MULTISCALE ANALYSIS OF EFFECTS OF ADDITIVE AND MULTIPLICATIVE NOISE ON DELAY DIFFERENTIAL EQUATIONS NEAR A BIFURCATION POINT*

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We study effects of noisy and deterministic perturbations on oscillatory solutions to delay differential equations. We develop the multiscale technique and derive amplitude equations for noisy oscillations near a critical delay. We investigate effects of additive and multiplicative noise. We show that if the magnitudes of noise and deterministic perturbations are balanced, then the oscillatory behavior persists for long times being sustained by the noise. We illustrate the technique and its results on linear and logistic delay equations.

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1. Introduction

Many physical phenomena are modeled by delay differential equations (DDE's); for example, these include lasers coupled face to face [1,2], optical communication systems [3], a system of coupled neurons [4], population dynamics [5], among others [6]. In such a setup, the value of the state variable at a previous moment of time affects the current rate of change and hence the current value of the state variable. Including the delay in the model often makes it more physically plausible. Solutions to DDE's exhibit many interesting properties; in particular, existence of periodic solutions makes an explanation of some experimentally observed phenomena possible when the oscillatory behavior is induced by a delay in the independent variable.

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Analysis of DDE's and stochastic DDE's (SDDE's) poses many challenges. Despite recent interest in SDDE's [7–15], many questions remain open. In particular, analytical tools and methods for such equations are not well established. The techniques developed for ordinary differential equations (ODE's) often do not apply to delay/functional equations. Various insights and intuition gained from ODE's cannot be easily translated into new techniques for DDE's since these equations and their periodic solutions do not exhibit symmetries present in ODE's. A classical nonlinear oscillator is described by a second order differential equation; on the other hand, a first order DDE may have periodic solution(s). A linear first order (backward) DDE is the simplest example of this class of equations, and plays a role similar to the harmonic oscillator in providing the understanding of basic properties and the development of new analytical methods.

In this paper, we consider the one-dimensional SDDE's

$$dx(t) = f(x(t), x(t-\tau))\, dt + \delta g(x(t))\, dw(t)\,, \tag{1}$$

where $\tau > 0$ is a delay parameter and $w(t)$ denotes standard Brownian motion. In particular, we take the linear f in (1)

$$dx(t) = (-\alpha x(t) + \beta x(t-\tau))\, dt + \delta g(x(t))\, dw(t)\,, \tag{2}$$

and the logistic equation

$$dx(t) = -r x(t)(1 + x(t-\tau))\, dt + \delta g(x(t))\, dw(t)\,. \tag{3}$$

We present a multiscale approach to analyze the effect of additive ($g(x(t)) \equiv 1$ in (1)–(3)) and multiplicative noise on a periodic solution to a SDDE. The assumption that the unperturbed system possesses a periodic solution is essential for our analysis. We derive dynamics of the perturbed oscillations under the assumption that the magnitude of the perturbations is small compared to the frequency of the periodic solution. Specifically, we discuss the case near the bifurcation point of the deterministic DDE; that is, in the absence of the perturbation, the delay parameter τ takes on a critical value τ_c for which the DDE has 2 linearly independent periodic solutions. The case of noisy DDE's away from the bifurcation point (*i.e.*, the case of noisy perturbations of an isolated periodic solution (a limit cycle) to a DDE) is to be presented elsewhere [16]. We illustrate this multiscale technique in detail for the linear DDE (2) and then show how it works in a nonlinear example of the logistic DDE (3).

We show how noisy and deterministic perturbations affect the amplitude of the oscillatory solution. If the strength of the noise and deviation of the delay parameter are balanced, then the oscillatory behavior of the solution persists for a long time. This fundamentally alters the dynamics of the system: in the absence of the noise, the exponential behavior governs the decay (or growth). Due to noise, the exponential decay is replaced by periodic behavior with stochastically varying amplitude. We derive the amplitude equations which combine effects of all perturbations. These equations are again SDDE's but with a small delay. (We note that the value of the delay parameter of the original problem is not assumed to be small.) The analysis of these equations shows that in the linear case the amplitude is a stationary process whose variance is proportional to the strength of the noise and inversely proportional to the proximity to the critical value of the delay parameter. In the nonlinear case, numerical calculations yield the same conclusion [17].

Previous studies of SDDE's in the physics literature have been limited to specific examples which include (but are not limited to) phase equations with specific forms of nonlinearity [14,15], systems with small delay [11], and population models which can be transformed to linear equations [10]. Tools used to study SDDE's combine linearization with numerical simulations [9, 10, 13, 15], or analytical approaches designed for particular problems and do not allow clear extensions [12,14]. In this paper, we outline the development of more systematic methods that can be applied in a general set-up.

Standard techniques used in studying SDE's based on generators and density evolution equations have limited applicability to the SDDE's. In Section 2, we show that the Fokker–Planck type equation for the transition probability density of the process defined by a SDDE involves a joint two-dimensional density calculated at times separated by the delay of the SDDE. In a nonlinear case, the solution to the Fokker–Planck equation for a SDE is not available in general. In the case of a SDDE, an additional difficulty stems from the fact that such a solution requires the nonlocal behavior of the density.

In the case of a linear SDDE, we can solve the evolution equation exactly. We derive the fundamental results on the covariance of the process given by a system of linear SDDE's by extending the results of [7].

In Section 3, we review standard multiscale techniques [18] and explain their applicability to our problem. Then we derive the amplitude equations for the linear SDDE's with additive and multiplicative noise. The latter case requires a generalization of projection techniques developed for the Duffing–van der Pol stochastic oscillator [19].

In Section 4, we illustrate this technique with the logistic DDE's. In Section 5, we summarize the results and discuss future directions of multiscale analysis of SDDE's.

2. Linear SDDE's: Are there any analogies to SDE's?

The theory of linear systems of SDE's has been well established for years [20]. Direct analysis of the SDE leads to rules for variance and covariance. The generator (the backward Kolmogorov operator) and the Fokker–Planck equation provide methods to study properties of the process given by a SDE. The form of the equivalence between a SDE's and its generator is well understood. A complete parallel theory for the SDDE's is still lacking. In this section we fill some of these voids.

