Vol. 35 (2004) ACTA PHYSICA POLONICA B No 5

COSMOLOGICAL CONSTANT AND RELATIVISTIC INVARIANCE: THE ZETA-FUNCTION APPROACH

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(Received December 1, 2003)

The problem of the correct computation of the vacuum energy contribution to cosmological constant is discussed in the context of the relativistic invariant zeta-function approach. This method is shown to yield the value of this quantity proportional to the fourth power of the (small) quantized field mass, while the dependence on the large mass scale is only logarithmic. This value is compared to the result obtained in the dimensional regularization scheme which also satisfies the relativistic invariance condition, and found to be the same up to irrelevant finite terms. The consequences of the renormalization group invariance are also briefly discussed.

PACS numbers: 11.10.–z, 11.10.Gh

The cosmological constant is known to contain the contributions of various origin, the explicit evaluation and "fine tuning" of which still remains to be an open problem [1]. Here, only one of them will be addressed — the contribution of the ground state energy of quantum fields. The Einstein equation may be written in the form:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -g_{\mu\nu} \Lambda_0 - 8\pi G T_{\mu\nu} \, , \tag{1}$$

where Λ is the "classical" part while the vacuum average of the energy-momentum tensor $\langle T_{\mu\nu} \rangle$ arises due to the quantum fluctuations. The condition of the relativistic invariance for this quantity may be formulated as [1,2]:

$$\langle T_{\mu\nu} \rangle = \varepsilon g_{\mu\nu} \, , \tag{2}$$

what means that the energy density $\varepsilon = \langle T_{00} \rangle$ and the vacuum pressure $p = \langle T_{ii} \rangle$ are related as:

$$\varepsilon = -p\,. \tag{3}$$

Using (2), Eq. (1) can be written in the form

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -g_{\mu\nu} \Lambda_{\rm eff}\,, \tag{4}$$

where

$$\Lambda_{\rm eff} = \Lambda_0 + 8\pi G \varepsilon\,, \tag{5}$$

can be treated as the effective cosmological term. The "standard" estimation of the energy density for a scalar field is based on the obvious field theoretical formula with the UV cutoff of the divergent integral at the Planck scale $M_{\rm P} \approx 10^{19}$ GeV:

$$\varepsilon = \frac{1}{2} \int \frac{d^3\vec{k}}{(2\pi)^3}\, \omega_k \to \int\limits_0^{M_{\rm P}} \frac{k^2 dk}{(2\pi)^2} \sqrt{\vec{k}^2 + m^2}\,, \tag{6}$$

producing the well-known huge value

$$\varepsilon_{\rm P} \approx 10^{71}\ {\rm GeV}^4\ , \tag{7}$$

that is about 120 orders larger than the observable one.

However, recently it was argued in Ref. [3] that the leading power-law terms in this naive evaluation do not satisfy the relativistic invariance condition (3). Instead, it was shown that the calculation of the zero-point energy in the dimensional regularization satisfies the condition (3) at all steps, and yields after removing the singular part within the $\overline{\rm MS}$ scheme:

$$\varepsilon^{(0)}_{\rm dim} = -p = -\frac{m^4}{64\pi^2} \left(\ln \frac{\Lambda^2}{m^2} + \frac{3}{2} \right)\,, \tag{8}$$

where m is the field mass, and Λ is the mass scale parameter. The dimensionally regulated, but non-renormalized expression contains neither quartic, nor quadratic divergent terms — the only logarithmic ones appear. In contrast, the four-dimensional cutoff performed for computation the vacuum averaged trace of the energy-momentum tensor does give the quadratic divergent term, being in the same time the relativistic invariant. In all these cases, it was shown that the vacuum energy density vanishes for a massless field [3].

On the other hand, a powerful method for calculation of the ground state energy for various configurations of quantum fields is provided by the ζ-function technique supplied, if necessary, with the heat-kernel expansion

[4–6]. Within the general framework of the zero-point energy computations, the classical cosmological term Λ_0 is treated as a "bare constant", and possible divergences appearing in calculation of the quantum contribution must be absorbed by the (infinite) redefinition of this parameter. One of the aims of the present paper is to check if this approach can be applied in a relativistic invariant fashion to the cosmological constant calculations, and compare the results with that ones obtained in the other invariant frameworks.

