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RADIATIVE CORRECTIONS TO NEUTRINO–NUCLEON QUASIELASTIC SCATTERING

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Dedicated to the Memory of Jan Kwieciński

Full one-loop radiative corrections are calculated for low energy neutrino–nucleon quasi-elastic scattering, $\bar{\nu}_e + p \longrightarrow e^+ + n$, which involves both Fermi and Gamow–Teller transitions, in the static limit of nucleons. The calculation is performed for both angular independent and dependent parts. We separate the corrections into the outer and inner parts à la Sirlin. The outer part is infrared and ultraviolet finite, and involves the positron velocity. The calculation of the outer part is straightforward, but that of the inner part requires a scrutiny concerning the continuation of the long-distance hadronic calculation to the short-distance quark treatment and the dependence on the model of hadron structure. We show that the logarithmically divergent parts do not depend on the structure of hadrons not only for the Fermi part, but also for the Gamow–Teller part. This observation enables us to deal with the inner part for the Gamow–Teller transition nearly parallel of that for the Fermi transition. The inner part is universal to weak charged-current processes and can be absorbed into the modification of the coupling constants up to the order of the inverse proton mass $O(1/m_p)$. The resulting $O(\alpha)$ corrections to the differential cross section take the form $[1+\delta_{\rm out}(E)]\left[(1+\delta_{\rm in}^{\rm F})\langle 1\rangle^2 + g_A^2(1+\delta_{\rm in}^{\rm GT})\langle \boldsymbol{\sigma}\rangle^2\right]$, where $\langle 1\rangle$ and $\langle \boldsymbol{\sigma}\rangle$ stand for the Fermi and Gamow–Teller matrix elements; the outer correction $\delta_{\rm out}$ is positron energy (E) dependent and takes different functional forms for the angular independent and dependent parts. All factors are explicitly evaluated.

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1. Introduction

The purpose of this paper is to present a calculation of the radiative correction to neutrino–nucleon quasielastic scattering (inverse beta decay),

$$\bar{\nu}_e + p \longrightarrow e^+ + n\,, \tag{1}$$

whose experimental accuracy now approaches the level that makes radiative corrections a practically important problem. The calculation is similar to that for neutron beta decay, which has a long history of no less than forty years [1–11].

The calculation of radiative corrections to neutron beta decay is not quite straightforward. There are a number of subtleties. They originate from the fact that, at low energies, one has to deal with the proton and neutron in four-Fermi theory while the loop integral is divergent: this divergence is made finite in the electroweak theory, but here we must treat quarks and continue the quark calculation to hadronic theory at an intermediate energy scale.

To clarify the issues and the scope of the present paper, we start with a brief survey of the work done in the past. The pioneering work of Kinoshita and Sirlin [1], long before the development of renormalizable gauge theories, evaluated photon exchange corrections to the neutron beta-decay within four-Fermi V–A theory of weak interactions. They showed the cancellation of infrared divergences and derived the correct velocity dependence of the final electron, taming the ultraviolet divergence with cut-off theory.

Repeating the calculation of Kinoshita and Sirlin, Berman and Sirlin [2] speculated that the logarithmic divergences are not affected by strong interactions. This was later proven by Abers *et al.* [3]. The theorem that the logarithmic divergences are universal is based on the conserved vector current with the use of the current algebra technique [12,13]. This theorem, however, applies only to the logarithmic divergent part of super-allowed Fermi transitions. More precisely speaking, it applies only to the purely vector-current contribution in the Fermi transitions: it does not apply to the logarithmic divergent part arising from the axial current which appears on the loop level by interference. The divergences due to the axial current (which are linear in the axial-vector coupling g_A) in general depend on the model of hadron structure and strong interaction. Abers *et al.* [3] proposed a model (A_1-exchange model) to evaluate such contributions.

Sirlin [4] has gone one step further to separate radiative corrections of neutron beta decay into outer and inner parts. The outer part corrections are both infrared and ultraviolet finite, gauge-independent, and contain full electron's velocity dependence, thus depending on specific nuclei that undergo beta decay. The outer part is not affected by strong interactions. The inner part is independent of electron's velocity, infrared finite but contains the ultraviolet divergences. This separation enables one to absorb the inner part into a universal multiplicative correction factor of the vector coupling constant that is relevant to all beta decays.