2.1. The Fokker–Planck-like equation

In this section, we take the SDDE (2) with additive noise, $g \equiv 1$. The transition probability density function (tpdf) $p(x,t) \equiv \frac{d}{dx} P(x(t) \leq x | x_0(t), t \in (-\tau, 0])$ satisfies the integro-differential equation [10,11]

$$p_t(x,t) = -\frac{\partial}{\partial x}\left(\int f(x,z) p(x,t,z,t-\tau) dz\right) + \frac{1}{2}\sigma^2 p_{xx}(x,t)\,, \tag{4}$$

where $p(x,t,z,t-\tau) = \frac{\partial^2}{\partial x \partial z} P(x(t) \leq x, x(t-\tau) \leq z | x_0(t))$ is a 2-dimensional tpdf.

If the initial condition $x_0(t)$ is a deterministic function or a Gaussian stochastic process, and $f(x,z) = -\alpha x + \beta z$, then the solution to (4) is a Gaussian process whose tpdf $p(x,t)$ has the form

$$p(x,t,z,t-\tau) = \frac{1}{2\pi\sqrt{\det \boldsymbol{C}}} \exp\left\{-\frac{1}{2}[x,z]^T \boldsymbol{C}^{-1}[x,z]\right\}\,, \tag{5}$$

with the covariance matrix $\boldsymbol{C}$

$$\begin{aligned} \boldsymbol{C} &= \begin{bmatrix} \mathrm{cov}(x(t)x(t)) & \mathrm{cov}(x(t)x(t-\tau)) \\ \mathrm{cov}(x(t)x(t-\tau)) & \mathrm{cov}(x(t-\tau)x(t-\tau)) \end{bmatrix} \\ &\equiv \begin{bmatrix} a(t,t) & a(t,t-\tau) \\ a(t,t-\tau) & a(t-\tau,t-\tau) \end{bmatrix}. \end{aligned} \tag{6}$$

Substituting (6) into (5), we obtain

$$\frac{d}{dt} a(t,t) = 2[-\alpha a(t,t) + \beta a(t,t-\tau)] + \sigma^2. \tag{7}$$

Eq. (7) relates the variance with the covariance at the lag of the delay τ. Eq. (7) is a special case of (12), below; and hence the solution can be written as (11) with $t_1 = t_2 = t$. In Section 2.2, we derive general results on the covariance of the process defined by a SDDE.

Given an initial function $x_0(t), t \in (-\tau, 0]$, (4) reduces to the Fokker–Planck equation for $t \in (0, \tau]$. On successive time intervals of length τ the solution to (4) can be obtained by conditioning on a particular realization on a preceding time interval, and averaging over all possible paths. Such an approach requires further development. Here we focus on a different method based directly on the SDDE.

2.2. Correlations, stationary solutions

In this section, we consider a linear system of SDDE's

$$d\boldsymbol{X}(t) = \boldsymbol{P}\,\boldsymbol{X}(t)\,dt + \boldsymbol{Q}\,\boldsymbol{X}(t-\bar{\tau})\,dt + \boldsymbol{D}\,d\boldsymbol{w}(t)\,. \tag{8}$$

First we state general results on the covariance of the process $\boldsymbol{X}(t)$ in (8), and then in Section 3 we use these results to study the amplitude equation for the noisy oscillations.

We define by $\boldsymbol{\Phi}$ the fundamental solution to the deterministic system

$$\begin{aligned} \frac{d}{dt}\boldsymbol{\Phi}(t) &= \boldsymbol{P}\,\boldsymbol{\Phi}(t) + \boldsymbol{Q}\,\boldsymbol{\Phi}(t-\bar{\tau})\,, \\ \boldsymbol{\Phi}(0) &= \boldsymbol{I}, \quad \boldsymbol{\Phi}(t) = \boldsymbol{0} \text{ for } t < 0\,. \end{aligned} \tag{9}$$

Explicitly, we write

$$\boldsymbol{\Phi}(t) = \begin{cases} \sum_{j=0}^{J} \mathrm{e}^{\boldsymbol{P}(t-j\bar{\tau})}\,\boldsymbol{Q}^j \frac{(t-j\bar{\tau})^j}{j!} & t \geq 0 \\ 0 & t < 0 \end{cases}\,, \tag{10}$$

where $J \equiv \max\{j : t - j\bar{\tau} \geq 0\}$. Then the covariance $\boldsymbol{K}(t_1, t_2)$ of the process $\boldsymbol{X}(t)$ is given by

$$\boldsymbol{K}(t_1,t_2) = \int_0^{\min(t_1,t_2)} \boldsymbol{\Phi}(t_1-s)\boldsymbol{D}\,\boldsymbol{D}^T\boldsymbol{\Phi}^T(t_2-s)\,ds + \boldsymbol{\Phi}(t_1)\mathrm{cov}(\boldsymbol{X}(0),\boldsymbol{X}(0))\,\boldsymbol{\Phi}^T(t_2)\,. \tag{11}$$

In particular, if $t_1 = t_2 = t$, then from (11) we obtain

$$\frac{d}{dt}\boldsymbol{K}(t,t) = \boldsymbol{D}\,\boldsymbol{D}^T + \boldsymbol{P}\,\boldsymbol{K}(t,t) + \boldsymbol{Q}\,\boldsymbol{K}(t-\bar{\tau},t) + \boldsymbol{K}(t,t)\boldsymbol{P}^T + \boldsymbol{K}(t,t-\bar{\tau})\,\boldsymbol{Q}^T. \tag{12}$$

If all roots to the characteristic equation

$$\det(\boldsymbol{P} + \boldsymbol{Q}\mathrm{e}^{-\lambda\bar{\tau}} - \lambda\boldsymbol{I}) = 0 \tag{13}$$

have negative real parts, then the process $\boldsymbol{X}(t)$ is stationary (if an initial condition $\boldsymbol{X}(0)$ is suitably chosen or as $t \to \infty$), so that $\boldsymbol{K}(t_1, t_2) = \boldsymbol{K}(t_1 - t_2)$ and from Eq. (12) (since $\boldsymbol{K}(t_1 - t_2) = \boldsymbol{K}^T(t_2 - t_1)$) we obtain

$$\boldsymbol{D}\,\boldsymbol{D}^T + \boldsymbol{P}\,\boldsymbol{K}(0) + \boldsymbol{Q}\,\boldsymbol{K}^T(\bar{\tau}) + \boldsymbol{K}(0)\boldsymbol{P}^T + \boldsymbol{K}(\bar{\tau})\,\boldsymbol{Q}^T = 0\,. \tag{14}$$