It can be easily shown that the straightforward application of the ζ-function regularization [7]:

$$\int \frac{d^3\vec{k}}{(2\pi)^3}\ \omega_k \to \int \frac{d^3\vec{k}}{(2\pi)^3}\ \omega_k^{1-\epsilon} \tag{9}$$

does not meet the relativistic invariance condition (3). In order to use this approach properly, let us consider the following definition of the one-loop vacuum energy of the free quantum field [4, 8, 9]:

$$\varepsilon = \frac{1}{V_4} \ln Z(\varphi)\,, \tag{10}$$

where V_4 is the (Euclidean) infinite space-time volume, the partition function for the scalar field $\varphi(x)$ reads

$$Z(\varphi) = \int \mathcal{D}\varphi \exp\left(-\frac{i}{2}\int dx\ \sqrt{-g}\varphi(x) A_{\mathrm{S}}\varphi(x)\right)\,, \tag{11}$$

and the second-order operator $A_{\mathrm{S}} = g_{\mu\nu}\partial_\mu\partial^\nu + m^2$ has the eigenvalues λ_k:

$$A_{\mathrm{S}}\varphi_k = \lambda_k \varphi_k\,. \tag{12}$$

The generalized dimensionless ζ-function corresponding to this operator is defined as the infinite sum

$$\zeta(s) = \mu^{2s} \sum_k \lambda_k^{-s}\,, \tag{13}$$

where μ is an arbitrary mass scale providing the correct dimension. In case of the continuous spectrum λ_k (which we are actually dealing with), the sum in the Eq. (13) is replaced by the integral with the corresponding measure

$$\sum_n \to V_4 \int \frac{d^4k}{(2\pi)^4}\,. \tag{14}$$

The function (13) converges in (3+1)D space-time for Re $(z) > 2$, and can be analytically continued to a meromorphic function having the poles only at $s = 1$ and $s = 2$ [4]. At the origin ($s = 0$) it is regular.

Then, from the expressions

$$Z(\varphi) = (\det\ A_{\rm S})^{-1/2} = \exp\left(\sum_k \frac{m}{\lambda_k^{1/2}}\right) \quad , \quad \ln Z(\varphi) = -\frac{1}{2}\sum_k \ln\frac{\lambda_k}{m^2} \, , \tag{15}$$

evaluating the derivative of the ζ-function at $s = 0$

$$\zeta'(0) = \sum_k \ln\frac{\mu^2}{\lambda_k} \, , \tag{16}$$

one gets the formula for the zero-point energy

$$\varepsilon = \frac{1}{V_4}\ln Z(\varphi) = \frac{1}{2V_4}\left(\zeta'(0) - \zeta(0)\ln\frac{\mu^2}{m^2}\right) \tag{17}$$

for the scalar field φ with the mass m (for comparison, see, *e.g.*, [8]). Now, using the continuous spectrum of eigenvalues λ_k in the momentum space

$$\lambda_k = -k^2 + m^2 \, , \tag{18}$$

we evaluate (replacing the infinite sum by the corresponding integral and performing the Wick rotation) explicitly the corresponding ζ-function (by virtue of simplicity of the expression for λ_k, we can do this. In general case, the eigenvalues are unknown, and the heat-kernel technique should be applied [4–6, 10]. In some situations, the direct large-k expansion may also be useful [11]):

$$\zeta(s) = V_4\mu^{2s}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+m^2)^s} = V_4\frac{m^4}{16\pi^2}\left(\frac{\mu^2}{m^2}\right)^s \frac{1}{s^2 - 3s + 2} \, . \tag{19}$$

Then we have

$$\zeta(0) = \frac{V_4 m^4}{32\pi^2} \quad , \quad \zeta'(0) = \frac{3}{4}\frac{V_4 m^4}{16\pi^2} \, , \tag{20}$$

and the energy density is given by

$$\varepsilon_\zeta = -\frac{m^4}{64\pi^2}\left(\ln\frac{\mu^2}{m^2} - \frac{3}{2}\right) \, . \tag{21}$$

