

## NOISE ENHANCED STABILITY\*

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The noise can stabilize a fluctuating or a periodically driven metastable state in such a way that the system remains in this state for a longer time than in the absence of white noise. This is the noise enhanced stability phenomenon, observed experimentally and numerically in different physical systems. After shortly reviewing all the physical systems where the phenomenon was observed, the theoretical approaches used to explain the effect are presented. Specifically the conditions to observe the effect in systems: (a) with periodical driving force, and (b) with random dichotomous driving force, are discussed. In case (b) we review the analytical results concerning the mean first passage time and the nonlinear relaxation time as a function of the white noise intensity, the parameters of the potential barrier, and of the dichotomous noise.

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## 1. Introduction

The escape from metastable states continues to attract increasing interest since the Kramers' seminal paper [1]. It occurs in a wide variety of natural systems such as chemical systems, spin systems, quantum liquids, polymers and in problems of transport in complex systems, such as glasses and proteins. Specifically the noise activated escape in systems with

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metastable states and fluctuating barriers is important to describe the dynamics of complex nonequilibrium systems such as the molecular dissociation in strongly coupled chemical systems, electron transport in a quantum double-well structure, crystal growth, glasses, microstructures, lasers, Josephson junction devices, ratchet models, migration of ligands in proteins and biological systems [2–4]. A common peculiarity of all these systems is that they are open systems with internal nonlinear dynamics, and interacting with a noisy environment, which is responsible for noise induced phenomena. In these complex nonstationary nonequilibrium systems the continuous time-translation symmetry is broken, in contrast to the phenomenon of stochastic resonance characterized by deterministic barrier modulations [5].

Noise activated escape from a metastable state with oscillating or fluctuating barriers has recently attracted increasing attention [6–10]. In many situations the system is driven away from thermal equilibrium by an additional periodical driving force or by some external random perturbations. While important from both fundamental and applied point of view, analytical progress in the theory of oscillating barrier crossing is rather difficult. In the weak noise regime an interesting phenomenon appears: the enhancement of stability by thermal noise in systems with a metastable state and a periodically driven potential [11, 12]. This noise enhanced stability phenomenon (NES) was observed experimentally and numerically in various physical systems [8–23]. By varying the value of the thermal noise intensity we can lengthen or shorten the mean lifetime of the metastable state of our physical system. The enhancement of stability implies that the system remains in the metastable state for a longer time than in the absence of noise.

The paper is organised as follows. In the first section we shortly review all the physical systems where the effect was observed. The theoretical approaches that we used to explain the effect are presented in second and third sections. Specifically the conditions for the NES effect: (a) with periodical driving force, and (b) with random dichotomous driving force, are discussed. In the final section we review for the case (b) the analytical results concerning the mean first passage time (MFPT) and the nonlinear relaxation time (NLRT) as a function of the white noise intensity, the parameters of the potential barrier, and of the dichotomous noise.

## 2. Enhancement of stability in physical systems

The mean first passage time of a Brownian particle moving in potential fields with metastable and unstable states normally decreases with noise intensity growth according to the Kramers' formula [1]

$$\tau_k = A e^{\Delta U/q} \quad (1)$$

or some universal scaling function of the system parameters [24]. In equation (1)  $A$  is a pre-factor, which depends on the curvature of the potential at the metastable state and at the top of the potential barrier of height  $\Delta U$ , and  $2q$  is the noise intensity. However, the dependence of the MFPT for unstable or oscillating metastable states, was revealed to have resonance character with a nonmonotonic behaviour as a function of the noise intensity. This is the NES phenomenon: the noise can modify the stability of the system. Under the action of noise a system remains in the unstable or in the oscillating metastable state for a longer time than in the deterministic case and the escape time as a function of noise intensity has a maximum. The NES phenomenon has been observed in different physical systems, which we review in this section.

Hirsch *et al.* [19] first noted this nonmonotonic dependence in studying the onset of intermittent chaotic behaviour for one-dimensional maps just before a tangent bifurcation. They considered the effect of external noise on the regular path length and obtained an interesting result: for some values of the system parameters the average length of the laminar regions may be enhanced by the presence of a given finite amount of noise. Hirsh and coauthors considered a Langevin equation with the potential corresponding to the unstable state and reduced the average path length of the laminar regime problem to the mean first passage time problem. They found that *a small amount of noise increase the average time of passage, contrary to what one might have expected.* A simple model which exhibits the same phenomenon was studied by Agudov and Malakhov [25]. They considered a logistic map with a piece-wise linear function and studied this effect in detail.

