

# PATTERN FORMATION AND SPATIAL CORRELATION INDUCED BY THE NOISE IN TWO COMPETING SPECIES\*

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We analyze the spatio-temporal patterns of two competing species in the presence of two white noise sources: an additive noise acting on the interaction parameter and a multiplicative noise which affects directly the dynamics of the species densities. We use a coupled map lattice (CML) with uniform initial conditions. We find a nonmonotonic behavior both of the pattern formation and the density correlation as a function of the multiplicative noise intensity.

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## 1. Introduction

We present a stochastic model for spatial distribution of two competing species. Our theoretical model could be useful to describe biological systems, where the presence of fluctuations, such as random variability of temperature, can modify strongly the dynamics of an ecosystem [1, 2]. We focus on the role played by the noise on the transient dynamics of the spatial distributions of two competing species belonging to an ecosystem described by generalized Lotka–Volterra equations [3] in the presence of multiplicative noise. We find nonmonotonic behaviors for the pattern formation and the density correlation of the species as a function of the multiplicative noise intensity. The theoretical results could contribute to select environmental and periodical driving forces using the proper space and time scales to develop physical models of population dynamics useful to interpret spatial patterns in the abundance of the species [4, 5].

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## 2. The model

To study the spatial effects due to the presence of noise sources we consider a discrete time evolution model, which is the discrete version of the Lotka–Volterra equations with diffusive terms, namely a coupled map lattice [6].

The time evolution of the spatial distribution for the two species is given by the following equations

$$x_{i,j}^{n+1} = \mu x_{i,j}^n (1 - x_{i,j}^n - \beta^n y_{i,j}^n) + \sqrt{\sigma_x} x_{i,j}^n X_{i,j}^n + D \sum_{\gamma} (x_{i,j}^n - x_{i,j}^n), \quad (1)$$

$$y_{i,j}^{n+1} = \mu y_{i,j}^n (1 - y_{i,j}^n - \beta^n x_{i,j}^n) + \sqrt{\sigma_y} y_{i,j}^n Y_{i,j}^n + D \sum_{\gamma} (y_{i,j}^n - y_{i,j}^n), \quad (2)$$

where  $x_{i,j}^n$  and  $y_{i,j}^n$  denote respectively the densities of prey 1 and prey 2 in the site  $(i, j)$  at the time step  $n$ ,  $\mu$  is proportional to the growth rate,  $D$  is the diffusion constant,  $\sum_{\gamma}$  indicates the sum over the four nearest neighbors.

The random terms are modeled by independent Gaussian variables denoted by  $X_{i,j}^n$ ,  $Y_{i,j}^n$  with zero mean and variance unit and  $\beta^n$  takes into account for the interaction between the species. In Eqs. (1) and (2),  $\sigma_x$  and  $\sigma_y$  are the intensities of the multiplicative noise which models the interaction between the species and the environment.

### 2.1. Stochastic resonance

It is known that for  $\beta < 1$  a coexistence regime takes place, that is both species survives, while for  $\beta > 1$  an exclusion regime is established, that is one of the two species vanishes after a certain time [7, 8]. Coexistence and exclusion of one of the two species correspond to stable states of the Lotka–Volterra’s deterministic model [3]. Real ecosystems are immersed in a noisy nonstationary environment, so also the interaction parameter is affected by the noise and some other deterministic periodical driving such as the temperature. The change in the competition rate between exclusion and coexistence occurs randomly because of the coupling between the limiting resources and the noisy environment. A random variation of limiting resources produces a random competition between the species. The noise, therefore, together with the periodic force determines the crossing from a dynamical regime ( $\beta < 1$ , coexistence) to the other one ( $\beta > 1$ , exclusion). To describe this continuous and noisy behavior of the interaction parameter  $\beta(t)$  we consider an Ito stochastic differential equation with a bistable potential and a periodical driving force

$$\frac{d\beta(t)}{dt} = -\frac{dU(\beta)}{d\beta} + \gamma \cos(\omega_0 t) + \xi_{\beta}(t), \quad (3)$$

where  $U(\beta)$  is the bistable potential (see Fig. 1)

$$U(\beta) = h(\beta - (1 + \rho))^4/\eta^4 - 2h(\beta - (1 + \rho))^2/\eta^2, \quad (4)$$

the periodical driving mimics the climatic temperature oscillations, and  $\xi_\beta(t)$  is a white Gaussian noise with  $\langle \xi_\beta(t) \rangle = 0$  and  $\langle \xi_\beta(t)\xi_\beta(t') \rangle = \sigma_\beta \delta(t - t')$ . Since the dynamics of the species strongly depends on the value of the interaction parameter, we initially consider the time evolution of  $\beta$  for  $\sigma_\beta = 0$  (see Fig. 2).

