COMPACTIFICATION IN DECONSTRUCTED GAUGE THEORY WITH TOPOLOGICALLY NON-TRIVIAL LINK FIELDS

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We investigate the mass spectrum of a scalar field in a world with latticized and circular continuum space where background fields take a topological configuration. We find that the mass spectrum is related to the characteristic values of Mathieu functions. The gauge symmetry breaking in a similar spacetime is also discussed.

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1. Introduction

The most exotic approach to realize the unification of fundamental forces is based on assuming higher-dimensional gauge theory. The field in higher dimensions brings the corresponding Kaluza–Klein (KK) spectrum in four dimensional spacetime provided that the extra space is a compact manifold [1].

Recently, there appears a novel scheme to describe higher-dimensional gauge theory, which is known as deconstruction [2–4]. A number of copies of a four-dimensional theory linked by a new set of fields can be viewed as a single gauge theory. The resulting theory may be almost equivalent to a higher-dimensional theory with discretized, or, latticized extra dimensions.

In the continuum spacetime, if the compactification involves a topologically non-trivial configuration of gauge fields, the mass spectrum of charged fields becomes radically changed from that of the conventional compactification [5]. The cases with non-trivial field strength on a flat torus have also been considered [6]. The mass spectrum affects the low mass degree of freedom as well as the (Casimir-like) quantum energy density.

It is also known that, for higher-dimensional non-Abelian theory, a similar topological configuration gives rise to a symmetry breaking [7,8]. Such an alternative candidate to the Higgs mechanism is worth studying in higher-dimensional theories including string(-inspired) theories.

In deconstructed theories, the link fields can form a topological configuration as a whole in the presence of another compact dimension. Such a 'hybrid' compactification has been studied [9] in the other context. Of course, the original motivation of the deconstruction scheme, which gives good UV behavior of the theory, must be ignored in the present case. On the other hand, however, we can regard the hybrid compactification as a continuum limit of the deconstructed theory.

In the present paper, we investigate a U(1) gauge theory with a latticized circle, assuming another circular continuum dimension and topologically non-trivial background fields. We explicitly show the mass spectrum in the background fields, which has a certain limit of the continuum theory.

In Sec. 2, we examine the topologically non-trivial configuration of the link fields in the compactified spacetime. The mass spectrum of a charged scalar field in this background field is studied in Sec. 3. The mass spectrum of the Yang–Mills field in the same background is studied in Sec. 4, where symmetry breaking in this situation is shown. The final section, Sec. 5, is devoted to conclusion.

2. Topologically non-trivial configuration

We begin with the Lagrangian for deconstructing $(D+1)$-dimensional pure U(1) gauge theory [3]:

$$\mathcal{L}_V = \sum_{k=1}^{N} \frac{1}{e^2} \left[-\frac{1}{4} F_k^{\mu\nu} F_{k\ \mu\nu} - (D^\mu U_k)^\dagger D_\mu U_k \right] , \tag{1}$$

where e is a gauge coupling,

$$F_k^{\mu\nu} = i \left[\partial^\mu - i\tilde{A}_k^\mu \,, \partial^\nu - i\tilde{A}_k^\nu \right] , \tag{2}$$

and

$$D^\mu U_k = \partial^\mu U_k - i\tilde{A}_k^\mu U_k + iU_k \tilde{A}_{k+1}^\mu \,. \tag{3}$$

The labels of the fields are considered as periodic modulo N, *e.g.*, $U_0 \equiv U_N$, $U_{N+1} \equiv U_1$, and so on. Further we assume that all U_k have a common absolute value $|U_k| = f/\sqrt{2}$.

This theory is invariant under the following gauge transformation: The transformation of gauge fields is

$$\tilde{A}_k^\mu \to \tilde{A}_k^\mu + iW_k \partial^\mu W_k^\dagger \,, \tag{4}$$

while the link fields U_k are transformed as

$$U_k \to W_k U_k W_{k+1}^\dagger \,, \tag{5}$$

where absolute values of W_k's are unity.

It is known that when all U_k are assumed to equal $f/\sqrt{2}$, the mass spectrum of the D-dimensional gauge field reads [3]

$$4f^2 \sin^2\left(\frac{\pi p}{N}\right) \,, \qquad p \text{ is an integer} \,. \tag{6}$$

Now we examine the case with another compact dimension. We suppose that the z-direction is periodical, or the following identification is assumed:

$$z \sim z + 2\pi R \,, \tag{7}$$

where R can be considered as a radius of a circle.

If the circular dimension exists, there are other solutions to the equation of motion for U_k with vanishing A_k^μ's. Then the part of Lagrangian density $(D^\mu U_k)^\dagger D_\mu U_k$ can be rewritten as $\partial^\mu \chi_k \partial_\mu \chi_k/2$, when we set $U_k \equiv \exp(i\chi_k/f)$. The equation of motion leads to $\partial^2 \chi_k = 0$.

