random variables are the asymptotic distributions of an appropriately constructed scaled minima of a random variables sequence. In order to find a scheme corresponding to the studied case, we consider a sequence of independent and identically distributed (i.i.d) positive random variables $\{Y_i\}_{1\leq i\leq N}$. When Y_i is exponentially distributed, *i.e.*, $Y_i = \Gamma_1$, then the random variable $S(N)$ given by expression $S(N) = [\min(Y_1, Y_2, \ldots, Y_N)]^{-1}$ fulfills the following relation

$$\begin{aligned}\Pr\left(\frac{N}{S(N)} > x\right) &= \Pr\left(\min(Y_1, Y_2, \ldots, Y_N) > \frac{x}{N}\right) \\ &= \prod_{i=1}^{N} \Pr\left(Y_i > \frac{x}{N}\right) = \Pr(\Gamma_1 > x)\end{aligned} \tag{11}$$

for each N. Hence

$$\Pr\left(\frac{S(N)}{N} \leq x\right) = \Pr\left(\frac{1}{\Gamma_1} \leq x\right). \tag{12}$$

We must notice also that negative-binomial random variable M_t, which density is of the form

$$\Pr(M_t = n) = \frac{\Gamma(n+c)}{n!\Gamma(c)} (p_t)^c (1-p_t)^n, \quad 0 < p_t < 1; \quad n = 0, 1, 2, \ldots \tag{13}$$

has a very useful property, namely, if $p_t \stackrel{t\to\infty}{\longrightarrow} 0$ then $p_t M_t \stackrel{t\to\infty}{\longrightarrow} \Gamma_c$. Indeed, the moment-generating function for the rescaled negative-binomial distribution

$$\begin{aligned}\phi(z)_{p_t NB} &= E\left[z^{p_t M_t}\right] = \sum_{k=1}^{\infty} \Pr(M_t = k) (z^{p_t})^k \\ &= \frac{p_t^c}{[1-(1-p_t)z^{p_t}]^c}, \qquad z = \mathrm{e}^{i\omega},\end{aligned} \tag{14}$$

gives for $t \to \infty$ characteristic function of gamma Γ_c random variable, *i.e.*, $(1-i\omega)^{-c}$. Now we can make use of the following theorem [25]:

Let us assume that $\{S(n)\}_{n\geq 1}$ is a sequence of the independent and identically random variables such that

$$\Pr\left(\frac{S(n)}{cn^{\rho}} \leq x\right) \to Y(x), \tag{15}$$

where $Y(x)$ is a limiting distribution for $n \to \infty$; $\{M_t\}_{t>0}$ is independent on $S(n)$ family of random variables which fulfills

$$\Pr\left(\frac{M_t}{dt^{\delta}} \leq x\right) \to Z(x) \tag{16}$$

with limiting distribution $Z(x)$ for $t \to \infty$, where c, ρ, d, δ are positive constants. Then for $t \to \infty$, we have

$$\Pr\left(\frac{S(M_t)}{cd^\rho t^{\rho\delta}} \leq x\right) \to \Pr\left(Z^\rho Y \leq x\right), \tag{17}$$

where Z and Y are independent random variables distributed with $Z(x)$ and $Y(x)$, respectively.

So, in particular, the Tsallis statistics (9) may be constructed as

$$(1+Bx)^{-c} = \lim_{t\to\infty} \Pr\left(p_t^{-1} \min\left(Y_1, Y_2, \ldots, Y_{M_t}\right) > x\right), \tag{18}$$

where M_t represents the random number of Gibbsian (exponential) contributions to the generalized statistics. The number M_t is distributed with the negative-binomial law (13) and $Y_1, Y_2, \ldots$ is the sequence of independent and exponentially distributed random variables. In fact, to derive the limit in (18) it is not necessary to know the detailed nature of the distribution of Y_i. The limit is determined only by the behavior of the distribution for small x. The necessary and sufficient condition reads

$$F_Y(x) = Pr\left(Y_i \leq x\right) \propto x \quad \text{for} \quad x \to 0\,. \tag{19}$$

