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# MEANDERING BROWNIAN DONKEYS*

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We consider a Brownian particle whose motion is confined to a "meandering" pathway and which is driven away from thermal equilibrium by an alternating external force. This system exhibits absolute negative mobility, *i.e.* when an external static force is applied the particle moves in the direction opposite to that force. We reveal the physical mechanism behind this "donkey-like" behavior, and derive analytical approximations that are in excellent agreement with numerical results.

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## 1. Introduction

According to our daily-life experience we "know" that, when pulling or pushing something at rest, it responds with a motion in the direction of that external static force. However, this type of "knowledge" does not necessarily hold true any more, if we go beyond the realm of daily life, considering *e.g.* Brownian particles suspended in a fluid at room temperature or some other small (but still classical) systems where thermal fluctuations play a dominant role. It turns out that the response behavior of such systems to an external static perturbation is indeed as naively expected, namely a permanent motion (or "particle current") in the direction of the static force, on condition that the system at rest is at thermodynamic equilibrium. Any other behavior could be exploited to construct a *perpetuum mobile* of the second kind and thus would contradict the second law of thermodynamics.

Away from thermodynamic equilibrium, however, various "alternative" response behaviors are possible. As a first example we mention the well-known ratchet effect [1–4] which is characterized by the existence of a non-

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vanishing current *without* any external static perturbation. The reason behind this behavior is an intrinsic asymmetry of the system, *e.g.* in the form of a periodic, asymmetric "ratchet"-potential.

Another, completely different response phenomenon is known under the term *absolute negative mobility* (ANM): Upon application of an external static perturbation of whatever direction the (non-equilibrium) system at rest reacts with a permanent (average) motion which always runs into the direction *opposite* to that of the static force. In particular, the average current is zero when there is no static perturbation applied. ANM has been described in various *quantum mechanical* non-equilibrium systems [5–13] as a genuine quantum mechanical effect, in models of *interacting* Brownian particles [14–20] as a result of *collective* processes, and, more recently, has also been shown to exist in *classical, single-particle* systems [21–26]. In the latter publications, the single Brownian particle either has some kind of internal memory [21], or else it moves in a potential landscape that is two-dimensional [22–25] or that dynamically switches between several different states [25, 26].

In the present contribution, we investigate the behavior of a single Brownian particle which moves along a quasi *one-dimensional* structured pathway (embedded in the two-dimensional plane) and which is driven away from thermodynamic equilibrium by an alternating external force. We show that this particle motion exhibits ANM, we explain the underlying physical mechanism, and we derive theoretical results for the average particle current that are in excellent agreement with numerical findings.

## 2. Model

We consider the overdamped motion of a Brownian particle which is confined to a symmetric, periodically "meandering pathway", as shown in Fig. 1. The particle is coupled to a thermal environment with temperature $T$, and it is subjected to an externally applied, spatially homogeneous, time-dependent force $\vec{F}_{\rm tot}(t)$. The resulting probability density $\rho(\vec{s},t)$ of finding the particle at position $\vec{s}=(s_1,s_2)$ on the meandering path after time $t$ has elapsed is governed by the Fokker–Planck equation [27]

$$\frac{\partial}{\partial t}\rho(\vec{s},t)+\frac{\partial}{\partial\vec{s}}\vec{J}(\vec{s},t)=0\,, \tag{1}$$

with the probability current ($\eta$ is the viscous friction coefficient and $k_{\rm B}$ is Boltzmann's constant)

$$\vec{J}(\vec{s},t)=\left\{\frac{\vec{F}_{\rm tot}(t)}{\eta}-\frac{k_{\rm B}T}{\eta}\frac{\partial}{\partial\vec{s}}\right\}\rho(\vec{s},t)\,, \tag{2}$$

and where the boundary condition

$$\vec{J}(\vec{s},t) \cdot \vec{n}(\vec{s}) = 0 \tag{3}$$

must be fulfilled. Here, $\vec{n}(\vec{s})$ is defined as the normal vector to the meandering path if the point $\vec{s}$ is located on that path, and is chosen to be the null vector elsewhere. The boundary condition (3) thus guarantees that the particle motion is confined for all times to the meandering path. Of course, already the initial density $\rho(\vec{s},0)$ has to be chosen such that it is located entirely on the meandering pathway.