The most general background solution is $\chi_k/f = nz/(NR) + \varphi$, or

$$U_k = \frac{f}{\sqrt{2}} \exp\left[i\left(\frac{nz}{NR} + \varphi\right)\right] \,, \tag{8}$$

where n is an integer and φ is independent of z. Possible arbitrary phases are gauged away by transformations (5), except for a common phase φ. For $n \neq 0$, it is irrelevant to the mass spectrum because the common phase implies only the translation in the z-direction. The field corresponding to the common phase, the existence of which does not require compact continuum dimensions, was studied by Hill and Leibovich [3, 4]. We adopt only the non-zero n in the present paper.

We show how $U_k = \frac{f}{\sqrt{2}} \exp\left(i\frac{nz}{NR}\right)$ is taken for a single-valued function with respect to z. When we choose the transformation (5) with

$$W_k = \exp\left(-2\pi i \frac{nk}{N}\right) \,, \tag{9}$$

the single-valuedness is satisfied as follows:

$$U_k|_{z=2\pi R} = \frac{f}{\sqrt{2}} \exp\left(2\pi i \frac{n}{N}\right) \to \frac{f}{\sqrt{2}} = U_k|_{z=0} \,. \tag{10}$$

The periodicity in the latticized dimension, such as $W_N = W_0$, holds when n is an integer. n is a topological number as in the case with the magnetic flux in the two dimensional compact space. Actually, in the limit of $N \to \infty$ and $N/f =$const., our model corresponds to the model with the constant magnetic field $B = nf/(NR)$ on the two-torus [6].

3. Scalar field

The Lagrangian for deconstructing $(D+1)$-dimensional scalar field theory is

$$\begin{aligned} \mathcal{L}_\phi &= \sum_{k=1}^{N} \left[-(D^\mu \tilde{\phi}_k)^\dagger D_\mu \tilde{\phi}_k \right] \\ &\quad + f \sum_{k=1}^{N} \left(\sqrt{2} \tilde{\phi}_k^\dagger U_k \tilde{\phi}_{k+1} + \sqrt{2} \tilde{\phi}_k^\dagger U_{k-1}^\dagger \tilde{\phi}_{k-1} - 2f \tilde{\phi}_k^\dagger \tilde{\phi}_k \right) \,, \end{aligned} \tag{11}$$

where

$$D^\mu \tilde{\phi}_k = \partial^\mu \tilde{\phi}_k - i \tilde{A}_k^\mu \tilde{\phi}_k \,. \tag{12}$$

This Lagrangian is invariant under the transformation (5) with

$$\tilde{\phi}_k \to W_k \tilde{\phi}_k \,. \tag{13}$$

We investigate the mass spectrum of the scalar field when the z-direction is periodic, $z \sim z + 2\pi R$, and the link fields take the topologically non-trivial form,

$$U_k = \frac{f}{\sqrt{2}} \exp\left(i \frac{nz}{NR} \right) \,, \tag{14}$$

with an integer n.

To obtain the eigenfunctions associated with the spectrum, we expand the scalar field as

$$\tilde{\phi}_k = \frac{1}{\sqrt{N}} \sum_{p=1}^{N} \phi_p \exp\left[2\pi i \frac{pk}{N} \right] \,. \tag{15}$$

Then the equation of motion for the charged scalar field reduces to

$$-\partial_\mu^2 \phi_p + 2f^2 \left[1 - \cos\left(\frac{2\pi p}{N} + \frac{nz}{NR} \right) \right] \phi_p = 0 \,. \tag{16}$$

Now we have to find the eigenfunction of the following eigenvalue equation

$$M^2\phi_p = -\partial_z^2\phi_p + 2f^2\left[1-\cos\left(\frac{2\pi p}{N}+\frac{nz}{NR}\right)\right]\phi_p\,, \tag{17}$$

to obtain the spectrum of the mass M for scalar fields in $(D-1)$-dimensional spacetime.

The possible eigenfunctions turn out to be the Mathieu functions [10]. They are given by

$$\phi_{0,m,p}(z) = \mathrm{ce}_{\frac{2mN}{n}}\left(\frac{nz}{2NR}+\frac{\pi p}{N}+\frac{\pi}{2},\frac{4N^2f^2R^2}{n^2}\right) \qquad m=0,1,2,\ldots\,, \tag{18}$$

and

$$\phi_{1,m,p}(z) = \mathrm{se}_{\frac{2mN}{n}}\left(\frac{nz}{2NR}+\frac{\pi p}{N}+\frac{\pi}{2},\frac{4N^2f^2R^2}{n^2}\right) \qquad m=1,2,3,\ldots\,, \tag{19}$$

up to some normalization constants omitted here. Their eigenvalues M^2 are given by

$$M^2_{0,m} = \frac{1}{R^2}\left[\frac{n^2}{4N^2}a_{\frac{2mN}{n}}(4N^2f^2R^2/n^2)+2(fR)^2\right] \qquad m=0,1,2,\ldots\,, \tag{20}$$

and

$$M^2_{1,m} = \frac{1}{R^2}\left[\frac{n^2}{4N^2}b_{\frac{2mN}{n}}(4N^2f^2R^2/n^2)+2(fR)^2\right] \qquad m=1,2,3,\ldots\,, \tag{21}$$

respectively, where $a_r(q)$ $[b_r(q)]$ is a characteristic value which yields an even [odd] periodic solution of the Mathieu's equation[1].