This value coincides with the one obtained in Ref. [3] provided that the arbitrary mass μ is re-scaled to give the same log-independent term. This coincidence may be formally explained as follows (for a comparison, see also Ref. [12]). Computing the vacuum average of the traced energy-momentum tensor

$$\langle 0|T_{\mu\mu}|0\rangle = m^2\langle 0|\varphi^2|0\rangle \, , \tag{22}$$

with account of the condition (3) we find [3]:

$$\varepsilon = \frac{m^2}{4}\int \frac{d^4k}{(2\pi)^4}\;\frac{i}{k^2-m^2+i\varepsilon}\,. \tag{23}$$

The equivalence of the definition (6) and the expression (23) stems from Eqs. (6) and (3):

$$\begin{aligned}\langle 0|T_{\mu\mu}|0\rangle &= 4\varepsilon = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_k}\left(\omega_k^2-\vec{k}^2\right) \\ &= m^2\int\frac{d^4k}{(2\pi)^4}\;2\pi\delta_+\left(k^2-m^2\right) = m^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\varepsilon}\,.\end{aligned} \tag{24}$$

This integral can be calculated in the dimensional regularization scheme what gives:

$$\varepsilon^{(1)}_{\mathrm{dim}} = -\frac{m^4}{64\pi^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{\varepsilon}\left(\frac{1}{\varepsilon}+1-\gamma_E\right)\,. \tag{25}$$

On the other hand, evaluating the vacuum energy density according the definition (10) in the same regularization scheme one gets:

$$\varepsilon^{(2)}_{\mathrm{dim}} = -\frac{1}{2}\int\frac{d^nk}{(2\pi)^n}\,\ln\frac{m^2}{k^2-m^2+i\varepsilon} = -\frac{m^4}{64\pi^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{\varepsilon}\left(\frac{1}{\varepsilon}+\frac{3}{2}-\gamma_E\right)\,, \tag{26}$$

what coincides exactly with the result obtained in Ref. [3] (in this paper Eq. (8)) and differs by irrelevant log-independent term with (25). Therefore, one can see that the dimensional and ζ-function regularization schemes satisfy the relativistic invariance condition and yield the similar result. In these cases, the power-law divergences are absent (in contrast to the four-dimensional cutoff method, as well as any other relativistic invariant scheme with an UV cutoff: here the quadratic term survives [3]), and the finite log-independent difference between them can be eliminated by means of the corresponding re-scaling of the arbitrary mass μ.

The corresponding result for the fermion field $\psi(x)$ can also be obtained from the fermionic partition function

$$Z_{\mathrm{F}}(\bar{\psi},\psi) = \int\mathcal{D}\bar{\psi}\mathcal{D}\psi\;\exp\left(i\int dx\sqrt{-g}\bar{\psi}(x)A_{\mathrm{F}}\psi(x)\right)\,, \tag{27}$$

where

$$A_{\mathrm{F}} = i\hat{\partial} - m_f\,. \tag{28}$$

By means of the similar considerations, one gets the fermion field contribution to the zero-point energy:

$$\varepsilon_f = \frac{1}{V_4}\ln Z_{\mathrm{F}}(\bar{\psi},\psi) = -\frac{4}{V_4}\left(\zeta_f'(0)-\zeta_f(0)\ln\frac{\mu}{m_f}\right) \tag{29}$$

with the following relations between the scalar and fermion ζ-functions:

$$\zeta_f(0, m_f) = \zeta(0, m) \ , \ \ \zeta_f'(0, m_f) = \tfrac{1}{2}\zeta'(0, m) \, , \tag{30}$$

and the factor of two added to take into account the anti-particle states. One sees now that the fermionic and scalar zero-point energy densities satisfy:

$$\varepsilon_f = -4\varepsilon \, , \tag{31}$$

as it should be.