The nonnegative definite solution of (14) can be written as

$$\boldsymbol{K}(u) = \int_0^\infty \boldsymbol{\Phi}(u+s)\boldsymbol{D}\,\boldsymbol{D}^T\,\boldsymbol{\Phi}^T(s)ds\,. \tag{15}$$

Since $\boldsymbol{\Phi}(t)$ satisfies Eq. (9), it follows that $\boldsymbol{K}(u)$ in Eq. (15) can be written as

$$\frac{d}{du}\boldsymbol{K}(u) = \boldsymbol{P}\,\boldsymbol{K}(u) + \boldsymbol{Q}\,\boldsymbol{K}(u-\bar{\tau}) \quad \text{for } u \geq 0\,. \tag{16}$$

In particular, from Eqs. (14) and (16), we obtain

$$\frac{d}{du}\boldsymbol{K}(0) = -\frac{1}{2}\boldsymbol{D}\,\boldsymbol{D}^T\,, \tag{17}$$

while from Eqs. (14) and (16), for $u \in [0, \bar{\tau}]$ we have

$$\frac{d^2}{du^2}\boldsymbol{K}(u) = \boldsymbol{P}\boldsymbol{K}(u)\boldsymbol{P}^T - \boldsymbol{Q}\boldsymbol{K}(u)\boldsymbol{Q}^T + \boldsymbol{P}\frac{d}{du}\boldsymbol{K}(u) - \frac{d}{du}\boldsymbol{K}(u)\boldsymbol{P}^T\,. \tag{18}$$

We note that in the case when $\boldsymbol{Q} \equiv \boldsymbol{0}$, Eqs. (12) and (14), and (16) reduce to the well known results for linear stochastic systems [20]. In the case of a linear SDDE, Eqs. (17) and (18) reduce to results obtained in [7].

To calculate the variance of the process $\boldsymbol{X}(t)$, it is enough to solve a system of second order linear differential equations (18) with initial conditions (14) and (17). Longer than $\bar{\tau}$ time correlation involves the fundamental solution (10) through (15).

3. Multiscale approach to noisy linear DDE

We begin with the linear model, considering both additive and multiplicative noise. The underlying general assumption states that the unperturbed system exhibits a periodic solution. To capture the time evolution of this periodic solution under noisy and deterministic perturbations we extend the classical multiscale technique to SDDD's. We develop and illustrate this approach on a linear SDDE's and then in Section 4 we extend it to a nonlinear problem. Our multiscale analysis is not limited to small delays.

3.1. Classical multiscale method

In the setting of a Hopf bifurcation, a multiscale approximation explicitly employs the natural frequency, say ω, of the oscillation of the unperturbed system [18]. To capture the behavior of the solution near the bifurcation point, the following form of the solution is postulated:

$$x \sim A(T)\cos\omega t + B(T)\sin\omega t, \quad T = \varepsilon^2 t\,, \tag{19}$$

where ε^2 is the parameter measuring the proximity to the bifurcation. Here $A(T)$ and $B(T)$ are functions of a slow time T and they are treated as constants with respect to the fast oscillations with frequency ω on the t time scale. The approximate behavior of the solution is captured in the amplitude (envelope) equations for $A(T)$ and $B(T)$.

The method treats x as a function of two independent times t and T, $x = x(t, T)$. Then a perturbation expansion $x \sim x_0 + \varepsilon x_1 + \ldots$ is used, with x_0 given by (19) and derivative x_t replaced by $x_t + \varepsilon^2 x_T$. Proceeding with the perturbation expansion, the equation for the higher order contributions x_j for $j > 0$ are subject to solvability conditions, which give envelope equations for $A(T)$ and $B(T)$. These solvability conditions are often in the form of conditions of orthogonality to the oscillatory modes $\cos\omega t$ and $\sin\omega t$. The benefits of analyzing the envelope equations are that they are often relatively simple compared to the original model and that they allow an analysis or computation on the long time scale.

In the context of DDE's, the applicability of the multiscale method can be easily seen in the example of a first order differential equation

$$\frac{dx}{dt} = -\alpha x(t) + \beta x(t-\tau)\,. \tag{20}$$

If the delay τ takes on a critical value τ_c, then (20) has periodic solutions $\{\cos bt, \sin bt\}$ where $b = \sqrt{\beta^2 - \alpha^2}$, $\beta\cos b\tau_c = \alpha$, and $b = -\beta\sin b\tau_c$, so that $x(t) = \tilde{A}\cos bt + \tilde{B}\sin bt$ (where $\tilde{A}$ and $\tilde{B}$ are arbitrary constants). If $\tau < (>)\tau_c$ then the oscillatory solution to (20) decays to zero (grows to infinity) at an exponential rate. Specifically, if $\tau = \tau_c + \varepsilon^2\tau_2$ then the exact solutions to (20) behaves like $\{\exp(\lambda\varepsilon^2 t)\cos bt, \exp(\lambda\varepsilon^2 t)\sin bt\}$ where λ is given in (31). That is, we write $x = x(t,T) = \tilde{A}\exp(\lambda T)\cos bt + \tilde{B}\exp(\lambda T)\sin bt \equiv A(T)\cos bt + B(T)\sin bt$. Hence, the amplitude evolves on the slow time scale $T = \varepsilon^2 t$ and the exact solution to (20) has the form (19). The bifurcation parameter measures the proximity of the delay τ to its critical value τ_c and the decay or growth rate is determined by the sign of τ_2.

3.2. Additive noise

We first consider a linear SDDE

$$dx = (-\alpha x(t) + \beta x(t-\tau))dt + \delta dw\,. \tag{21}$$

Here w is Brownian motion and we take $\delta \ll 1$, in order to examine sensitivity to small noise. We assume the parameter values to be such that the system is just below the threshold for growth or decay of oscillatory solutions such as $x = \cos bt$. Without loss of generality, we define the threshold as $\tau_{\rm c}$, keeping the other parameters fixed. We set $\tau = \tau_{\rm c} + \varepsilon^2\tau_2$; the parameter ε measures the proximity to this threshold, and we assume that $1 \gg \varepsilon > 0$.