Let us note that this condition is fulfilled if the distribution is of the exponential form $F_Y(x) = 1 - \mathrm{e}^{-x}$. If instead of (19), one imposes a more general, "fractal" condition

$$\Pr\left(Y_i \leq x\right) \propto x^\alpha \quad \text{for} \quad x \to 0 \quad \text{and} \quad \alpha > 0 \tag{20}$$

then the extreme-value scheme may lead to a generalization of the Tsallis statistics [26]. Namely, if the condition (20) is assumed then the limit minima is distributed with the Weibull law [22]

$$\lim_{N\to\infty} \Pr\left(N^{1/\alpha} \min\left(Y_1, Y_2, \ldots, Y_N\right) > x\right) = \mathrm{e}^{-A_0 x^\alpha}, \quad x \geq 0\,. \tag{21}$$

This result helps us to find a natural generalization of the Tsallis statistics (18) if $\alpha \in (0,1)$

$$\left[1 + \frac{1}{c}(A_0 x)^\alpha\right]^{-c} = \lim_{t\to\infty} \Pr\left(p_t^{-1/\alpha} \min\left(Y_1, Y_2, \ldots, Y_{M_t}\right) > x\right). \tag{22}$$

As it is easy to check, the new statistics in the limit case of $c \to \infty$ tends to the stretched exponential form $\exp[-(A_0 x)^\alpha]$, where A_0 is a positive constant. If, moreover, $\alpha \to 1$ we recover the BG statistics. The above result can be written as the Laplace transform

$$\left[1 + \frac{1}{c}(Ax)^\alpha\right]^{-c} = \int_0^\infty \mathrm{e}^{-bx} f_{\mathrm{ML}}\,(b)\, db\,, \tag{23}$$

of the pdf $f_{\rm ML}(b)$ which is known as the generalized Mittag–Leffler pdf and can be given in the series expansion only (for details see [26])

$$f_{\rm ML}(b) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(c+k)}{k!\Gamma(c)\Gamma[\alpha(c+k)]} \left(\frac{b}{A}\right)^{\alpha(c+k)-1} . \tag{24}$$

In general, the explicit forms of the superstatistics, resulting from the extreme value scheme, depends on the assumed properties of the distributions of the random variable Y_i and the number of contributions M_t.

The class of negative-binomial distributions has a long-time tradition in biological [27, 28] and physical modeling [29, 30]. In biology the negative-binomial distribution has been applied to model growth of populations and physical examples cover, *e.g.*, random walk models of the scattered electromagnetic field in granular materials or the clan structure of the high energy reactions. In general, the negative-binomial distribution is very often a first choice as an alternative [31] when it is felt that the counting Poisson distribution might be inadequate, *i.e.*, when the strict randomness requirements for the Poisson distribution, particularly independence, are not approximated sufficiently closely. Note, the Poisson distribution approaches the total number of successes in N ($N \to \infty$) independent trials, in each the probability of a success (an occurrence of a zero-one-type outcome) equals to $p \in [0, 1]$ and $Np \to$ const.

The model leading to the negative-binomial number of contributions may be a birth-and-death immigration model where $M_t = M(t)$ is a Markov process with the time parameter t being a continuous quantity, whereas the state space consists of non-negative integers. The state at time t is the value of $M(t)$ and the state changes can only happen between neighboring states. In particular, for the birth-and-death immigration process the transition probabilities in the time interval $(t, t + \Delta t)$ are given by the following conditions

$$\Pr(M(t+\Delta t) = n+1 | M(t) = n) = (\lambda n + \nu)\Delta t + o(\Delta t) , \tag{25}$$

$$\Pr(M(t+\Delta t) = n-1 | M(t) = n) = \mu n \Delta t + o(\Delta t) , \tag{26}$$

$$\Pr(M(t+\Delta t) = n \quad | M(t) = n) = [1 - ((\lambda+\mu)n + \nu)\Delta t] + o(\Delta t) , \tag{27}$$

and

$$\Pr(M(t+\Delta t) > n+1 \vee M(t+\Delta t) < n-1 | M(t) = n) = o(\Delta t) . \tag{28}$$