In a theoretical study of the transient dynamics of an overdamped Brownian particle in a time dependent cubic potential with metastable state (see Fig. 1), Dayan *et al.* [11] showed numerically that the stability of the system is enhanced for a wide choice of values of the control parameter. The initial intuition of authors was that an increase in the amplitude of the noise should decrease the mean escape time because noise forces the particle to sample more of the available space than without noise. However their simulation studies showed that at an appropriately chosen frequency of the periodical driving force *the escape time increases when the intensity of the noise increases.* This means that the noise can modify the stability of the system in a counter-intuitive way. This effect was named by Mantegna and Spagnolo [12] as Noise Enhanced Stability (NES) and gives a nonmonotonic behaviour of the average escape time as a function of the noise intensity. The NES phenomenon was experimentally observed by investigating the escape time from a time modulated physical system: the tunnel diode [12]. The system can be deterministically overall stable or overall unstable. In the presence of noise

the overall-stable regime becomes metastable, the stability of the system is higher for lower values of noise amplitude. In the overall-unstable regime, however, a finite amount of *noise increase the stability* of the otherwise deterministically unstable system. A related phenomenon was revealed and investigated by Agudov and Malakhov in Refs. [13, 14] for different kinds of fixed potential profiles. This is the noise delayed decay (NDD) of unstable nonequilibrium states. Previous investigations showed that noise accelerated the decay of any unstable state [24, 26]. Agudov and Malakhov by analysing the influence of the potential profile shape and initial conditions on the NDD effect showed that the decay time of unstable states, under some conditions, can be increased considerably by the external noise. In other words, *the external additive noise can delay the decay of unstable states*.

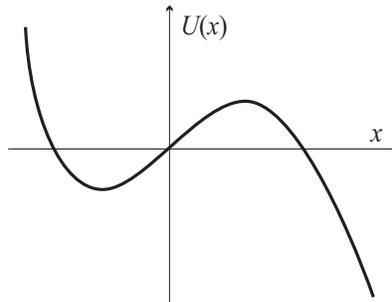


Fig. 1. Cubic potential with a metastable state.

We note that the NDD and NES effects are two different aspects of the same noise induced phenomenon occurring in nonlinear physical systems, but with some peculiarities. The NDD effect concerns the delay of the decay of unstable nonequilibrium initial states in fixed potential profiles [13, 14]. The NES effect appears in potential profiles with metastable state in the presence of a strong driving force. The dynamical regime is characterized by the absence of the potential barrier for some short time interval; that is, the system is *deterministically overall unstable* [12] and in this time interval we have the same physical situation as for NDD effect. After this time interval the Brownian particle, because of the interplay between the noise and the time dependent driving force, can return into the potential well and the mean lifetime of the metastable state increases with the noise intensity, in comparison with the dynamical lifetime [8, 11, 12]. When we consider fixed potentials with *metastable state* and with *initial unstable positions*, we can refer to both NDD or NES effect. In this case we have a nonmonotonic behaviour of the average escape time as a function of noise intensity and a new interesting dynamical regime, characterized by a *divergency* of the average escape time similar to an exponential Kramers behaviour [8, 9].

By investigating the influence of thermal fluctuations on the superconductive state lifetime and the turn-on delay time for a single Josephson element with high damping, Malakhov and Pankratov [20] found that fluctuations may both decrease and increase the turn-on delay time. Specifically they found that for low noise intensities and for current values greater than the critical current, which characterize the onset of the resistive state from a superconductive state, *an increase of fluctuations intensity causes increasing of the metastable state lifetime.*

Wackerbauer analysed in detail the influence of dynamical noise on switching processes in one-dimensional discontinuous maps [17], namely the piece-wise linear map, the Lorenz map and the piece-wise linear Lorenz map. The main result is that the switching dynamics of all Lorenz-type maps is significantly reduced by dynamical noise. This reduction is mainly caused by a noise-induced escape of a typical trajectory into a less frequently visited part of the attractor. This causes a *noise-induced stabilization, i.e. an enhancement of the mean passage time.* Several properties found in the noisy Lorenz system are related to findings in the transient dynamics of a modulated metastable system, which shows the NES phenomenon [12].