In order to capture the influence of the noise over a long time, we seek a periodic solution which has an amplitude that varies stochastically on a slow time scale $T = \varepsilon^2 t$. The choice of slow time is motivated by the form of the exact solution to the linear deterministic equation (20). We postulate the form (19) for the solution to (21), but now $A(T)$ and $B(T)$ evolve in time stochastically, and we assume their form as

$$\left[\begin{array}{c} dA \\ dB \end{array}\right] = \left[\begin{array}{c} \psi_A \\ \psi_B \end{array}\right] dT + \boldsymbol{\sigma} \left[\begin{array}{c} d\xi_A(T) \\ d\xi_B(T) \end{array}\right]. \tag{22}$$

Here $d\xi_A(T)$ and $d\xi_B(T)$ denote independent white noises and $\boldsymbol{\sigma}$ is a diffusion matrix. The relation between the noises $d\xi_A(T)$ and $d\xi_B(T)$ in the ansatz (22) and the noise $dw(t)$ in the original equation (21) is going to be determined in the ensuing analysis. The same analysis will determine the form of the drift coefficients ψ_A and ψ_B. In this section we postulate that $\boldsymbol{\sigma}$ is a diagonal matrix with unknown entries on the diagonal, σ_A and σ_B.

First, we calculate the differential dx on the left-hand side of (21) using Itô's formula and the ansatz (19) with (22). Then, on the right-hand side of (21), we substitute the assumed form of the solution and compare the two representations of the solution.

From Itô's formula we obtain

$$dx = \frac{\partial x}{\partial t}dt + \frac{\partial x}{\partial A}dA + \frac{\partial x}{\partial B}dB + \frac{\sigma_A^2}{2}\frac{\partial^2 x}{\partial A^2}dT + \frac{\sigma_B^2}{2}\frac{\partial^2 x}{\partial B^2}dT\,. \tag{23}$$

In the linear case, the second derivatives of x with respect to A and B vanish, and with the representation (22) of dA and dB (23) becomes

$$\begin{aligned} dx \;&=\; (-bA(T)\sin bt + bB(T)\cos bt)dt + (\psi_A\cos bt + \psi_B\sin bt)dT \\ &\quad +\sigma_A\cos bt d\xi_A(T) + \sigma_B\sin bt d\xi_B(T)\,. \end{aligned} \tag{24}$$

Second, substituting (19) into the right-hand side of (21) gives

$$\begin{aligned} dx \;&=\; [-\alpha(A(T)\cos bt + B(T)\sin bt) \\ &\quad + \beta\left\{A(T-\varepsilon^2\tau)(\cos bt\cos b\tau + \sin bt\sin b\tau)\right. \\ &\quad \left.+B(T-\varepsilon^2\tau)(\sin bt\cos b\tau - \cos bt\sin b\tau)\right\}\big]\, dt + \delta dw(t)\,. \end{aligned} \tag{25}$$

Now we equate these two expressions (24) and (25) for dx, and thus determine the coefficients ψ_A, ψ_B, σ_A, and σ_B. To account for the noise effects, we rewrite the Brownian motion as

$$dw(t) = \cos bt\, dw_1(t) + \sin bt\, dw_2(t)\,, \tag{26}$$

where $dw_j(t)$, $j = 1, 2$ are independent Brownian motions. To account for the time evolution of the amplitude equations (22) on the T time scale we change the time scale in the white noises $dw_j(t) = dw_j(T)/\varepsilon$ ($j = 1, 2$) in (26), and consequently in (25).

We neglect the $O(\varepsilon^4)$ terms and obtain the drift and diffusion coefficients ψ_A, ψ_B, σ_A and σ_B in the equations for A and B by projecting these equations onto $\cos bt$ and $\sin bt$, while treating functions of T as independent of t. We find $\sigma_A = \sigma_B = \delta/\varepsilon$, $dw_1(T) = d\xi_A(T)$, $dw_2(T) = d\xi_B(T)$, and

$$\begin{aligned}
\psi_A &= b\tau_2\left(-\alpha B(T) + bA(T)\right) \\
&\quad + \frac{B(T-\varepsilon^2\tau) - B(T)}{\varepsilon^2} + \alpha\frac{A(T-\varepsilon^2\tau) - A(T)}{\varepsilon^2}\,, \\
\psi_B &= b\tau_2\left(\alpha A(T) + bB(T)\right) \\
&\quad - b\frac{A(T-\varepsilon^2\tau) - A(T)}{\varepsilon^2} + \alpha\frac{B(T-\varepsilon^2\tau) - B(T)}{\varepsilon^2}\,.
\end{aligned} \tag{27}$$

The drift $[\psi_A, \psi_B]$ correspond to the long time dynamics obtained using a multiscale analysis for the deterministic problem $\delta = 0$. This correspondence also holds for the non-linear logistic example considered in Section 4.

Given these expressions for ψ_A and ψ_B, the system (22) can be written

$$\begin{bmatrix} dA(T) \\ dB(T) \end{bmatrix} = \left\{ \boldsymbol{P} \begin{bmatrix} A(T) \\ B(T) \end{bmatrix} + \boldsymbol{Q} \begin{bmatrix} A(T-\varepsilon^2\tau) \\ B(T-\varepsilon^2\tau) \end{bmatrix} \right\} dT + \boldsymbol{D} \begin{bmatrix} d\xi_A(T) \\ d\xi_B(T) \end{bmatrix}, \tag{28}$$

where

$$\boldsymbol{P} = \begin{bmatrix} b^2\tau_2 - \alpha/\varepsilon^2 & -\alpha b\tau_2 - b/\varepsilon^2 \\ \alpha b\tau_2 + b/\varepsilon^2 & b^2\tau_2 - \alpha/\varepsilon^2 \end{bmatrix}, \qquad \boldsymbol{Q} = \begin{bmatrix} \alpha/\varepsilon^2 & b/\varepsilon^2 \\ -b/\varepsilon^2 & \alpha/\varepsilon^2 \end{bmatrix}$$

and

$$\boldsymbol{D} = \frac{\delta}{\varepsilon} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{29}$$

The characteristic equation (13) with matrices in (29) becomes

$$\left[b^2\tau_2 - \frac{\alpha}{\varepsilon^2}\left(1 - \mathrm{e}^{-\lambda\tau\varepsilon^2}\right) - \lambda\right]^2 + \left[\alpha b\tau_2 + \frac{b}{\varepsilon^2}\left(1 - \mathrm{e}^{-\lambda\tau\varepsilon^2}\right)\right]^2 = 0\,. \tag{30}$$

For $\varepsilon \ll 1$ roots of (30) are given by

$$\lambda = \frac{1}{1+2\alpha\tau_{\mathrm{c}}+\beta^2\tau_{\mathrm{c}}^2}\left(b^2\tau_2 \pm i\left[(1+\alpha\tau_{\mathrm{c}})\tau_2\alpha b + b^3\tau_{\mathrm{c}}\tau_2\right]\right) + O(\varepsilon^2)\,, \quad (31)$$

so that if $\tau_2 < 0$ then roots (31) have negative real parts. Starting with arbitrary initial conditions, the process approaches its stationary realization at a rate on the T scale determined by the real part of these eigenvalues. If $\tau_2 > 0$, then the process (28) is not stationary, and the multiscale approximation is valid only for short times; the original model (21) is dominated by exponential growth.