The expression for the vacuum energy (21) has been obtained within the relativistic covariant formulation from the very beginning, and we observe that the properly applied ζ-function regularization does not break the invariance. Note also, that within this approach the expressions for the energy density are finite at all steps of calculations, containing no divergent terms. This is the well-known feature of the ζ-function method, and follows directly from the analytical properties of the generalized ζ-function [4, 6].

The question remains how to deal with the logarithmic dependence on the arbitrary mass scale μ. One might treat it as a high energy (UV) boundary of order of the Planck scale $M_{\rm P}$, but this is not a satisfactory way since the scale μ is completely arbitrary and its fixing at a certain value has not any physical basis. Another possible way is to demand the total effective constant $\Lambda_{\rm eff}$ in Eq. (5) to be independent of this (now treated as "unphysical") scale parameter. This requirement gives the renormalization invariance equation

$$\mu \frac{d}{d\mu} \Lambda_{\rm eff} = 0 \, , \tag{32}$$

what means that the renormalized classical cosmological constant Λ_0 (see Eq. (1)) becomes a "running constant" having the logarithmic dependence on the scale μ [13, 14]:

$$\Lambda_0(\mu) = \Lambda_0(\mu_0) + \frac{Gm^4}{4\pi} \ln \frac{\mu^2}{\mu_0^2} \, , \tag{33}$$

where the value $\Lambda_0(\mu_0)$ gives the boundary condition for the solution of the differential equation (32). One may think that the running parameter $\Lambda_0(\mu)$ depends on the initial value $\Lambda_0(\mu_0)$ as well as on μ_0 itself, but this is not the case. Indeed, it should not depend on the starting point, what is provided by the renormalization invariance condition (32). Then it is convenient to express the running constant in terms of a single variable (thus excluding an extra parameter):

$$M = \mu_0 \exp \left[-\frac{4\pi}{Gm^4} \Lambda_0(\mu_0) \right] \, , \tag{34}$$

and write

$$\Lambda_0(\mu) = \frac{Gm^4}{4\pi}\ln\frac{\mu}{M}\,. \tag{35}$$

The mass parameter M plays now a role of the fundamental energy scale analogous to $\Lambda_{\rm QCD}$, while the renormalized classical cosmological constant Λ_0 becomes a running parameter, the value of which can be extracted from the data at certain energy scale μ. The expression for the renormalized effective cosmological constant then reads:

$$\Lambda_{\rm eff} = \frac{Gm^4}{4\pi}\left[\ln\frac{m}{M} + \frac{3}{2}\right]\,, \tag{36}$$

and is controlled by the forth power of the (small) mass m. The dependence on the experimentally detectable parameter M, which now can be identified with some high energy scale, such as the Planck mass, is only logarithmic and affects slightly the result.

To summarize, it is confirmed that the proper relativistic invariant calculation in the regularization scheme which does not use explicitly any UV cutoff (such as dimensional or ζ-function regularization) of the quantum zero-point energy contribution to the cosmological constant in the flat space-time yields no power-law divergences (just logarithmic ones in the dimensional regularization, and no divergences at all in the ζ-function regularization), and yields the result which is determined by the fourth power of the elementary quanta mass rather than the large mass scale. The dependence on the latter appears to be only logarithmic and its influence on the result is not so important. In this simple study we neglected the possible curvature of the space-time what might change the results significantly. Also we did not take into account the possible presence of different sorts of quantum fields which can contribute to the vacuum energy [13].

As compared to the other possible regularization schemes, the ζ-function method based on the formulas (10),(11) seems to have the following advantages:

(i) By virtue of the analytical properties of the generalized ζ-function, the divergences do not appear at any step of the calculations and hence, from the formal point of view, no (infinite) renormalization is required.

(ii) This method starts with the covariant expression (11) what can be easily generalized to the case of the curved space-time. This is important since the effects of the curvature may be nontrivial [4, 5].

The author is most grateful to E.Kh. Akhmedov for fruitful discussions, critics and useful explanations, and to H. Stefancic for discussion and pointing out some errors in results. The work was partially supported by the Russian Federation President's Grant 1450-2003-2. The hospitality and financial support of the Abdus Salam ICTP in Trieste is gratefully acknowledged.

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