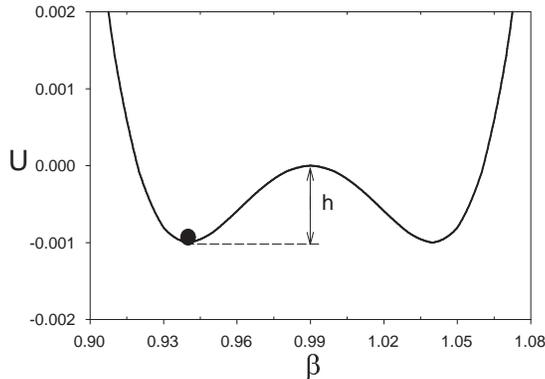


Fig. 1. The bistable potential  $U(\beta)$  of the interaction parameter  $\beta(t)$ . The potential  $U(\beta)$  is centered on  $\beta = 0.99$ . The parameters of the potential are  $h = 6.25 \times 10^{-3}$ ,  $\eta = 0.05$ ,  $\rho = -0.01$ .

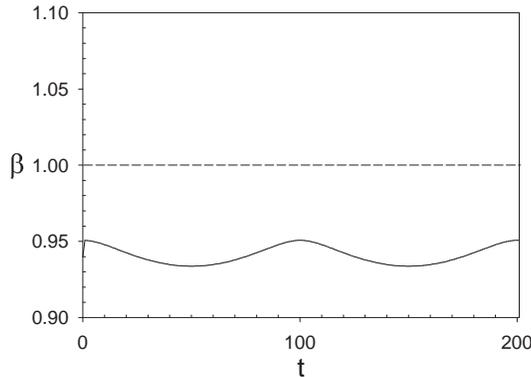


Fig. 2. Time evolution of the interaction parameter  $\beta$  for  $\sigma_\beta = 0$  and initial value  $\beta(0) = 0.94$ . The values of the parameters are:  $\gamma = 1.5 \cdot 10^{-1}$ ,  $\omega_0/(2\pi) = 10^{-2}$ . In the absence of noise  $\beta(t)$  oscillates below the critical value  $\beta_c = 1$ . The transient behavior near the initial point  $\beta(0) = 0.94$  is due to the choice of cosine function in Eq. (3) as deterministic periodical driving.

We note that in the absence of the additive noise  $\xi_\beta(t)$ ,  $\beta(t)$  shows a periodical evolution but its values always remain below  $\beta = 1$ , *i.e.* in the coexistence regime. The noise can synchronize with the periodical driving force. In this case a Stochastic Resonance (SR) [9, 10] effect appears which affects strongly the dynamics of the system [7, 11, 12]. Therefore we fix the additive noise intensity at the value  $\sigma_\beta = 2.65 \times 10^{-3}$  corresponding to a competition regime with  $\beta$  periodically switching from coexistence to exclusion regions [12] (see Fig. 3). The SR effect in the dynamics of interaction parameter  $\beta$  induces SR phenomenon in the dynamics of two competing species. This produces noise-induced anticorrelated periodic oscillations in time evolution of the two species densities (see Ref. [12]).

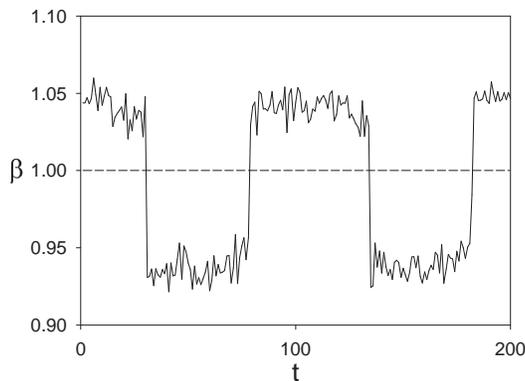


Fig. 3. Time evolution of the interaction parameter  $\beta(t)$  for  $\sigma_\beta = 2.65 \times 10^{-3}$ . The values of the parameters and the initial value  $\beta(0)$  are the same as in Fig. 2. For this level of noise a synchronization appears (Stochastic Resonance) and  $\beta(t)$  oscillates quasi-periodically below and above the critical value  $\beta_c = 1$ .