The mobility of an overdamped particle in a periodic potential tilted by a constant external field and moving in a medium with periodic friction coefficient shows noise induced slowing down [21]. For large values of the constant external field, for which the potential barrier disappears, the mobility decreases as the intensity of the thermal noise increases from zero temperature, where one would have expected the particle to become more mobile as the temperature is increased from zero. The presence of *noise slows down the motion of deterministically overall unstable states* in an appropriate range of potential parameters, contrary to what one might have expected. This is somewhat akin to the phenomenon of noise enhanced stability of unstable states [12].

The overdamped motion of a Brownian particle in an asymmetric bistable fluctuating potential shows *noise induced stability* of the state which has most of the time the higher energy [18]. For intermediate fluctuation rates the mean occupancy of minima with energy above the absolute minimum is enhanced. The less stable minimum most of the time is metastable and nevertheless it can be highly occupied.

Yoshimoto showed in recent papers [22] that one type of *noise-induced order*, in one-dimensional return map of the Belousov–Zhabotinsky reaction, takes place in the intermittent chaos, when the length of the laminar region was increased by the noise. Finally Xie and Mei studied in Ref. [23] the transient properties of a bistable kinetic system driven by two correlated noises: an additive noise and a multiplicative coloured noise. They found *one-peak structure in the mean first passage time* (MFPT) as a function of



For *fixed potential* ( $F(t) = 0$ ), when the particle is within the potential well, the decay time of metastable state increases exponentially in the limit  $q \rightarrow 0$ , according to Kramers' formula (1). The MFPT from initial position  $x_0$  to boundary  $b$  is

$$\tau(x_0, b) = \frac{1}{q} \int_{x_0}^b e^{\frac{U(x)}{q}} dx \int_0^x e^{-\frac{U(y)}{q}} dy. \quad (4)$$

If the starting position of the particle is between the maximum and the right boundary of the potential ( $L < x_0 < b$ ), then the initial state is unstable. Specifically if the initial position of the particle is between the maximum and the crossing point of the potential with the  $x$ -axis  $x_c = (L + E/k)$ , then the average escape time (MFPT) rises to infinity when  $q \rightarrow 0$ , while for zero noise we obtain a finite deterministic decay time  $(b - x_0)/k$ . In fact from Eq. (4), for  $\Delta E < E$  and  $q \rightarrow 0$ , we obtain

$$\tau(x_0, b) \simeq \frac{q}{kh} e^{\frac{(E-\Delta E)}{q}} \longrightarrow \infty, \quad \text{for } q \longrightarrow 0, \quad (5)$$

where  $\Delta E = k(x_0 - L)$ . The average escape time has therefore a singularity at  $q = 0$  for the following range of starting positions of the particle:  $L < x_0 < x_c$ . When the initial position is between the crossing point and the absorbing boundary ( $x_c < x_0 < b$ ), then  $\Delta E > E$  (see Fig. 2) and the average escape time has a nonmonotonic behaviour, with a maximum, as a function of noise intensity [8]. The qualitative mechanism of this phenomenon is as follows: a small quantity of noise can push the particle into potential well, then the particle will be trapped there for a long time because the well is deep with respect to the noise intensity considered. As a consequence the NES effect appears for a fixed potential with a metastable state if the initial position of the particle is within the range  $L < x_0 < b$  [8].

We consider now the initial state at  $x(0) = 0$  and *dichotomous driving force*  $F(t) = \pm a$  with period  $T$ . When  $F(t) = -a$ , the initial state  $x(0) = 0$  is metastable, while for  $F(t) = +a$ , it becomes unstable. Potential  $U(x)$  is defined by Eq. (3) where  $h = 0$ . We choose  $F(t) = +a$  ( $0 \leq t < T/2$ ), *i.e.* the potential barrier is absent for the first half of a period. In the absence of noise Eq. (2) has a periodical solution in the deterministic regime for  $T < 2L/a$ , and the particle always remains trapped in the metastable state ( $x(t) < L$ ). This is the overall stable regime. We have overall unstable regime when the period of the driving force is

$$T > \frac{2L}{a}. \quad (6)$$

In this case the particle surmounts the region  $[0, L]$  at time  $t = L/a$  and reaches some point  $x_1$ , between  $L$  and boundary  $b$ , at time  $t = T/2$ , and then crosses the absorbing boundary. If we add a small quantity of noise into the system, the position of the particle at time  $t = T/2$  is almost the same:  $x_1(q) \approx x_1(0)$ . The decay time for an initial position  $x(0) = 0$  is therefore  $\tau(0, q) \approx T/2 + \tau(x_1, q)$ , and  $\tau(x_1, q) \gg \tau(x_1, 0)$  because the potential barrier, which appears at  $t = T/2$ , makes the average escape time very large just for  $q \rightarrow 0$ , in accordance with Eq. (5). This means that the particle at time  $t = T/2$  is in an unstable position with a potential well on the left (see Fig. 2). All the trajectories that put the particle into the metastable state contribute to increase the average escape time  $\tau$  with respect to the dynamical time, producing a nonmonotonic behaviour of  $\tau$  as a function of noise intensity. The decay time  $\tau(0, q)$  will increase with small  $q$  and the NES phenomenon appears. Thus, the NES effect will occur, if the position  $x$  of the particle at time  $t = T/2$  is in the following range