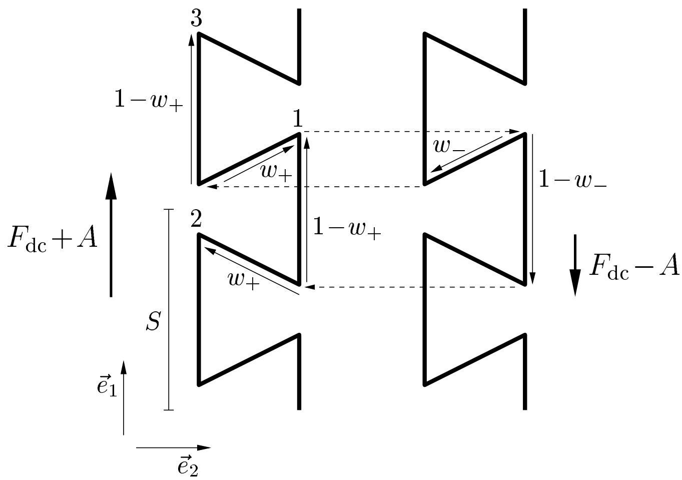


Fig. 1. Left bold curve: Piecewise linear example of a meandering pathway, to which the quasi one-dimensional motion of the Brownian particle is confined. Along the $\vec{e}_1$-direction, the meandering path is periodic with spatial period $S$. The “diagonal” and “vertical” segments are connected by sharp “corners”. In the concrete example which we discuss in the main text these “corners” will be slightly smoothened out, see Eq. (12) and Fig. 2 (inset). The static force $F_{\mathrm{dc}}$ and the time-periodic force $F_{\mathrm{ac}}(t)$ having the value $+A$ add up to the total force $F_{\mathrm{dc}} + A$ along the $\vec{e}_1$ axis as indicated by the bold arrow. Left bold curve: Same but when the time periodic force $F_{\mathrm{ac}}(t)$ has the value $-A$. Thin arrows: Different possible routes of the particle along the meandering pathway during one time-period $P$ of the driving. Jumps of the total force between $F_{\mathrm{dc}} + A$ and $F_{\mathrm{dc}} - A$ are indicated by the dashed arrows. The solid arrows mark different possible routes the particle can follow after a jump of the force, namely either along a “diagonal” or along a “vertical” linear segment. The respective probabilities for the different alternatives are indicated at the solid arrows; see also main text. A particle that starts, *e.g.*, in “corner” 1 ends up after one period of the external forcing in corner 1 or 2 or 3, with probabilities $w_-w_+ + (1-w_-)(1-w_+)$, $(1-w_-)w_+$, and $w_-(1-w_+)$, respectively.

The total external force $\vec{F}_{\rm tot}(t)$ in (2) consists of an unbiased time-dependent driving $F_{\rm ac}(t)$ and a static component $F_{\rm dc}$, both pointing "along" the meandering path, *i.e.*

$$\vec{F}_{\rm tot}(t) = (F_{\rm dc} + F_{\rm ac}(t))\,\vec{e}_1\,. \tag{4}$$

The time-dependent forcing $F_{\rm ac}(t)$ constantly drives the system away from thermal equilibrium by switching periodically between the two values $\pm A$ with period $P$, *cf.* Fig. 1. It thus vanishes on average, and it is clear from the symmetries of the meandering pathway (Fig. 1) that without a static component, $F_{\rm dc} = 0$, the particles perform no net motion after averaging over one time-period $P$ and over a statistical ensemble of realizations. The aim of the present contribution is to study the response behavior of this non-equilibrium system at rest to external static perturbations $F_{\rm dc}$. Specifically, we are interested in the dependence of the average particle current $\vec{v} = (v_1, v_2)$ along the meandering path on the bias $F_{\rm dc}$, where $v_1$ is defined as [2]