## 2.2. Spatial distributions

We consider the time evolution of the spatial distribution of the ecosystem, described by Eqs. (1) and (2), in the SR dynamical regime. The interaction parameter  $\beta^n$  of Eqs. (1), (2) corresponds to the value of continuous  $\beta(t)$  of Eq. (3) taken at the step  $n$ . So we fix the additive noise at the value  $\sigma_\beta = 2.65 \times 10^{-3}$  and we vary both the intensities of multiplicative noise. In Figs. 4 and 5 we report the spatio-temporal patterns of the two species for different values of the multiplicative noise intensity  $\sigma = \sigma_x = \sigma_y$ , namely  $\sigma = 10^{-12}, 10^{-8}, 10^{-4}, 10^{-1}$  with  $\mu = 2$ ,  $D = 0.05$ ,  $\gamma = 1.5 \times 10^{-1}$ ,  $\omega_0/(2\pi) = 10^{-2}$ ,  $\beta(0) = 0.94$  and  $x_{i,j}^0 = y_{i,j}^0 = 0.5$  at all sites  $(i, j)$ . We see that for very low noise intensity (see Fig. 4(a)) an average correlation on the considered lattice ( $N = 100 \times 100$ ) between the species is observed. For

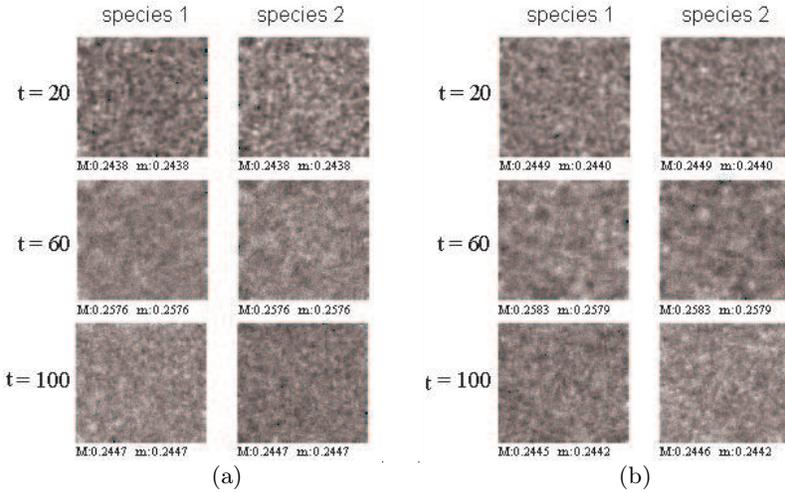


Fig. 4. Spatial distributions at different times for (a)  $\sigma = 10^{-12}$  and (b)  $\sigma = 10^{-8}$ . The value of the additive noise is fixed at  $\sigma_\beta = 2.65 \times 10^{-3}$ . The values of the parameters are:  $\mu = 2$ ,  $D = 0.05$ ,  $\gamma = 1.5 \times 10^{-1}$ ,  $\omega_0/(2\pi) = 10^{-2}$ ,  $N = 100 \times 100$ . The initial values are  $x_{i,j}^0 = y_{i,j}^0 = 0.5$  for all sites  $(i, j)$  and  $\beta(0) = 0.94$ .