$$L < x\left(\frac{T}{2}\right) < b. \quad (7)$$

This condition can be rewritten as follows [8]

$$\frac{L}{a} < \frac{T}{2} < \frac{ab + kL}{a(a + k)}. \quad (8)$$

This inequality, together with the condition  $a < k$ , gives the area on the parameters region  $(T, a)$ , where the NES effect takes place for small noise intensity with respect to the barrier height (see Fig. 2). Below the lower boundary we obtain Kramers behaviour. The magnitude of the NES effect decreases from the lower to the upper boundary. This is because near the upper boundary the potential barrier is absent or the reverse potential barrier “seen” from the particle becomes very high during the noise induced escape process.

### 3.2. Sinusoidal driving

In the case of sinusoidal force  $F(t) = a \sin \omega t$  we consider the same fixed potential profile  $U(x)$  of Eq. (3) with  $h = 0$ . The solution in the deterministic regime is  $x(t) = (a/\omega)(1 - \cos \omega t)$  for  $\omega > 2a/L$ . We have overall stable regime,  $x(t) < L$  for any  $t$ , and the particle always remains in the metastable state. If the frequency is  $\omega < 2a/L$ , the particle surmounts the region  $[0, L]$ , and the solution reads

$$x(t) = \begin{cases} (a/\omega)(1 - \cos \omega t), & 0 < t < \theta_1, \quad 0 < x(t) < L, \\ k(t - \theta_1) + (a/\omega)(1 - \cos \omega t), & t > \theta_1, \quad L < x(t) < b, \end{cases} \quad (9)$$

where  $\theta_1$  is the time at which the particle crosses the point  $x = L$ . Since the mechanism of NES effect is the same as for dichotomous driving, we can apply the same condition (7) for the effect occurrence. For the sinusoidal driving this condition can be rewritten as follows

$$\frac{k}{b} \left[ \pi - \arccos \left( 1 - \frac{\omega L}{a} \right) \right] + \frac{2a}{b} < \omega < \frac{2a}{L}. \tag{10}$$

This inequality, together with the condition  $a < k$ , give the area on the parameters region  $(a, \omega)$ , where the NES effect takes place for small noise intensity with respect to the barrier height (see Fig. 3). So the particle, after

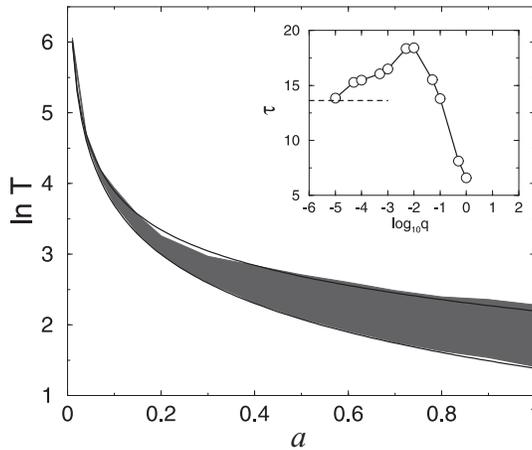


Fig. 3. The shaded area, obtained by numerical simulations, is the region of the plane  $(\ln T, a)$ , where the NES effect appears for a dichotomous driving force. The lower and upper continuous lines correspond to the left and to the right sides of inequality (8). The average escape time is greater than 10%, above the deterministic escape time, near the lower boundary. The parameters are:  $b = 7, k = 1, L = 2$ . Inset: the average escape time *versus* the noise intensity for  $a = 0.3$  and  $T = 13.5$ . The dashed line indicates the deterministic escape time.