$$v_1 := \lim_{\tau\to\infty} \frac{1}{P} \int\limits_{\tau}^{\tau+P} \mathrm{d}t \int\limits_{-\infty}^{+\infty} \mathrm{d}\vec{s}\; J_1(\vec{s}, t) \tag{5}$$

and $v_2$ is obviously zero. Fig. 2 depicts a typical example of such a $v_1$-$F_{\rm dc}$-characteristics for our system, with ANM as its most outstanding feature.

## 3. Physical mechanism and theoretical description of ANM

The origin of ANM in Fig. 2 can be understood as follows: As a result of the combined action of the non-equilibrium driving $F_{\rm ac}(t)$ and the static perturbation $F_{\rm dc}$, the total force (4) in (1) adopts periodically the two values $(F_{\rm dc} \pm A)\vec{e}_1$ with sojourn times $P/2$ for each of them. We assume that $0 \leq F_{\rm dc} < A$ (due to symmetry reasons we can restrict ourselves to $A > 0$ and $F_{\rm dc} > 0$), such that the two associated total forces $(F_{\rm dc} + A)\vec{e}_1$ and $(F_{\rm dc} - A)\vec{e}_1$ point in opposite directions. Consequently, a particle that reaches one of the "corners" of the meandering path due to the drift imposed by the instantaneous total force $\vec{F}_{\rm tot}(t)$ is "trapped" there, until $\vec{F}_{\rm tot}(t)$ reverses its direction. During its trapping time in the corner the particle actually still diffuses (a little) around that corner due to the ambient thermal noise; its position within the trapping region is thus given by some probability distribution. At the "releasing" sign-change of $\vec{F}_{\rm tot}(t)$ the particle is located according to this distribution either on the "vertical" segment or on the "diagonal" segment that merge at the (former)

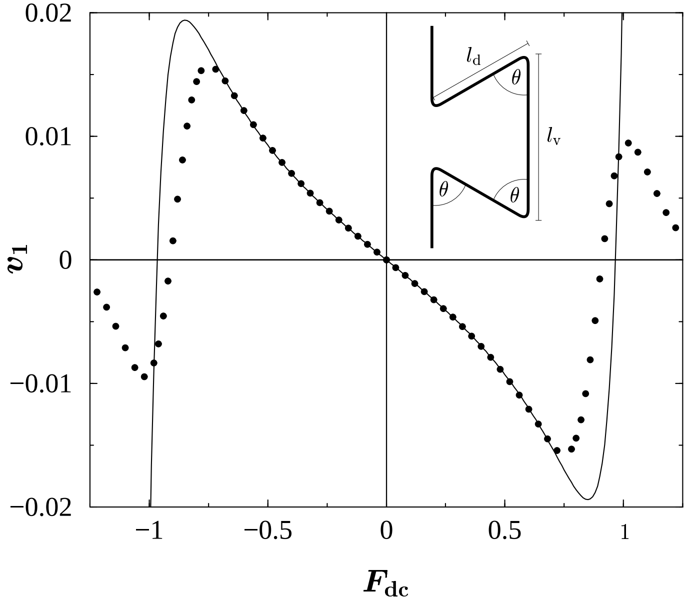


Fig. 2. Typical $v_1$-$F_{\mathrm{dc}}$-characteristics for the meandering pathway specified in (12). Dots: Numerical solution of the Fokker–Planck equation. Solid line: Theoretical result (6), (9). The dynamical parameters are $\eta = 1$, $k_{\mathrm{B}}T = 0.01$, $A = 1$, $P = 10$. The parameters for the meandering path are chosen as $l_{\mathrm{d}} = 0.4$, $l_{\mathrm{v}} = 0.6$, $\Delta l = 0.04$, and $\theta = 60°$. One period of the corresponding pathway is shown to scale in the inset. Note, in particular, that the corners at the merging points of diagonal and vertical segments are "rounded off" by tiny pieces of parabolas, see also Eq. (12).