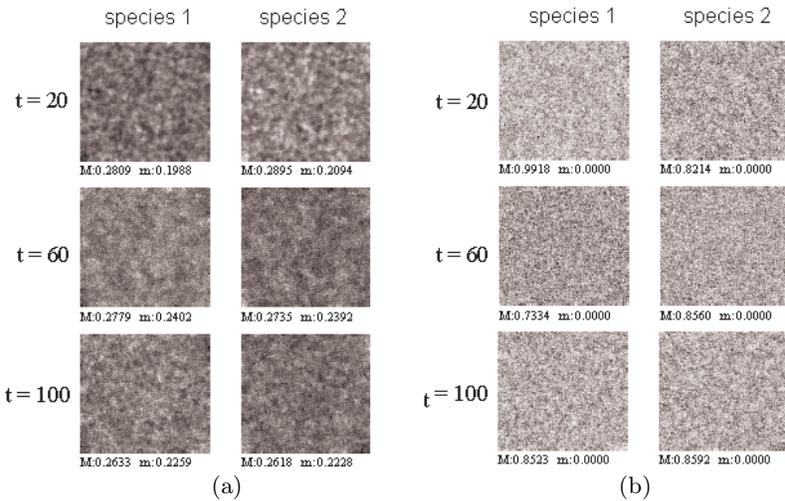


Fig. 5. Spatial distributions at different times for (a)  $\sigma = 10^{-4}$  and (b)  $\sigma = 10^{-1}$ . The additive noise intensity, the values of the parameters and the initial conditions are the same as in Fig. 4.

higher noise intensities (see Fig. 4(b), 5(a)) an anticorrelation between the two species is observed: the two species tend to occupy different positions. The anticorrelation is more evident in Fig. 5(a). By increasing the multiplicative noise the anticorrelation is strongly reduced (see Fig. 5(b)). Further

increase of the noise ( $\sigma = 10^{+3}$ ) causes the anticorrelation to disappear and the two species densities become uncorrelated. We note also that the average size of the patterns increases with the noise intensity non-monotonically: at very low noise intensity ( $\sigma = 10^{-14}$ ) the spatial distribution is almost uniform, by increasing the noise intensity spatial patterns arise (see Figs. 4(b), 5(a)) and a further increase of the noise intensity reduces the average size of the patterns (see Fig. 5(b)).

### 2.3. Spatial correlation

In order to evaluate the spatial correlation between the two species for the noise intensities considered we calculate, at the time step  $n$ , the correlation coefficient  $\langle c^n \rangle$  defined on the lattice as

$$\langle c^n \rangle = \frac{\text{COV}_{xy}^n}{s_x^n s_y^n} \quad (5)$$

with

$$\text{cov}_{xy}^n = \frac{\sum_{i,j} (x_{i,j}^n - \bar{x}^n)(y_{i,j}^n - \bar{y}^n)}{N}, \quad (6)$$

where  $\bar{x}^n$ ,  $s_x^n$ ,  $\bar{y}^n$ ,  $s_y^n$  are the mean value and the root mean square, respectively, of species 1 and species 2, obtained over the whole spatial grid at the time step  $n$ ,  $\text{cov}_{xy}^n$  is the corresponding covariance and  $N = 100 \times 100$  the number of sites which compose the grid. The behavior of the correlation coefficient  $\langle c^n \rangle$  as a function of the time for different levels of the multiplicative noise has been reported in Fig. 6. We observe a nonmonotonic behavior of  $\langle c^n \rangle$  as a function of the multiplicative noise intensity. In fact for low noise intensities  $\sigma = 10^{-12}$ ,  $\langle c^n \rangle$  shows weak oscillation around 1, that is strong correlation between the two species. For higher levels of the noise  $\sigma = 10^{-10}$ ,  $\langle c^n \rangle$  is affected by fluctuations and its values vary strongly as a function of the time. A further increase of the multiplicative noise, *i.e.*  $\sigma = 10^{-8}$  and  $\sigma = 10^{-4}$ , determines an oscillation of  $\langle c^n \rangle$  around a negative value, that is anticorrelation between the two species, with the frequency of the periodical forcing. For higher intensities of the noise  $\sigma = 10^{-1}$ , the value of the correlation coefficient  $\langle c^n \rangle$  increases and it vanishes for  $\sigma = 10^{+3}$ . Finally to show clearly the nonmonotonic behavior of  $\langle c^n \rangle$ , we calculate the time average of the correlation coefficient  $\langle c^n \rangle_t$  and we report it as a function of the multiplicative noise intensity in Fig. 7. A clear minimum is shown, which corresponds to the anticorrelated oscillations shown in the time evolution of two competing species in each point of our spatial grid [12]. We note therefore the different role of the two noise sources in the ecosystem dynamics. The additive noise determines the conditions of the dynamical