$t = T/2$ , has the potential well on the left as in previous case (see Fig. 2), and as a result, the average escape time will increase with  $q$ , and the NES phenomenon takes place. When the frequency  $\omega$  and the amplitude  $a$  are chosen exactly on the left hand boundary of Eq. (10), the maximal height of induced potential barrier is zero, and the effect is very small. When we move to the right hand boundary of Eq. (10), the maximal barrier increases and the NES phenomenon too. After we cross the right boundary the deterministic decay time becomes infinite and the NES effect disappears. Then, in the presence of noise we get Kramers behaviour.

### 4. Dichotomous random force

#### 4.1. Mean first passage time

We consider now a randomly switching potential profile with reflecting boundary at  $x = 0$  and absorbing boundary at  $x = b$ . In Eq. (2)  $U(x)$  is a fixed potential and  $F(t) = a\eta(t)$ , where  $\eta(t)$  is a Markovian dichotomous process which takes the values  $\pm 1$  with the mean rate of switchings  $\nu$ . Exact results of MFPT for non-Markovian processes driven by two-state noise, without thermal diffusion ( $q = 0$ ), have been obtained in Ref. [27] and then generalized by various authors (see, for example, [28]). Exact equations for MFPTs for Brownian diffusion in switching potentials were first derived in [29].

From the backward Fokker–Plank equation, we obtain the following coupled differential equations

$$\begin{aligned} qT_+'' + [a - U'(x)] T_+' + \nu(T_- - T_+) &= -1, \\ qT_-'' - [a + U'(x)] T_-' + \nu(T_+ - T_-) &= -1. \end{aligned} \tag{11}$$

Our boundary conditions are

$$T_\pm'(0) = 0, \quad T_\pm(b) = 0. \tag{12}$$

Here  $T_+(x)$  and  $T_-(x)$  are the mean first passage times for initial values  $\eta(0) = +1$  and  $\eta(0) = -1$ , respectively. Introducing two auxiliary functions

$$T = \frac{T_+ + T_-}{2}, \quad \theta = \frac{T_+ - T_-}{2} \tag{13}$$

we can write the boundary conditions (12) in the form:  $T'(0) = \theta'(0) = 0$ ,  $T(b) = \theta(b) = 0$ . So that from Eqs. (11) and (13) we obtain

$$\begin{aligned} qT'' - U'(x)T' + a\theta' &= -1, \\ q\theta'' - U'(x)\theta' + aT' - 2\nu\theta &= 0. \end{aligned} \tag{14}$$

After removing  $T(x)$  from Eqs. (14) we obtain a third-order linear differential equation for the variable  $\theta(x)$

$$\theta''' - \frac{2U'(x)}{q}\theta'' + \left[ \frac{U'^2(x)}{q^2} - \frac{U''(x)}{q} - \gamma^2 \right] \theta' + \frac{2\nu U'(x)}{q^2}\theta = \frac{a}{q^2}, \tag{15}$$

where

$$\gamma = \sqrt{\frac{a^2}{q^2} + \frac{2\nu}{q}}. \tag{16}$$

We analyze here the piece-wise linear potential (3) with  $h = 0$ . We have a metastable state for  $\eta(t) = -1$ , and an unstable state for  $\eta(t) = +1$ . Let us focus on  $T_+(0)$  corresponding to a finite deterministic escape time. We solve Eqs. (14) and (15) separately for regions  $0 \leq x \leq L$ , and  $L \leq x \leq b$ . Using the boundary conditions and the continuity conditions at the point  $x = L$ , we obtain for small noise intensity

$$T_+(q) \simeq T_+(0) + \frac{q}{a^2} f(\beta, \omega, s) + o(q), \tag{17}$$

where

$$f(\beta, \omega, s) = \frac{\beta^3 [2 + s(1 + \beta^2)]}{(1 + \beta)(1 - \beta^2)} e^{-s} + \frac{\beta(1 - \beta^2 - 2\beta^3)}{2(1 - \beta^2)} (1 - e^{-s}) - \frac{5 + \beta}{2(1 + \beta)} + 2\omega \left( \frac{1}{1 - \beta^2} - \frac{3}{\beta} \right) - \frac{2\omega^2}{\beta^2} \tag{18}$$

with dimensionless parameters  $\beta$ ,  $\omega$  and  $s$

$$\beta = \frac{a}{k}, \quad \omega = \frac{\nu L}{k}, \quad s = \frac{2\nu(b - L)}{k(1 - \beta^2)} = \frac{2\omega}{1 - \beta^2} \left( \frac{b}{L} - 1 \right), \tag{19}$$

and

$$T_+(0) = \frac{2L}{a} + \frac{\nu L^2}{a^2} + \frac{b - L}{k} \left[ 1 - \frac{\beta}{s(1 + \beta)} (1 - e^{-s}) \right] \tag{20}$$

is the MFPT  $T_+(0)$  of initially unstable state in the absence of thermal diffusion.