The assumptions made to derive (28) imply our results are valid for $\delta = O(\varepsilon)$ or smaller. That is, the noise strength of the original problem cannot exceed (the square root of) the deviation of the delay from its critical value. If the strength of the noise is small, then the oscillatory behavior of the solution to the original SDDE (21) persists for long times. The deterministic periodic solutions of the unperturbed equation serve as carriers, with their amplitudes evolving stochastically. The long time behavior of the amplitude processes $[A(t), B(T)]$ can be approximated by a stationary Gaussian process whose statistics can be explicitly determined using the formulas derived in Section 2.

The delay parameter τ of the original problem (21) is not necessarily small, but the parameters characterizing the system are near their critical values. This causes the delay in the amplitude equations (28) to be small, and proportional to the proximity to the critical values of the parameters.

Since the delay in (28) is small, it is tempting to replace the quotient terms in (27) which involve the small delay by derivatives. We investigate the validity of such an approximation by comparing first order statistics of the original process (28) and its non-delay approximation, the process $[A^0(T), B^0(T)]$.

We replace the quotient terms in (27) by the derivatives as

$$\begin{aligned} A(T) - A(T-\varepsilon^2\tau)]/\varepsilon^2 &\sim \tau A'(T)\,,\\ B(T) - B(T-\varepsilon^2\tau)]/\varepsilon^2 &\sim \tau B'(T)\,, \end{aligned}$$

to obtain the system for $[A^0(T), B^0(T)]$

$$\begin{aligned} d\begin{bmatrix} A^0(T)\\ B^0(T)\end{bmatrix} &= \frac{1}{1+2\alpha\tau_{\mathrm{c}}+\beta^2\tau_{\mathrm{c}}^2}\\ &\times\left(\begin{bmatrix} \tau_2 b^2 & -\gamma\\ \gamma & \tau_2 b^2\end{bmatrix}\begin{bmatrix} A^0(T)\\ B^0(T)\end{bmatrix} dT + \frac{\delta}{\varepsilon}\begin{bmatrix} 1+\alpha\tau_{\mathrm{c}} & -b\tau_{\mathrm{c}}\\ b\tau_{\mathrm{c}} & 1+\alpha\tau_{\mathrm{c}}\end{bmatrix}\begin{bmatrix} dw_1\\ dw_2\end{bmatrix}\right)\,, \end{aligned} \quad (32)$$

where $\gamma \equiv -(1+\alpha\tau_{\rm c})\tau_2\alpha b - b^3\tau\tau_2$. The process $[A^0(T), B^0(T)]$ is a 2-dimensional Ornstein–Uhlenbeck process. We calculate its variance $\boldsymbol{K}^0(0)$ by evaluating (15) with $\boldsymbol{\Phi}(t) = \exp(\boldsymbol{P}t)$ where $\boldsymbol{P}$ is the matrix of system (32) and $\boldsymbol{D}$ is the diffusion matrix of the same system. We find

$$\boldsymbol{K}^0(u) = -\frac{\delta^2}{\varepsilon^2\tau_2 b^2}\boldsymbol{I}\,, \tag{33}$$

and in particular, from (33) we obtain

$$\operatorname{var}\left(A^0(T)\cos(bt) + B^0(T)\sin(bt)\right) = -\frac{\delta^2}{\varepsilon^2}\frac{1}{\tau_2 b^2}\,. \tag{34}$$

Using (18) with conditions (14) and (17), we calculate the variance of the process $x(t)$ in (21) (again assuming that $\alpha > 0$ and $\tau_2 < 0$, so that the process $x(t)$ is stationary)

$$\operatorname{var} x(t) = \frac{\delta^2}{2b}\frac{b - \beta\sin[b(\tau_{\rm c}+\varepsilon^2\tau_2)]}{\alpha - \beta\cos[b(\tau_{\rm c}+\varepsilon^2\tau_2)]} = -\frac{\delta^2}{\varepsilon^2\tau_2 b^2} + \frac{\delta^2}{12\tau_2}\varepsilon^2 + O(\varepsilon^6)\,. \tag{35}$$

Hence, the variance in Eq. (34) is the leading term of (35).

Eqs. (33) and (35) show that the variances of the process $[A(T), B(T)]$ and its delay-free approximation $[A^0(T), B^0(T)]$ agree to the leading order in ε. Correlations over longer time intervals of the process $[A(T), B(T)]$ contain a combination of exponentials and polynomials as indicated by (10) and (15). In contrary, the covariance of the delay-free approximation contains only exponential terms. In general, approximation of a small delay by a derivative may not be correct to any order. It may result in the removal of oscillatory behavior generated by a delay. Also, when the oscillatory behavior includes sharp front the replacement of the delay terms by derivatives may lead to a solution which blows up in finite time [21]. While the delay-free approximation may provide some local description of the process, it does not capture the correct long time response of the full SDDE's; this fact can be illustrated by numerical simulations [17].