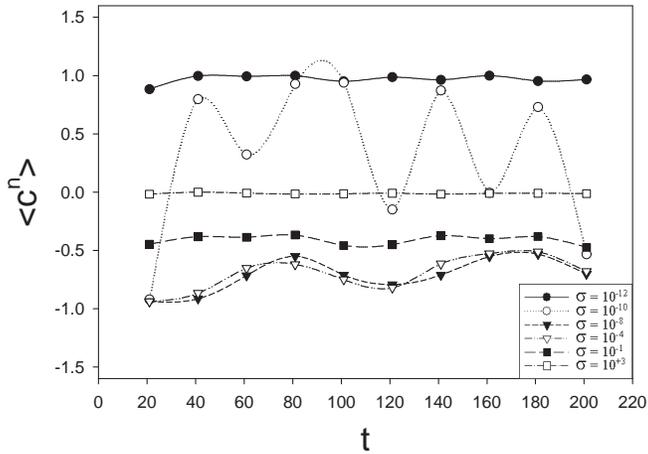


Fig. 6. Correlation coefficient  $\langle c^n \rangle$  as a function of time. For low levels of the multiplicative noise ( $\sigma = 10^{-12}$ ) the species are strongly correlated and  $\langle c^n \rangle$  is approximately constant. By increasing the intensity of the multiplicative noise ( $\sigma = 10^{-10}$ )  $\langle c^n \rangle$  shows big fluctuations. A further increase of the noise ( $\sigma = 10^{-8}$ ,  $\sigma = 10^{-4}$ ) causes strong anticorrelation between the two species with  $\langle c^n \rangle$  oscillating at the frequency of the periodical forcing. For very high levels of noise, the anticorrelation is reduced ( $\sigma = 10^{-1}$ ) and finally it disappears ( $\sigma = 10^{+3}$ ), that is the species are totally uncorrelated.

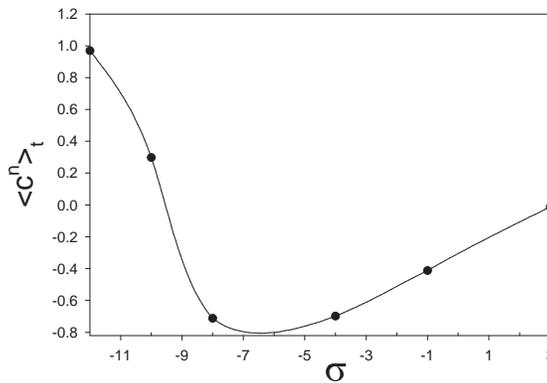


Fig. 7. Time average of the correlation coefficient  $\langle c^n \rangle_t$  as a function of the multiplicative noise in semilog scale.

regime, the multiplicative noise produces a coherent response of the system (see Ref. [12]), which is responsible for the appearance of anticorrelation behavior in the spatial patterns of the species.

### 3. Conclusions

We present a study on the role of the noise in the spatial distributions of two interacting species. The main result is that noise plays a very important role in population dynamics and cannot be neglected. Noise can have a constructive role and it is responsible for the enhancement of the response of the system to a driving force producing stochastic resonance. By using a discrete time evolution model, which is the discrete version of the Lotka–Volterra equations with diffusive terms, in the presence of a multiplicative noise and with a random interaction parameter, we analyze the temporal behaviors of the spatial distributions for an ecosystem consisting of two species. The noise induces spatio-temporal behaviors which are absent in the deterministic dynamics, *i.e.* pattern formation with the same periodicity of the deterministic force. Moreover, appearance of temporal oscillation is observed in the correlation coefficient between the two species. We find a nonmonotonic behavior of the time average correlation coefficient as a function of the multiplicative noise. In fact at low levels of the noise intensity the species densities are almost uniform and a strong correlation appears in the spatial distributions. By increasing the noise intensity we observe pattern formation and anticorrelated behavior, which exhibit the same periodicity of the deterministic driving force and correspond to the minimum of Fig. 7. For higher values of the noise intensity no patterns or correlations appear. Our model could be useful to explain spatio-temporal behaviors of populations, whose dynamics is strongly affected by the noise and by the environmental physical variables.

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