The condition to observe the NES effect can be expressed by the following inequality

$$f(\beta, \omega, s) > 0. \tag{21}$$

Let us analyze the structure of NES phenomenon region on the plane  $(\beta, \omega)$ . At very slow switching  $\nu \rightarrow 0$  ( $\omega \rightarrow 0, s \rightarrow 0$ ), in accordance with Eq. (18), the inequality (21) takes the form

$$\beta > 0, 802; \quad \omega < \frac{2\beta^2(1 - \beta) - 5\beta(1 - \beta^2)^2/2}{6(1 - \beta^2)^2 - 2\beta(1 - \beta^2) + \beta^2(3\beta^2 - 1)(b/L - 1)}. \tag{22}$$

In the case of  $\beta \simeq 1$  we obtain from Eqs. (18), (21) and (22)

$$\omega < \frac{1 - \beta}{b/L - 1}, \quad \frac{1}{2} + \frac{5}{2}(1 - \beta) < \omega < \frac{1}{2(1 - \beta)}. \tag{23}$$

In Fig. 4 are shown two NES regions (shaded area) on the plane  $(\beta, \omega)$ . The NES effect occurs at the values of  $\beta \simeq 1$ , *i.e.* at very small steepness

$k - a = k(1 - \beta)$  of the reverse potential barrier for the metastable state. For such a potential profile, a small noise intensity can move back Brownian particles into potential well, after they crossed the point  $x = L$ , increasing the MFPT.

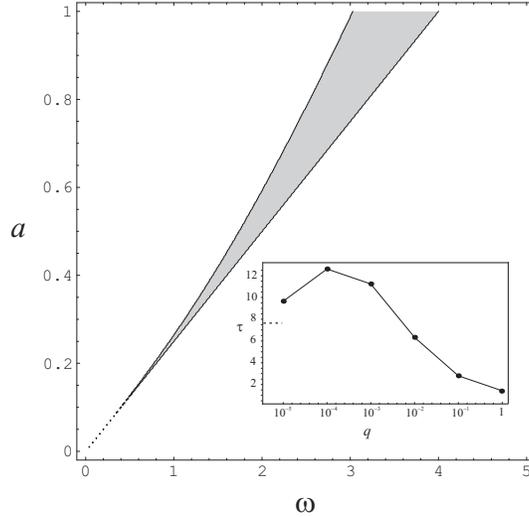


Fig. 4. The shaded area is the region of the plane  $(a, \omega)$  where the NES effect takes place for a sinusoidal driving force. The parameters are:  $b = 1, k = 1, L = 0.5$ . Inset: the average escape time *versus* the noise intensity for  $a = 0.1$  and  $\omega = 0.39$ . The dashed line indicates the deterministic escape time.

#### 4.2. Nonlinear relaxation time

Now we consider the nonlinear relaxation time (NLRT) for the system with randomly fluctuating potential of the previous paragraph. The NLRT implies to take into account the inverse probability current through the boundary, or equivalently, to consider the absorbing boundary at infinity [13]. By considering the well-known expression for probability density of the process  $x(t)$

$$W(x, t) = \langle \delta(x - x(t)) \rangle$$

and using the auxiliary function

$$Q(x, t) = W(x, t) \langle \eta(t) \mid x(t) = x \rangle$$

we obtain the following set of closed equations for the functions  $W(x, t)$  and  $Q(x, t)$  [10, 30]

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial x} [U'(x) W] - a \frac{\partial Q}{\partial x} + q \frac{\partial^2 W}{\partial x^2},$$

$$\frac{\partial Q}{\partial t} = \frac{\partial}{\partial x} [U'(x) Q] - a \frac{\partial W}{\partial x} + q \frac{\partial^2 Q}{\partial x^2} - 2\nu Q. \tag{24}$$

The initial conditions for these functions are:  $W(x, 0) = \delta(x - x_0)$ ,  $Q(x, 0) = 0$ . We consider potential profiles  $U(x) \pm ax$ , with a reflecting boundary at  $x = 0$  and an absorbing boundary at  $x \rightarrow +\infty$ . We assume that the potential profile  $U(x) + ax$  corresponds to a metastable state, and  $U(x) - ax$  corresponds to an unstable state. The average escape time from metastable state within the interval  $(L_1, L_2)$  is defined as follows

$$\tau(x_0) = \int_0^{+\infty} dt \int_{L_1}^{L_2} W(x, t | x_0, 0) dx. \tag{25}$$