trapping corner. It then moves (mainly deterministically but also a little bit diffusively) along the meandering path, until it eventually gets trapped again in one of the neighboring corners at the "end" of these path segments, see Fig. 1. We denote by $w_\pm$ the probability that the particle is trapped in the corner of the diagonal segment by the force $(F_{\mathrm{dc}} \pm A)\vec{e}_1$. Accordingly, the probability to end up in the corner of the vertical segment is $1 - w_\pm$ (*cf.* Fig. 1). As detailed below, these probabilities result from the "starting distribution" within the trapping region and the so-called splitting probabilities [28, 29], that the particle reaches the corner of the diagonal segment or the corner of the vertical segment when starting from a given position close to the former trap (*cf.* the Appendix). From Fig. 1 we can read off the average traveling distance during one period $P$ of the driving as $-(S/2)\cdot(1-w_-)w_+ + 0\cdot[(1-w_-)(1-w_+) + w_-w_+] + (S/2)\cdot w_-(1-w_+) = (S/2)(w_- - w_+)$, where $S$ is the spatial period of the meandering path along the $\vec{e}_1$-direction, see Fig. 1. For the current (5) we thus find

$$v_1 = \frac{S}{2P}(w_- - w_+)\,. \tag{6}$$

As already mentioned, the probabilities $w_\pm$ are obtained from so-called splitting probabilities [28, 29] and the particle distribution within a small trapping region near some corner. In the Appendix, the splitting probabilities are calculated for a Brownian particle moving in a one-dimensional potential. The results (22) can be applied to our system, based on the observation that the particle motion along the meandering pathway under the influence of the external force $\vec{F}_{\mathrm{tot}}(t)$ corresponds to a one-dimensional motion in the auxiliary potential

$$\phi(l,t) := -\int\limits_{\vec{\gamma}(l_*)}^{\vec{\gamma}(l)} \mathrm{d}\vec{s}\cdot\vec{F}_{\mathrm{tot}}(t)\,. \tag{7}$$

Here, $\vec{\gamma}(l)$ represents the meandering path parametrized by the auxiliary coordinate $l$ (for an explicit example see Eq. (12) below). The periodicity of the original pathway is captured by the property $\vec{\gamma}(l+L) = \vec{\gamma}(l) + S\vec{e}_1$. The integral in (7) goes along the meandering path $\vec{\gamma}(l)$ starting at an arbitrary initial point $\vec{\gamma}(l_*)$. The minima of the potential (7) represent those corners of the original meandering pathway where the particle can be trapped by the actual state of the forcing $\vec{F}_{\mathrm{tot}}(t)$, whereas the potential maxima represent the respective "non-trapping" corners; the roles of minima and maxima are interchanged with every reversal of $\vec{F}_{\mathrm{tot}}(t)$. Therefore, the particle distribution acquired in a corner of the meandering pathway during the trapped phase can be expressed by a corresponding distribution $\rho_\pm^{\mathrm{trap}}(l)$ of the particle in a minimum of (7); the subscripts $\pm$ refer to the actual trapping force $(F_{\mathrm{dc}} \pm A)\vec{e}_1$, *i.e.* to the actual value of $\vec{F}_{\mathrm{tot}}(t)$ in (7). Labeling also the splitting probabilities from the Appendix by subscripts $\pm$ for the forcings $\vec{F}_{\mathrm{tot}}(t) \equiv (F_{\mathrm{dc}} \pm A)\vec{e}_1$ in the potential (7), the probabilities $w_\pm$ can be written as