These eigenfunctions do not necessarily satisfy the periodic condition in the z-direction. Therefore the degree of freedom for each eigen value is determined by the possible combinations in the form (15) which satisfy the periodic boundary condition.

First we consider the case that the topological number n is a divisor of N, or $n=1$. In this case, the linear combination

$$\tilde{\phi}_{a,m,\ell,k}(z) = \sum_{p'=1}^{N/n}\phi_{a,m,\ell+p'n}(z)\exp\left[2\pi i\frac{(\ell+p'n)k}{N}\right] \qquad (a=0,1)\,, \tag{22}$$

[1] In the limit $q\to 0$, both $a_r(q)$ and $b_r(q)$ approach to r^2.

has the correct boundary condition. To see this, we notice that $\phi_{a,m,p}(z + 2\pi R) = \phi_{a,m,p+n}(z)$ when N/n is an integer. Then one can find

$$\tilde{\phi}_{a,m,\ell,k}(z + 2\pi R) = \tilde{\phi}_{a,m,\ell,k}(z) \exp\left[-2\pi i \frac{nk}{N}\right] \qquad (a = 0, 1)\,, \tag{23}$$

and this is gauge equivalent to $\tilde{\phi}_{a,m,\ell,k}(z)$ via the transformation (13) with

$$W_k = \exp\left(2\pi i \frac{nk}{N}\right)\,. \tag{24}$$

The degeneracy of each mass eigenvalue is n, which corresponds to $\ell = 1, \ldots, n$. This degeneracy is the same as the counterpart of continuum theory [6].

In Fig. 1, the mass-squared eigenvalues of the scalar field are exhibited against fR for $n = N$. For general values of n, similar dependence on fR can be found. One notices that, in the limit of $fR \to 0$, the mass-squared spectrum approaches the KK spectrum with a circular dimensions, *i.e.* m^2/R^2 (m: integer). This limit means that the discrete dimension becomes degenerate. Oppositely, in the limit of $fR \to \infty$, the mass-squared spectrum behaves as $(2p+1)nf/(NR)$ (p: integer). This spectrum coincides with that of the continuum theory [6]. One can find the boundary of the behavior of the mass level in Fig. 1. This lies on $M^2 = 4f^2$, which indicates how the mass scale f of the discrete compactification affects the mass spectrum. Note that although the mass spectrum has both continuum and discrete compactifications, that is not a mere sum of each spectrum.

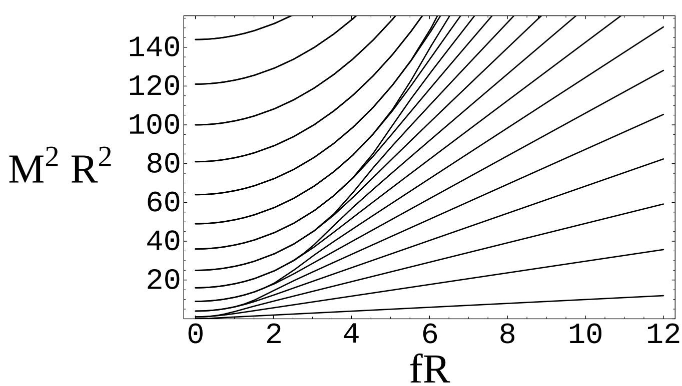

Fig. 1. The mass-squared eigenvalues of the scalar field are plotted as functions of fR for $n = N$.

Next, we consider the case that n is not a divisor of N. Let g be the greatest common divisor of n and N. In this case, not all m are permitted but only $m = m'n/g$ ($m' = 0, 1, 2, \ldots$). Then the eigenfunction is proportional to

$$\tilde{\phi}_{a,m'n/g,\ell,k}(z) = \sum_{p'=1}^{N/g} \phi_{a,m'n/g,\ell+p'g}(z) \exp\left[2\pi i \frac{(\ell+p'g)k}{N}\right] \qquad (a = 0, 1)\,, \tag{25}$$

and the degeneracy of each eigenvalue is given by g. The restriction on the eigenstates looks very similar to the case with orbifold compactification. Of course, the present case includes the previous case where n is the divisor of N, as a special case.