3.3. Multiplicative noise

In this section, we outline the procedure to find the stochastic amplitude equations when the DDE is acted on by multiplicative noise

$$dx = (-\alpha x(t) + \beta x(t-\tau))dt + \delta g(x(t))\,dw\,. \tag{36}$$

Again we postulate the form of the solution as in (19) and (22), so that $x(t)$ is to be replaced by a function of 2 time scales $x(t, T)$, but now the matrix $\boldsymbol{\sigma} = \boldsymbol{\sigma}(A, B)$ is a full matrix. To determine how the noise $g(x(t))\,dw$ is distributed between the two variables A and B, or equivalently how to find its projection on $\{\cos bt, \sin bt\}$, we write the white noise in the form of a Fourier series with noisy coefficients. These coefficients are independent white noises on the slow time scale T; that is, we have

$$\delta g(x(t,T))\,dw(t) = \frac{\delta}{\varepsilon} g(A(T)\cos bt + B(T)\sin bt) \times \left(\sum_{j=0}^{\infty} k_j^c \cos(jbt)\,dw_j^c(T) + \sum_{j=1}^{\infty} k_j^s \sin(jbt)\,dw_j^s(T) \right). \tag{37}$$

Projection of (37) on $\{\cos bt, \sin bt\}$ identifies the matrix $\boldsymbol{\sigma}$ as follows

$$\begin{aligned} &\boldsymbol{\sigma}(A,B) \begin{bmatrix} d\xi_A(T) \\ d\xi_B(T) \end{bmatrix} \\ &\equiv 2 \sum_{j=0}^{\infty} \frac{b}{2\pi} k_j^c \begin{bmatrix} \int_0^{2\pi/b} g(x(t,T)) \cos(jbt)\cos(bt) dt \\ \int_0^{2\pi/b} g(x(t,T) \cos(jbt)\sin(bt) dt \end{bmatrix} dw_j^c(T) \\ &+2 \sum_{j=1}^{\infty} \frac{b}{2\pi} k_j^s \begin{bmatrix} \int_0^{2\pi/b} g(x(t,T)) \sin(jbt)\cos(bt) dt \\ \int_0^{2\pi/b} g(x(t,T) \sin(jbt)\sin(bt) dt \end{bmatrix} dw_j^s(T) \,. \end{aligned} \tag{38}$$

(The factor 2, on the right-hand side, is included here to make the representation (22) consistent with the results of the projection procedure. This procedure gives the factor $1/2 = b/(2\pi)\int_0^{2\pi} \cos^2(bt)dt = b/(2\pi)\int_0^{2\pi} \sin^2(bt)dt$ in front of dA and dB and here we account for the elimination of this factor.) The objective of our analysis is to find unknown coefficients $k_0^c, k_j^{c(s)}$, $j = 1, 2, \ldots$, rather than identify the matrix $\boldsymbol{\sigma}$ directly. To determine these coefficients we look at the generator (the backward Kolmogorov operator) of the noise $\delta g(x(t,T))\,dw(t)$ averaged over the fast time scale, so it becomes the generator for the process on the slow time scale. That is, we calculate

$$\begin{aligned} &\frac{b}{2\pi} \int_0^{2\pi/b} g^2(x) \frac{\partial^2}{\partial x^2}\,dt \\ &= \frac{b}{2\pi} \int_0^{2\pi/b} g\left(A(T)\cos bt + B(T)\sin bt\right)^2 \left(\cos bt \frac{\partial}{\partial A} + \sin bt \frac{\partial}{\partial B} \right)^2 dt \end{aligned}$$

$$= \left(\frac{b}{2\pi} \int_0^{2\pi/b} g^2 \cos^2 bt \, dt \right) \frac{\partial^2}{\partial A^2} + \left(\frac{b}{\pi} \int_0^{2\pi/b} g^2 \cos bt \sin bt \, dt \right) \frac{\partial^2}{\partial A \partial B}$$

$$+ \left(\frac{b}{2\pi} \int_0^{2\pi/b} g^2 \sin^2 bt \, dt \right) \frac{\partial^2}{\partial B^2} , \tag{39}$$

and identify the diffusion matrix which must agree with calculations in (38). Specifically, we obtain

$$\begin{aligned} &\boldsymbol{\sigma}(A,B)\boldsymbol{\sigma}^T(A,B) = \mathcal{D} \\ &\equiv 2\frac{b}{2\pi} \begin{bmatrix} \int_0^{2\pi/b} g^2 \cos^2 bt \, dt & \int_0^{2\pi/b} g^2 \cos bt \sin bt \, dt \\ \int_0^{2\pi/b} g^2 \cos bt \sin bt \, dt & \int_0^{2\pi/b} g^2 \sin^2 bt \, dt \end{bmatrix} . \end{aligned} \tag{40}$$

(Again the factor 2 in (40) results from averaging out the fast time scale in the generator.) According to results in [19], Eq. (40) can be solved for the coefficients $k_j^{c(s)}$. An arbitrary function g yields an infinite system of equations; a function g which is a polynomial in x gives a finite system of equations. We illustrate this procedure for linear and quadratic multiplicative couplings.

First we take $g(x) = x$ in (36). Evaluating integrals in (38) we see that only 3 modes $\{1, \cos 2bt, \sin 2bt\}$ give non-zero contributions:

$$\begin{aligned} &2 \left(\frac{b}{2\pi} \int_0^{2\pi/b} x \, dw(T) \begin{bmatrix} \cos bt \\ \sin bt \end{bmatrix} dt \right) \\ &= k_0^c \begin{bmatrix} A \\ B \end{bmatrix} dw_0^c(T) + \frac{1}{2} k_2^c \begin{bmatrix} A \\ -B \end{bmatrix} dw_2^c(T) + \frac{1}{2} k_2^s \begin{bmatrix} B \\ A \end{bmatrix} dw_2^s(T). \end{aligned} \tag{41}$$

Evaluating (40), we find the diffusion matrix $\mathcal{D}$ and obtain the system of equation for k_0^c, k_2^c, k_2^s

$$\begin{aligned} \mathcal{D} &= \frac{1}{4} \begin{bmatrix} 3A^2 + B^2 & 2AB \\ 2AB & A^2 + 3B^2 \end{bmatrix} \\ &= (k_0^c)^2 \begin{bmatrix} A^2 & AB \\ AB & B^2 \end{bmatrix} + \frac{(k_2^c)^2}{4} \begin{bmatrix} A^2 & -AB \\ -AB & B^2 \end{bmatrix} + \frac{(k_2^s)^2}{4} \begin{bmatrix} B^2 & AB \\ AB & A^2 \end{bmatrix}, \end{aligned} \tag{42}$$

which yields $k_0^c = 1/\sqrt{2}$ and $k_2^c = k_2^s = 1$. After combining all the terms in the slow time amplitude equations and simplifying their coefficients, we obtain

$$\begin{bmatrix} dA(T) \\ dB(T) \end{bmatrix} = \begin{bmatrix} \psi_A \\ \psi_B \end{bmatrix} dT$$
$$+\frac{1}{\sqrt{2}} \begin{bmatrix} A \\ B \end{bmatrix} dw_0^c(T) + \frac{1}{2} \begin{bmatrix} A \\ -B \end{bmatrix} dw_2^c(T) + \frac{1}{2} \begin{bmatrix} B \\ A \end{bmatrix} dw_2^s(T). \tag{43}$$