To obtain the escape time we generalize the method, proposed in [31], for fluctuating potentials. As it is shown in that work the escape time (25) can be expressed in terms of the function  $Z_1(x, x_0)$

$$\tau(x_0) = \int_{L_1}^{L_2} Z_1(x, x_0) dx,$$

where  $Z_1(x, x_0)$  is the linear coefficient of the expansion of the function  $sY(x, x_0, s)$  in a power series in  $s$ , and  $Y(x, x_0, s)$  is the Laplace transform of the conditional probability density  $W(x, t | x_0, 0)$ . By Laplace transforming the auxiliary function  $Q(x, t)$  in  $R(x, x_0, s)$ , and expanding the function  $sR(x, x_0, s)$  in similar power series, we obtain from Eq. (24) and the conditions of zeroth probabilistic flow at reflecting boundary  $x = 0$

$$[aQ - U'(x)W - qW']_{x=0} = 0, \quad [aW - U'(x)Q - qQ']_{x=0} = 0,$$

the following coupled integro-differential equations for the functions  $Z_1(x, x_0)$  and  $R_1(x, x_0)$

$$\begin{aligned} qZ_1' + U'(x)Z_1 - aR_1 &= -1(x - x_0), \\ qR_1' + U'(x)R_1 - aZ_1 &= 2\nu \int_0^x R_1(y, x_0) dy. \end{aligned} \tag{26}$$

Here  $R_1(x, x_0)$  is the linear coefficient of the expansion of the function  $sR(x, x_0, s)$  and  $1(x)$  is the step function. We consider now the same piecewise linear potential profile (3) with  $h = 0$  and  $b \rightarrow +\infty$ . The initial position of the Brownian particles is  $x_0 = 0$ . Solving the set of Eqs. (26), with the

continuity conditions at the point  $x = L$ , we obtain the final expression for the lifetime of metastable state ( $L_1 = 0, L_2 = L$ )

$$\tau(0) = c_1 \left( \frac{\sinh \gamma L}{\gamma L} + \frac{2\nu q}{a^2} \right) + c_2 (\cosh \gamma L - 1) - \frac{\nu L^2}{q^2 \gamma^2}, \quad (27)$$

where  $c_1, c_2$  have complicated expressions in terms of the system parameters, and  $\gamma$  is given by Eq. (16). The exact formula (27) was derived without any assumptions on the thermal noise intensity  $q$ , and the mean rate of switchings  $\nu$ . From Eq. (27) we obtain explicit expressions of the asymptotic behaviours of the average escape time as a function of the noise intensity  $q$  and the system parameters. Specifically for  $q \rightarrow \infty$  we find

$$\tau(0) = \frac{L}{k} + \frac{L^2}{2q} \left( 1 + \frac{a^2}{\nu k L} \right) + o(q^{-1}). \quad (28)$$

Thus, the average escape time decreases with  $q$  and tends to a constant value  $L/k$  at  $q \rightarrow \infty$ . For very high noise intensity the Brownian particle “does not see” the fluctuations of the potential, and moves as in a fixed potential profile:  $U(x) = -kx$ . In the opposite limiting case of very slow diffusion ( $q \rightarrow 0$ ), using truncated expansions and algebraic manipulations we obtain

$$\tau(0) = \tau_d + \frac{q}{a^2} \left[ \frac{a(2k - a)}{k^2 - a^2} - 3 + \frac{2\nu L}{a} \left( \frac{ka}{k^2 - a^2} - 3 \right) - \frac{2\nu^2 L^2}{a^2} \right] + o(q), \quad (29)$$

where

$$\tau_d = \frac{\nu L^2}{a^2} + \frac{1}{2\nu} + \frac{2L}{a}.$$

In the absence of thermal diffusion ( $q = 0$ ), at the limiting cases  $\nu \rightarrow 0$  and  $\nu \rightarrow \infty$ , the average escape time becomes infinite:  $\tau_d \rightarrow \infty$ . For  $\nu \rightarrow 0$ , the metastable state becomes stable and therefore is long-lived. For  $\nu \rightarrow \infty$  the switchings are so fast that Brownian particles remain practically in the initial point  $x_0 = 0$ . The lifetime is minimum when the mean rate of switchings is equal to  $a/(L\sqrt{2})$ . To obtain NES effect in the system investigated, the term in quadratic brackets in Eq. (29) must be positive. Introducing the same dimensionless parameters  $\beta$  and  $\omega$  (see Eq. (19)) we can write the condition for the NES phenomenon in the form of inequality

$$\omega < \frac{\beta}{2} \left[ \sqrt{\frac{1}{(1 - \beta^2)^2} - \frac{2\beta + 3}{1 - \beta^2} + 5} + \frac{\beta}{1 - \beta^2} - 3 \right], \quad \beta > \frac{\sqrt{7} - 1}{2}. \quad (30)$$