4. SU(2) Yang–Mills field

In this section, we will briefly describe how the symmetry breaking can occur in the Yang–Mills theory. For simplicity, we consider a deconstructed U(2) Yang–Mills theory. The action and the gauge symmetry on the fields are similar to the U(1) case in Sec. 2, provided that the fields A_k^μ and χ_k are matrix valued and some trace operations are attached. Whereas the link field U_k is transformed by $(\mathrm{SU}(2))_k$ and $(\mathrm{SU}(2))_{k+1}$, we assume that the background link field takes the following common U(2)-valued matrix form

$$U_k = \frac{f}{\sqrt{2}} \exp\left(i \frac{\tau_3}{2} \frac{nz}{NR}\right) = \frac{f}{\sqrt{2}} \begin{pmatrix} \mathrm{e}^{i\frac{nz}{2NR}} & 0 \\ 0 & \mathrm{e}^{-i\frac{nz}{2NR}} \end{pmatrix}. \tag{26}$$

The possible z-dependent term for the Yang–Mills field comes from the term $\mathrm{tr}[(D^\mu U_k)^\dagger D_\mu U_k]$ in the action. We expand the field as

$$\tilde{A}_k^\mu = \frac{1}{\sqrt{N}} \sum_{p=1}^{N} A_p^\mu \exp\left[2\pi i \frac{pk}{N}\right]. \tag{27}$$

Moreover, we can write $A_p^\mu = (A_p^\mu)^a (\tau^a/2)$. Among the three degrees of freedom in terms of U(2), $(A_p^\mu)^3$ has no z-dependent potential and thus one vector field has the mass spectrum

$$4f^2 \sin^2\left(\frac{\pi p}{N}\right) + \frac{m^2}{R^2}\,, \qquad p, m \text{ are integer}\,. \tag{28}$$

At this point, the massive scalar fields which come from fluctuations of $(\chi_p)^3$ in the link fields are absorbed in the vector field except for one massless degree of freedom, as in the case of deconstructed QED [3]. The rest two vector fields $(A_p^\mu)^1$ and $(A_p^\mu)^2$ have the same spectrum as the scalar field in Sec. 3, because the term $\mathrm{tr}[(D^\mu U_k)^\dagger D_\mu U_k]$ reduces to

$$\begin{aligned}\mathrm{tr}[(D^\mu U_k)^\dagger D_\mu U_k] &\rightarrow f^2 \left| \mathrm{e}^{\mp i \frac{nz}{2NR}} - \mathrm{e}^{i\frac{2\pi p}{N}} \mathrm{e}^{\pm i \frac{nz}{2NR}} \right|^2 \\ &= 2f^2 \left[1 - \cos\left(\frac{2\pi p}{N} \pm \frac{nz}{NR} \right) \right] . \end{aligned} \tag{29}$$

The z-component $(A_p^z)^a$ becomes three scalar fields which have the same mass spectrum as $(A_p^\mu)^a$. As a result, the massless sector in the $(D-1)$-dimensional theory contains a U(1) gauge field and two scalar fields.

5. Conclusion and discussion

In this paper, we have considered a topologically non-trivial configuration for the link fields in the deconstructed gauge theory with another compact continuous dimension. We have shown the bosonic spectrum in such a background fields. We have also examined the possibility of non-Abelian symmetry breaking by the background field by using a simple model.

A remarkable feature of the mass spectrum is as follows: the spectrum of mass squared has a nearly equal intervals at low levels, but another nearly equal intervals can be found in the spectrum of mass at higher levels.

It might be pointed out that the mass of the massive vector boson is of the order $nf/(NR)$ (or arbitrarily small according the choice of fR, N, and n) whereas that of the KK excited states in the gauge singlet is the smaller one of the order either $1/R$ or f/N. In addition, we can naturally take the sector which is independent of f. The KK excitation of such a sector has an order of $1/R$. In other words, we will be able to construct models with seemingly additional mass scales from a few scales in a similar manner.

The stability of the symmetry breaking vacuum should be studied. Classically the vacuum with the non-trivial background field has a positive finite energy density. The Casimir-like energy may lower the value of the vacuum energy [8]. For this purpose, we must consider also various types of matter fields and their quantum effects.

In this paper, we have treated only the bosonic field. Incidentally, the fermionic fields in the doubly latticized dimensions with the non-trivial background fields have been studied since almost three decades ago in the other context [11–13]. It is known that the eigenvalue equation becomes a type of the Harper equation (or almost Mathieu equation) [13, 14]. Therefore, the doubly latticized space, which is obtained if the z-direction in our present model is discretized, can also be applied to the symmetry breaking mechanism and will exhibit a new type of quantum effects of fermionic and bosonic fields. These subjects (on the Dirac fields and the doubly latticized extra space) will be discussed elsewhere.

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