Similarly, the quadratic multiplicative noise, $g(x) = x^2$, enters the slow time equation through 4 modes $\{\cos bt, \sin bt, \cos 3bt, \sin 3bt\}$ as

$$\boldsymbol{\sigma} \begin{bmatrix} d\xi_A(T) \\ d\xi_B(T) \end{bmatrix} = 2\frac{b}{2\pi} \int_0^{2\pi/b} x^2 \, dw \begin{bmatrix} \cos bt \\ \sin bt \end{bmatrix} dt$$
$$= \frac{1}{4} \left(k_1^c \begin{bmatrix} 3A^2 + B^2 \\ 2AB \end{bmatrix} dw_1^c(T) + k_1^s \begin{bmatrix} 2AB \\ A^2 + 3B^2 \end{bmatrix} dw_1^s(T) \right.$$
$$\left. + k_3^c \begin{bmatrix} A^2 - B^2 \\ -2AB \end{bmatrix} dw_3^c(T) + k_3^s \begin{bmatrix} 2AB \\ A^2 - B^2 \end{bmatrix} dw_3^s(T) \right). \tag{44}$$

The diffusion matrix $\mathcal{D}$ is calculated as

$$\mathcal{D} = \frac{1}{8} \begin{bmatrix} 5A^4 + B^4 + 6A^2B^2 & 4(A^3B + AB^3) \\ 4(A^3B + AB^3) & A^4 + 6A^2B^2 + 5B^2 \end{bmatrix}, \tag{45}$$

so that the solution to (40) for $k_{1(3)}^{c(s)}$ with $\boldsymbol{\sigma}$ and $\mathcal{D}$ given by (44) and (45) is by $k_1^{c(s)} = k_3^{c(s)} = 1$ and that the amplitudes evolve on the slow time scale according to

$$\begin{bmatrix} dA(T) \\ dB(T) \end{bmatrix} = \begin{bmatrix} \psi_A \\ \psi_B \end{bmatrix} dT + \frac{1}{4} \left(\begin{bmatrix} 3A^2 + B^2 \\ 2AB \end{bmatrix} dw_1^c \right.$$
$$\left. + \begin{bmatrix} 2AB \\ A^2 + 3B^2 \end{bmatrix} dw_1^s + \begin{bmatrix} A^2 - B^2 \\ -2AB \end{bmatrix} dw_3^c + \begin{bmatrix} 2AB \\ A^2 - B^2 \end{bmatrix} dw_3^s \right). \tag{46}$$

Terms of the Fourier series representation of the noise $g(x(t,T))dw$ which give nonzero contribution to the amplitude equation correspond to resonances with the oscillations of the unperturbed system. In the case of linear multiplicative noise, $g(x) = x$, it is necessary to include 3 Fourier terms which imply that 3 independent white noises are needed to describe the stochastic effects in the amplitude equations. Similarly, in the case of the quadratic g, 4 independent white noises are needed in the amplitude equations. The representation of the total noise effects by independent modes makes simulations of such a system efficient. It is not surprising that multiplicative noise in the original equation affects the amplitude equations in a multiplicative way by the same functional rule; that is, a linear g makes the noise in each of the amplitude equations proportional to A and B, while a quadratic $g = x^2$ generates proportionality factors in the form A^2, B^2, and AB. We reiterate that to determine all coefficients of the noise in the amplitude equations, we compare the averaged noise representation to the averaged noise generator. The equivalence between two such descriptions of diffusion processes implies consistency of the method. We note that we adapt the approach of [19] to SDDE's. In the original setting, the projection of a two-dimensional motion of the Duffing–van der Pol equation on a two-dimensional basis of the linear motion is analyzed. Both analyzes lead to system of type (40) for the coefficients of the Fourier series. Results of [19] show that this system has a unique solution for a given function g. Non-polynomial g leads to an infinite system of equations which imply that an infinite collection of independent white noises is necessary in the amplitude equation. As illustrated in examples, a polynomial g induces a finite number of independent noises in the averaged equations. If $g \equiv 1$ and the noise is additive, and this procedure can be used to rederive results of Section 3.2.

4. Logistic equation and noise near bifurcation point

We consider the logistic SDDE

$$dx = (rx(s)(1 - x(s - \tau)))ds + \delta dw \tag{47}$$

to illustrate the multiscale technique applied to a nonlinear equation near the bifurcation point. We first review some results for the deterministic case $\delta = 0$ [22]. We consider values of r, $r > 0$, where the solution $x = 0$ is unstable. The oscillations about $x = 1$ decay for subcritical delays ($\tau < \tau_c$) and are sustained for supercritical values ($\tau > \tau_c$), where $r\tau_c = \pi/2$. Here we consider the supercritical case; we investigate the effects of the noisy and nonlinear deterministic perturbations when their magnitudes are balanced on the critical periodic solution. This periodic solution is defined by the critical delay $r\tau_c = \pi/2$. (The subcritical case is discussed elsewhere [17].) As in the case of a classical nonlinear oscillator (described by a second order ODE), the amplitude and frequency of a periodic solution to the nonlinear DDE are related. Here we show their interdependence for the logistic DDE. We write the problem for $\varepsilon y(t) = x(t/\omega(\varepsilon)) - 1$, and seek a periodic solution to the problem for $y(t)$; that is, while keeping $\varepsilon \ll 1$, we derive a small amplitude periodic solution near $x(t) = 1$. On the time scale $s = t/\omega$ we obtain

$$\omega dy = -ry(t - \omega\tau)(1 + y(t))\, dt + \delta dW(t)\,, \tag{48}$$

where $W(t) = w(t/\omega)$. Using the strained coordinates technique, we find the relation between the frequency $\omega(\varepsilon)$, the delay τ, and the amplitude for the periodic solution of (48) to exist [18, 22]. Specifically, we have

$$\omega(\varepsilon) = 1 - \frac{3}{20}\varepsilon^2(A^2+B^2) + \ldots \equiv 1 + \varepsilon^2\omega_2 + \ldots \tag{49}$$

$$\tau = \frac{\pi}{2r} + \varepsilon^2\left(-\frac{1}{20r} + \frac{2}{30}\frac{\pi}{2r}\right)(A^2+B^2) + \ldots \equiv \tau_c + \varepsilon^2\tau_2 + \ldots \tag{50}$$

$$\begin{aligned} y(t) &= A\cos rt + B\sin rt \\ &\quad + \varepsilon\left[\frac{1}{5}\left(A^2 - AB - B^2\right)\cos 2rt + \frac{1}{10}\left(A^2 + 4AB - B^2\right)\sin 2rt\right] + \ldots \end{aligned} \tag{51}$$