The NES effect occurs mainly at the values of  $\beta$  near 1, *i.e.* at very small steepness  $k - a = k(1 - \beta)$  of the reverse potential barrier beyond the metastable state, as in the case of MFPT (see Fig. 5).

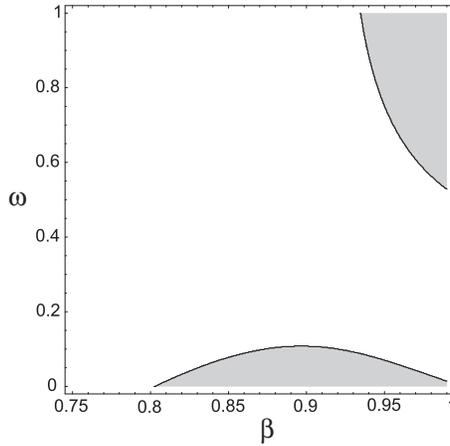


Fig. 5. The shaded areas are the region of the plane  $(\omega, \beta)$ , where the NES effect takes place. The parameters are:  $L = 1$ ,  $b = 1.2$ ,  $k = 1$ .

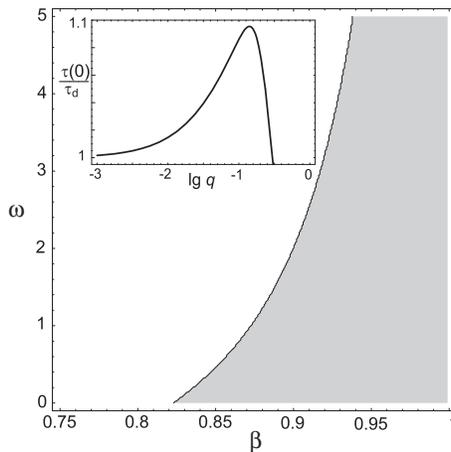


Fig. 6. The shaded area is the parameter region on the plane  $(\beta, \omega)$ , where the NES effect can be observed. The parameters are:  $L = 1$ ,  $k = 1$ . Inset: the average escape time *versus* the noise intensity for  $\beta = 0.97$  and  $\omega = 0.1$ .

Only Brownian particles that are put back into the potential well by a very small thermal noise intensity produce NES phenomenon. In Fig. 5 we plot in the inset the normalized average lifetime as a function of the noise intensity for  $\omega = 0.1$  and  $\beta = 0.97$ .

## 5. Conclusion

In this paper we have presented a short review of noise enhanced stability (NES) phenomenon. After shortly reviewing several physical systems, where the nonmonotonic behaviour or resonance-like phenomenon of the average escape time as a function of the noise intensity was observed, we have presented the theoretical approaches that we used to explain the NES effect, for systems with periodically driven or fluctuating metastable state. The variations of the potential are due to: (i) a periodical force, (ii) a Markovian dichotomous noise. For periodical driving force we obtained the conditions and the parameter region where the NES effect can be observed. Using the backward Fokker–Planck equation, and the Laplace-transform method we obtained the exact expressions of the MFPT and NLRT for randomly fluctuating metastable state in piece-wise linear potential profile with reflecting boundary at the origin. These expressions are valid for arbitrary noise intensity and for arbitrary fluctuation rate of the potential. The analysis at small thermal noise intensity allowed us to obtain analytically the region of NES phenomenon occurrence in case (ii). In contrast with the case of periodically driven metastable state, in the presence of a random dichotomous noise the NES effect can be observed only at very flattened potential profile beyond the potential well, *i.e.* in the absence of the reverse potential barrier for particles beyond the metastable state. Only in such a situation, Brownian particles, which are at large distances from the origin, can turn back into potential well by low noise intensity, producing an enhancement of stability of the metastable state of the system.

All the nonmonotonic behaviours, observed in different physical systems and related with the NES effect, allows us to conclude that this phenomenon provides a quite unexpected way to enhance the stability of metastable states.

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