$$w_+ \;=\; \int\limits_{l_+}^{l_+ + L/2} \mathrm{d}l\; \rho_-^{\mathrm{trap}}(l)\, w_+^{\mathrm{split}}(l \to l_+)\,, \tag{8a}$$

$$w_- \;=\; \int\limits_{l_-}^{l_- + L/2} \mathrm{d}l\; \rho_+^{\mathrm{trap}}(l)\, w_-^{\mathrm{split}}\left(l \to l_- + \frac{L}{2}\right)\,, \tag{8b}$$

where $l_+$ and $l_-$ denote the locations of a potential minimum of (7) for $\vec{F}_{\rm tot}(t) \equiv (F_{\rm dc} + A)\vec{e}_1$ and for $\vec{F}_{\rm tot}(t) \equiv (F_{\rm dc} - A)\vec{e}_1$, respectively. The "arrival points" $l_+$ in (8a) and $l_- + L/2$ in (8b) of the splitting probabilities thus correspond to those corners of the meandering path at the end of the diagonal segments. With the assumption that the distribution $\rho_\pm^{\rm trap}(l)$ of the trapped particle in a minimum of (7) can be approximated by the Boltzmann equilibrium distribution and with the general results (22) for the splitting probabilities, we obtain from (8) the final result

$$w_+ \;=\; \frac{1}{Z_+ z_-} \int\limits_{l_+}^{l_+ + L/2} \mathrm{d}l \; \mathrm{e}^{-\phi_-(l)/k_{\rm B}T} \int\limits_{l}^{l_+ + L/2} \mathrm{d}\lambda \; \mathrm{e}^{+\phi_+(\lambda)/k_{\rm B}T}\,, \tag{9a}$$

$$w_- \;=\; \frac{1}{Z_- z_+} \int\limits_{l_-}^{l_- + L/2} \mathrm{d}l \; \mathrm{e}^{-\phi_+(l)/k_{\rm B}T} \int\limits_{l_-}^{l} \mathrm{d}\lambda \; \mathrm{e}^{+\phi_-(\lambda)/k_{\rm B}T}\,, \tag{9b}$$

with the normalization constants

$$Z_\pm \;=\; \int\limits_{l_\pm}^{l_\pm + L/2} \mathrm{d}l \; \mathrm{e}^{+\phi_\pm(l)/k_{\rm B}T}\,, \tag{10a}$$

$$z_\pm \;=\; \int\limits_{l_\mp}^{l_\mp + L/2} \mathrm{d}l \; \mathrm{e}^{-\phi_\pm(l)/k_{\rm B}T}\,. \tag{10b}$$

In (9) and (10), we have introduced the abbreviations

$$\phi_\pm(l) := - \int\limits_{\vec{\gamma}(l_*)}^{\vec{\gamma}(l)} \mathrm{d}\vec{s} \cdot [F_{\rm dc} \pm A]\, \vec{e}_1\,. \tag{11}$$