Eqs. (49)–(51) indicate that for fixed amplitude (squared) A^2+B^2 and delay $\tau_c+\varepsilon^2\tau_2$, the frequency of the periodic solution is uniquely specified. Without any loss of generality, we normalize the amplitude to one, $A^2 + B^2 = 1$, fix $r > 0$, and define the small parameter ε as a measure of the proximity of the delay τ to the critical delay τ_c, so that with $\tau_2 > 0$ the frequency of the periodic solution is given by $\omega = 1 - \varepsilon^2(\tau_2 + 1/(20r))/\tau_c + O(\varepsilon^4)$ [22].

Now we turn to the stochastic problem. To determine the amplitude equation on the slow time scale, we postulate the form of the solution to (48) as

$$y = [A(T)\cos rt + B(T)\sin rt] + \varepsilon[C(T)\cos 2rt + D(T)\sin 2rt] + O(\varepsilon^2)\,. \tag{52}$$

Given the form of the oscillatory solution in (51), without any loss of generality we set $A(0) = 1$ and $B(0) = 0$. Functions $C(T)$ and $D(T)$ in (52) are quadratic functions of $A(T)$ and $B(T)$ as suggested by the form (51), $C(T) = (A(T)^2 - B(T)^2 - A(T)B(T))/5$ and $D(T) = ((A(T)^2 - B(T)^2 + 4A(T)B(T))/10$. Following the procedure developed for the linear case we calculate the differential dy on the left-hand side of (48) using Itô's formula (with x in (23) replaced by y). In the present case, second partial derivatives do not vanish. We substitute $y(t)$ as given in (52) on the right-hand side of (48), keeping terms up to $O(\varepsilon^2)$ in the drift coefficient. Writing the noise term as (26) and projecting the resulting equation onto $\cos rt$ and $\sin rt$, we obtain

$$\begin{aligned} \psi_A &= -\omega_2 rB + r^2(\tau_2 + \omega_2\tau_c)A + r\frac{B(T-\varepsilon^2\tau) - B(T)}{\varepsilon^2} \\ &\quad - \frac{r}{2}\Big[AD - BC\Big] + \frac{r}{2}\Big[AC + BD\Big]\,, \\ \psi_B &= +\omega_2 rA + r^2(\tau_2 + \omega_2\tau_c)B - r\frac{A(T-\varepsilon^2\tau) - A(T)}{\varepsilon^2} \end{aligned}$$

$$+\frac{r}{2}\Big[AC+BD\Big]-\frac{r}{2}\Big[-AD+BC\Big], \tag{53}$$

$$\sigma_A \;=\; \sigma_B = \frac{\delta}{\varepsilon}\,, \tag{54}$$

if the argument are omitted for A, B, C, and D, it is simply T.

In the absence of noise $\delta = 0$, there is a steady state solution to the equations $A'(T) = \psi_A$ and $B'(T) = \psi_B$, given by $A = 1$, $B = 0$ and ω given by (49). That is, the drift gives attracting dynamics to this steady state for A and B, and the noise gives fluctuations about this steady state. Linearization of (53) about the steady state gives a system of linear SDDE's with a small delay. This in turn can be analyzed using the tools of Section 2. Locally, the fluctuations can be described by an Ornstein–Uhlenbeck process. The long time steady state probability density of the amplitude equations (53) exhibits two peaks about two steady states of the deterministic system ((53) with $\delta = 0$) [17].

5. Discussion

We developed the multiscale method to analyze SDDE near the bifurcation point of the unperturbed system. The method derives a system of amplitude equations. This approach allows the study of the long time response of the original system to noisy and deterministic perturbations. The long time response of the system includes persistent oscillations. In the absence of the noise, the solution would decay exponentially fast. Oscillatory behavior of the solution to the stochastic equation is a result of the interaction between random and deterministic perturbations. The periodic solution serves as a carrier whose amplitude evolves stochastically. The amplitude satisfies a SDDE with a small delay. Exact analysis in the linear case shows that the variance of the steady state amplitude increases as the strength of the noise increases and the delay parameter decreases toward its critical value. This same conclusion applies to the local behavior of the amplitude of the nonlinear equation about its steady state. In the case of the logistic equation, the amplitude density is bimodal with peaks about the two steady points of the deterministic dynamics. The method applies both to the linear and nonlinear equations. The form of the averaged amplitude equation depends on the form of the nonlinearity and the type of the noise. Noise effects enter the equation through the resonant modes between the deterministic carrier and the coupling of the random term. In the case of the additive noise, there are two resonant modes; in the case of linearly multiplicative noise there are three resonant terms, and so on. Although we have expressed the evolution of the envelope $[A(T), B(T)]$, Itô's formula can be used to transform these equations into a stochastic equation for phase of the oscillatory modes. Hence the "phase diffusion" of the system can be determined as the covariance of the phase.

The assumption of being near the bifurcation point implies that the projection of the fast time scale is done on a two-dimensional space. An analysis of the DDE and SDDE away from the bifurcation point requires a different approach [16]. A periodic solution to a time autonomous first order DDE is determined up to phase. Linearization of the DDE about the periodic solution is again a first order DDE. The multiscale approach developed for this case differs from the classical method outlined in Section 3.1. The single resonant mode of the DDE is defined by the isolated solution. In a classical case of a nonlinear oscillator with an isolated periodic solution, the linearization of the second order ODE is a linear equation with two linearly independent solutions [18]. Solvability conditions which remove resonances use these 2 solutions. In the case of a DDE, the analogy to the nonlinear oscillator cannot be carried out. Since the isolated solution to the DDE is determined up to phase, the perturbations of the amplitude are observed as higher order effects; the amplitude equation is slaved to the phase equation. In the case of the SDDE's, this implies that additive noise in the DDE acts in multiplicative way in the amplitude equation [16].

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