The positive constants λ and μ may be interpreted as the rates at which each member of the process creates "an offspring" and perishes, respectively, whereas the parameter $\nu > 0$ stands for the constant stream of "immigrants" who arrive independently of the actual state. From the conditions (25)–(28) one derives the reccurence formula

$$\begin{aligned} P_n(t+\Delta t) &= (\lambda(n-1)+\nu)P_{n-1}(t)\Delta t + [1-((\lambda+\mu)n+\nu)\Delta t]P_n(t) \\ &\quad +\mu(n+1)P_{n+1}\Delta t + o(\Delta t) \end{aligned} \tag{29}$$

that corresponds to the differential equation of the form

$$\frac{dP_n(t)}{dt} = (\lambda(n-1)+\nu)P_{n-1}(t) - [(\lambda+\mu)n+\nu]P_n(t) + \mu(n+1)P_{n+1}(t)\,, \tag{30}$$

where $P_n(t)$ is the probability of finding the system in the state $n = 0, 1, 2, \ldots$ at time t and it is understood that $P_{-1}(t) \equiv 0$. Multiplying by s^n and summing over n from 0 to ∞ yields

$$\frac{\partial \phi(s,t)}{\partial t} = (\lambda s - \mu)(s-1)\frac{\partial \phi(s,t)}{\partial s} + \nu(s-1)\phi(s,t)\,, \tag{31}$$

where $\phi(s,t)$ is the moment generating function of $M(t)$ (compare with (14)). As a solution, under the assumption that the process starts in state 0, *i.e.*, $P_0(0) = 1$, one obtains

$$\phi(s,t) = \left[\frac{p_t}{1-s(1-p_t)}\right]^{\nu/\lambda}\,, \tag{32}$$

where

$$p_t = \frac{\lambda-\mu}{\lambda \mathrm{e}^{(\lambda-\mu)t} - \mu}\,. \tag{33}$$

Resulting distribution is clearly negative binomial, with parameters p_t and ν/λ (not necessarily an integer). This yields the following formula for individual probabilities

$$\Pr(M(t) = n) = \frac{\Gamma(n+\nu/\lambda)}{n!\Gamma(\nu/\lambda)}(p_t)^{\nu/\lambda}(1-p_t)^n. \tag{34}$$

Let us note that for negligible rate $\mu \approx 0$ the model becomes linear birth-immigration one and still leads to the negative-binomial distribution, *i.e.*,

$$\Pr(M(t) = n) = \frac{\Gamma(n+\nu/\lambda)}{n!\Gamma(\nu/\lambda)}(\mathrm{e}^{-\lambda t})^{\nu/\lambda}(1-\mathrm{e}^{-\lambda t})^n. \tag{35}$$

Again, in the special case of $\mu \approx \lambda$, p_t reduces (on the basis of the de L'Hôspital rule) to $\frac{1}{1+\lambda t}$ which implies

$$\Pr(M(t) = n) = \frac{\Gamma(n+\nu/\lambda)}{n!\Gamma(\nu/\lambda)}\left(\frac{1}{1+\lambda t}\right)^{\nu/\lambda}\left(1-\frac{1}{1+\lambda t}\right)^n. \tag{36}$$

The birth-and-death immigration model provides a new interpretation of the index $c = \frac{2-q}{q-1}$ (appearing in (8) and (9)), namely $c = \nu/\lambda$ is the ratio of the state-independent immigration rate and the state-dependent birth one.

On the other hand, the negative-binomial distribution may be regarded as a discrete compound Poisson one. A random variable is said to be compound Poisson distributed if it is of the form $\sum_{i=0}^{\tilde{N}} \xi_i$, where $\xi_0, \xi_1, \ldots$ is a sequence of i.i.d. discrete random variables and $\tilde{N}$ is the Poisson random variable independent of the ξ_i's . In particular, for the number of "groups" (clusters) of individuals having a Poisson distribution with expected value φ and the number of individuals per group having the logarithmic series distribution [32]

$$\Pr(\xi_i = k) = -\frac{1}{\ln p}(1-p)^k/k\,, \quad k = 1, 2, \ldots, \quad 0 < p < 1 \tag{37}$$

the probability that the total number of individuals equals n is negative binomial and reads

$$\frac{\Gamma(c+n)}{n!\Gamma(c)} p^c (1-p)^n\,, \tag{38}$$

where $c = -\varphi/\ln p$. Such an interpretation of the negative-binomial law leads to the concept of the cluster system. Therefore, it seems natural to replace the continuous parameter t in (13) with the integer-valued system-size-dependent parameter N. The cluster interpretation of the negative-binomial law provides also another meaning of the parameter c connecting it with the measure of aggregation in the system. If we consider the probability $P_M(n)$ to have n particles belonging to M clusters we will obtain the recurrence relation of the form