As a concrete example we consider the meandering pathway given by

$$\vec{\gamma}(l) = \begin{cases} [-l_\mathrm{d}/2\ \sin\theta]\,\vec{e}_2 + [l + l_\mathrm{d}(1+\cos\theta)]\,\vec{e}_1 \\ \qquad \text{for} \qquad -l_\mathrm{v} - l_\mathrm{d} \le l \le -l_\mathrm{v}/2 - l_\mathrm{d} - \Delta l\,, \\ [a_1(l+l_1)^2 - l_\mathrm{d}/2\ \sin\theta]\,\vec{e}_2 - [a_2(l+l_1)^2 - l - l_\mathrm{d}(1+\cos\theta)]\,\vec{e}_1 \\ \qquad \text{for} \quad -l_\mathrm{v}/2 - l_\mathrm{d} - \Delta l \le l \le -l_\mathrm{v}/2 - l_\mathrm{d} + \Delta l\,, \\ [l\ \sin\theta + (l_\mathrm{v}+l_\mathrm{d})/2\ \sin\theta]\,\vec{e}_2 - [l\ \cos\theta + l_\mathrm{v}(1+\cos\theta)/2]\,\vec{e}_1 \\ \qquad \text{for} \quad -l_\mathrm{v}/2 - l_\mathrm{d} + \Delta l \le l \le -l_\mathrm{v}/2 - \Delta l\,, \\ -[a_1(l+l_2)^2 - l_\mathrm{d}/2\ \sin\theta]\,\vec{e}_2 + [a_2(l+l_2)^2 + l]\,\vec{e}_1 \\ \qquad \text{for} \qquad -l_\mathrm{v}/2 - \Delta l \le l \le -l_\mathrm{v}/2 + \Delta l\,, \\ [l_\mathrm{d}/2\ \sin\theta]\,\vec{e}_2 + l\,\vec{e}_1 \\ \qquad \text{for} \qquad -l_\mathrm{v}/2 + \Delta l \le l \le l_\mathrm{v}/2 - \Delta l\,, \\ -[a_1(l-l_2)^2 - l_\mathrm{d}/2\ \sin\theta]\,\vec{e}_2 - [a_2(l-l_2)^2 - l]\,\vec{e}_1 \\ \qquad \text{for} \qquad l_\mathrm{v}/2 - \Delta l \le l \le l_\mathrm{v}/2 + \Delta l\,, \\ [-l\ \sin\theta + (l_\mathrm{v}+l_\mathrm{d})/2\ \sin\theta]\,\vec{e}_2 - [l\ \cos\theta - l_\mathrm{v}(1+\cos\theta)/2]\,\vec{e}_1 \\ \qquad \text{for} \qquad l_\mathrm{v}/2 + \Delta l \le l \le l_\mathrm{v}/2 + l_\mathrm{d} - \Delta l\,, \\ [a_1(l-l_1)^2 - l_\mathrm{d}/2\ \sin\theta]\,\vec{e}_2 + [a_2(l-l_1)^2 + l - l_\mathrm{d}(1+\cos\theta)]\,\vec{e}_1 \\ \qquad \text{for} \quad l_\mathrm{v}/2 + l_\mathrm{d} - \Delta l \le l \le l_\mathrm{v}/2 + l_\mathrm{d} + \Delta l\,, \\ [-l_\mathrm{d}/2\ \sin\theta]\,\vec{e}_2 + [l - l_\mathrm{d}(1+\cos\theta)]\,\vec{e}_1 \\ \qquad \text{for} \quad l_\mathrm{v}/2 + l_\mathrm{d} + \Delta l \le l \le l_\mathrm{v} + l_\mathrm{d}\,, \end{cases} \tag{12}$$

and

$$\vec{\gamma}(l+L) = \vec{\gamma}(l) + S\vec{e}_1\,, \tag{13}$$

with $a_1 = \sin\theta/(4\,\Delta l)$, $a_2 = (1+\cos\theta)/(4\,\Delta l)$, $l_1 = l_\mathrm{v}/2 + l_\mathrm{d} + \Delta l$, $l_2 = l_\mathrm{v}/2 - \Delta l$, $L = 2(l_\mathrm{v} + l_\mathrm{d})$, and $S = 2(l_\mathrm{v} - l_\mathrm{d}\cos\theta)$. This is a basically piecewise linear meandering path with diagonal segments of length $l_\mathrm{d}$ and vertical segments of length $l_\mathrm{v}$, and with an enclosed angle between those segments denoted by $\theta$ (see inset of Fig. 2). The sharp corners arising at the merging points of the different linear segments are smoothly "rounded off" in a vicinity $2\Delta l$ by parabolas. A typical $v_1$-$F_\mathrm{dc}$-characteristics for this example of a meandering pathway is shown in Fig. 2. The agreement between the numerical findings and the analytic prediction (6), (9) is excellent for not too large $|F_\mathrm{dc}|$-values, which, in fact, correspond to the range of validity of the theoretical results. Namely, our central theoretical results (6) and (9) rely on the following assumptions:

*(i)* The particle "equilibrates" to the Boltzmann distribution during the time-interval it is trapped in a corner of the meandering path. This assumption is justified if, first, the intra-well relaxation times in the corresponding potential minimum of (7) are much shorter than the sojourn time $P/2$ of the driving. This is fulfilled for typical meandering pathways, in particular for the one characterized by (12), in the range of not too large $|F_{\mathrm{dc}}|$. The latter can be inferred from the fact that the average traveling distance per driving period $P$, and thus the probabilities $w_{\pm}$, are independent of $P$ (data not shown). Second, the depth of the potential wells must be much larger than the thermal energy $k_{\mathrm{B}}T$ such that the particle can practically not escape from the potential minima during its trapped state. In other words, the mean time for escapes out of the potential wells (or corners of the meandering path, respectively) due to thermal fluctuations has to be much larger than the half-period $P/2$ of the driving.

*(ii)* The contributions to the current (6) due to thermally induced escapes out of the corners are neglected, which is consistent with the above assumption that $P/2$ is much larger than the respective mean escape time.

*(iii)* In order to obtain the expression (6) for the current (5), we tacitly assumed that the sojourn time $P/2$ is long enough such that the particle, when sliding down one of the segments of the meandering path after a switch of $\vec{F}_{\mathrm{tot}}(t)$, always arrives at the respective neighboring corner within $P/2$, irrespective of the actual potential state of the forcing $\vec{F}_{\mathrm{tot}}(t)$ (*cf.* Fig. 1).

## 4. Discussion

In this contribution, we have studied the quasi one-dimensional motion of a Brownian particle which is confined to a meandering pathway and which is driven away from thermal equilibrium by an alternating external force $F_{\mathrm{ac}}(t)$. This system exhibits the astonishing response phenomenon of ANM when perturbed by a (not too large) external static force $F_{\mathrm{dc}}$. As the underlying physical mechanism we identified the force-dependent trapping of the particle in the "corners" of the meandering path, and the subsequent "splitting" between the adjacent diagonal and vertical segments of the pathway after the release of the particle due to a sign change of the total external forcing. Based on this physical insight we have derived a theoretical description of the ANM-effect (see Eq. (6) and (9)) that is in excellent agreement with numerical results.

For the sake of simplicity, we have restricted ourselves to the case of a non-equilibrium driving $F_{\mathrm{ac}}(t)$ that jumps periodically between the two values $\pm A$. As well, the specific, nearly piecewise linear form of the meandering pathway, Eq. (12), has been chosen only for illustration purposes. The ANM-effect is expected to subsist for numerous generalizations of these choices, *e.g.* for a dichotomous noises $F_{\mathrm{ac}}(t)$, and for more general meandering pathways.

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## Appendix

In this Appendix we consider the following question: Suppose a particle that is moving in a one-dimensional potential $\phi(l)$ under the influence of thermal fluctuations (temperature $T$) has started its motion at an initial position $l_0$. What is the probability that it reaches a certain position $a$ before $b$, or the point $b$ before $a$ (where $a < l_0 < b$)? The answer is given in terms of so-called *splitting probabilities* $w^{\rm split}(l_0 \to a)$ and $w^{\rm split}(l_0 \to b)$, respectively. Of course, we have

$$w^{\rm split}(l_0 \to a) + w^{\rm split}(l_0 \to b) = 1\,. \tag{14}$$

To calculate $w^{\rm split}(l_0 \to a)$ and $w^{\rm split}(l_0 \to b)$, we start from the governing equation for the particle density $\rho(l,t)$ which is the Fokker–Planck equation [27]

$$\frac{\partial}{\partial t}\rho(l,t) + \frac{\partial}{\partial l}J(l,t) = 0 \tag{15}$$

with the probability current

$$J(l,t) = -\left\{\frac{\phi'(l)}{\eta} + \frac{k_{\rm B}T}{\eta}\frac{\partial}{\partial l}\right\}\rho(l,t)\,, \tag{16}$$

where $\eta$ is the viscous friction coefficient. The destination points $a$ and $b$ are represented by absorbing boundary conditions, *i.e.*

$$\rho(a,t) \equiv 0\,, \qquad \rho(b,t) \equiv 0\,. \tag{17}$$

Further, motivated by physical intuition we add "by hand" a particle source at the initial position $l_0$ of the particle. The splitting probabilities $w^{\rm split}(l_0 \to a)$ and $w^{\rm split}(l_0 \to b)$ are then related to the *stationary* probability currents $J(a)$ and $J(b)$ through the boundaries $a$ and $b$, respectively, according to

$$w^{\rm split}(l_0 \to a) = \frac{|J(a)|}{|J(a)| + |J(b)|}\,, \quad w^{\rm split}(l_0 \to b) = \frac{|J(b)|}{|J(a)| + |J(b)|}\,. \tag{18}$$

In other words, we now have to calculate the stationary probability currents from the solution of the time-independent equation

$$\frac{\partial}{\partial l}J(l) = \delta(l - l_0)\,, \qquad J(l) = -\left\{\frac{\phi'(l)}{\eta} + \frac{k_{\rm B}T}{\eta}\frac{\partial}{\partial l}\right\}\rho(l)\,, \tag{19}$$

with the Dirac-delta function $\delta(l)$. Solving this (inhomogeneous) ordinary differential equation is straightforward; the solution reads

$$\rho(l) = \begin{cases} \dfrac{1}{Z}\,\dfrac{\eta}{k_\mathrm{B}T}\,\mathrm{e}^{-\phi(l)/k_\mathrm{B}T} \displaystyle\int_{l_0}^{b} \mathrm{d}\lambda\,\mathrm{e}^{\phi(\lambda)/k_\mathrm{B}T} \int_{a}^{l} \mathrm{d}\lambda\,\mathrm{e}^{\phi(\lambda)/k_\mathrm{B}T} \\ \qquad\qquad\qquad \text{for} \quad a \le l < l_0\,, \\ \dfrac{1}{Z}\,\dfrac{\eta}{k_\mathrm{B}T}\,\mathrm{e}^{-\phi(l)/k_\mathrm{B}T} \displaystyle\int_{a}^{l_0} \mathrm{d}\lambda\,\mathrm{e}^{\phi(\lambda)/k_\mathrm{B}T} \int_{l}^{b} \mathrm{d}\lambda\,\mathrm{e}^{\phi(\lambda)/k_\mathrm{B}T} \\ \qquad\qquad\qquad \text{for} \quad l_0 \le l < b\,, \end{cases} \tag{20}$$

with the definition

$$Z := \int_{a}^{b} \mathrm{d}l\,\mathrm{e}^{\phi(l)/k_\mathrm{B}T}\,. \tag{21}$$

Finally, the result for the splitting probabilities is obtained from (18) and the second equation in (19):

$$w^{\text{split}}(l_0 \to a) = \frac{1}{Z}\int_{l_0}^{b} \mathrm{d}\lambda\,\mathrm{e}^{\phi(\lambda)/k_\mathrm{B}T}\,, \tag{22a}$$

$$w^{\text{split}}(l_0 \to b) = \frac{1}{Z}\int_{a}^{l_0} \mathrm{d}\lambda\,\mathrm{e}^{\phi(\lambda)/k_\mathrm{B}T}\,. \tag{22b}$$

These results are also derived in [28, 29] via alternative approaches.

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