$$\begin{aligned}
\frac{P_M(n)}{P_{M+1}(n)} &= \frac{\Pr\left(\sum_{i=0}^{M} \xi_i = n\right)\Pr(\tilde{M} = M)}{\Pr\left(\sum_{i=0}^{M+1} \xi_i = n\right)\Pr(\tilde{M} = M+1)} \\
&= \frac{\sum^* \Pr(\xi_1 = n_1, \ldots, \xi_M = n_M)}{\sum^* \Pr(\xi_1 = n_1, \ldots, \xi_{M+1} = n_{M+1})} \frac{M+1}{\varphi} \\
&= \frac{(-1/\ln p_M)^M}{(-1/\ln p_M)^{M+1}} \frac{\sum^* [(1-p_M)^{n_1}/n_1] \ldots [(1-p_M)^{n_M}/n_M]}{\sum^* [(1-p_M)^{n_1}/n_1] \ldots [(1-p_M)^{n_{M+1}}/n_{M+1}]} \frac{M+1}{\varphi} \\
&= \frac{-\ln p_M}{\varphi} \frac{\sum^* (n_1 \ldots n_M)^{-1}}{\sum^* (n_1 \ldots n_{M+1})^{-1}} (M+1) \\
&= c^{-1} \frac{\sum^* (n_1 \ldots n_M)^{-1}}{\sum^* (n_1 \ldots n_{M+1})^{-1}} \frac{(n!(M)!)^{-1}}{(n!(M+1)!)^{-1}}
\end{aligned} \tag{39}$$

which shows that c^{-1} may be regarded as an aggregation coefficient between clusters. The sum $\sum^*$ runs over all possible partitions of n, *i.e.*

$$\sum\nolimits^* = \sum_{n_1+\ldots+n_{M+1}=n;\quad n_i\geq 1} .$$

3. Conclusions

On the basis of limit theorems of probability theory we argued that the Tsallis generalization of the classical Boltzmann–Gibbs statistics can be represented by a distribution of an appropriately constructed scaled minima of a random variables sequence representing Gibbsian contributions to the superstatistics. The proposed formalism provides a natural framework of construction of the class of generalized statistics in which the Tsallis and the Boltzmann–Gibbs ones are special cases. As a consequence, the new interpretation of the entropic index q has been found. We have shown that $q = \frac{c+2}{c+1}$ as a function of the negative-binomial distribution parameter c is related to the number of random, exponentially distributed contributions to the effective statistics of a complex system. We have also shown that c^{-1} contains the information on the correlations that are characterized by aggregation of entities in the studied systems.

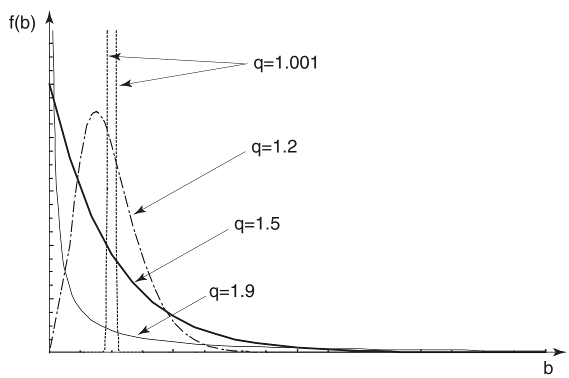

Fig. 1. The gamma probability density function for different values of the entropic index $1 < q < 2$. The density is unimodal with mode $\frac{3}{2-q}$ when $q < \frac{3}{2}$ and when $q > \frac{3}{2}$ it is monotone with an infinite peak at 0. For $q = \frac{3}{2}$ the gamma density becomes simply an exponential one.

The presented model introduces the gamma distribution of the parameter β in (7) as the limit of the negative-binomial distribution of the Gibbsian contributions. Thus the gamma pdf, underlying the Tsallis statistics, reflects the influence of the fluctuating number of random contributions to this statistics, each with property (19). The entropic parameter q changes the character of the gamma pdf (see Fig. 1). The gamma distribution tends to a degenerate one, *i.e.*, to the Dirac delta function as $q \to 1$. This corresponds to the situation when the number of contributions becomes deterministic and the random parameter β takes a particular value, say B_0, with probability 1.

Therefore, the presented analysis brings to light the origins of the randomness of the parameter β the averaging over which generates the superstatistics (23), in particular, the Tsallis' one (9). It also points on the role of random number of contributions as a main difference between the generalized statistics and the classical one.

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