Electroweak gauge theory renders the ultraviolet divergences of the inner part finite. The calculation [7,8] conventionally divides the integration region of the virtual gauge bosons into long- and short-distance parts:

$$(i) \quad 0 < |k|^2 < M^2\,, \qquad (ii) \quad M^2 < |k|^2 < \infty\,, \tag{2}$$

where k is the (Wick-rotated Euclidean) momentum of the virtual gauge bosons, and the mass scale M, introduced by hand, divides the low- and high-energy regimes and is supposed to lie between the proton–neutron masses (m_p and m_n) and the (W, Z) boson masses, m_W and m_Z. Old-fashioned four-Fermi interactions are applied to the proton and neutron in region (i), and the mass scale M is regarded as the ultraviolet cutoff of the QED (*i.e.*, purely photonic) correction. In region (ii), electroweak theory is used for quarks and leptons, and M is a mass scale that describes the onset of the asymptotic behaviour. The concern is whether the results in (i) and (ii) join together smoothly. In fact, the theorem mentioned above guarantees that the logarithmic terms coming from the pure vector current in the super-allowed Fermi transitions have the common coefficient in (i) and (ii), and the integrals continue smoothly; thereby M-dependence disappears[1].

Radiative corrections proportional to $f_V g_A$, on the other hand, require some intricate treatment. (Here $f_V = 1$ denotes the vector coupling constant parallel to g_A; we put f_V to trace the vector current contributions.) One conceivable way proposed by Sirlin [7] to deal with the axial current contribution is to use the free quark model for the short-distance contribution in the asymptotic regime, and to work explicitly with the proton and neutron for the long-distance part, by introducing nucleon's electromagnetic and weak form factors to render the logarithmic divergence milder; the mass scale M then remains as a lower cut-off for the asymptotic regime in the region (ii). This calculation is necessarily model-dependent. Marciano and Sirlin [9] (see also Towner [10]) evaluated the axial current contribution this way, and their calculation has widely been employed in $0^+ \to 0^+$ beta decay phenomenology.

All the developments after Kinoshita and Sirlin, however, are restricted to Fermi transitions, whose most important application is to $0^+ \to 0^+$ beta decays; little progress has been made regarding Gamow–Teller transitions, except for the early studies which showed that the outer correction of Sirlin is also applicable to Gamow–Teller transitions.

[1] Contrary to the logarithmically divergent part that is universal, the constant part may in principle receive correction from strong interactions.

The full calculation of the radiative correction to (1) requires not only the treatment of the Fermi transition, but also of the Gamow–Teller transition. QED radiative corrections to (1) were already calculated by Vogel [14] and Fayans [15] within the cut-off theory with the point nucleons. From these calculations one can extract the outer part whereas the inner part is basically left untouched, thus circumventing the problems concerning the axial current complications if the inner part is empirically evaluated from neutron life time. These calculations, that lack the evaluation of the inner part, however, would not completely elucidate the structure of radiative corrections in quasi-elastic neutrino scattering.

In this paper we attempt to calculate full one-loop radiative corrections to (1) for both angular independent and dependent parts, including those to the Gamow–Teller transition. We prove that the coefficient of the logarithmic divergences arising from the purely axial vector current are not affected by strong interactions, which is parallel to the theorem for the Fermi transition[2]. The constant term may receive extra contributions from non-conservation of the axial current, but our observation opens up a way to combine calculations in two regions in (2) smoothly for the Gamow–Teller part.

The vector current contribution to the Gamow–Teller transition is model dependent, just as much as the axial current part that contributes to the Fermi transition. Our treatments will be reciprocal between the Fermi and the Gamow–Teller transitions.

Our observations allow the full calculation of the radiative correction to the Gamow–Teller transition nearly parallel to that for the Fermi transition, though leaving somewhat more room for extra contributions from strong interactions to the constant term. We present the full expression of the outer and inner radiative corrections to neutrino–nucleon scattering in the static limit of nucleons. The application to other charged current processes is straightforward.

2. Preliminaries

We use the four-Fermi theory to derive radiative corrections in region (i). The matrix element of the Born term for the process (1) is given by

$$\mathcal{M}^{(0)} \equiv \frac{G_V}{\sqrt{2}} \left[\bar{v}_\nu(p_\nu)\gamma^\lambda(1-\gamma^5)v_e(\ell)\right] \left[\bar{u}_n(p_1)W_\lambda(p_1,p_2)u_p(p_2)\right] , \qquad (3)$$

[2] This proof is implicitly included in the work of Sirlin [16], which discussed the universality of the logarithmic divergence for the semi-leptonic processes in the electroweak theory. We thank Professor Sirlin for calling our attention to his work.

where $G_V = G_{\rm F}\cos\theta_{\rm C}$ with the universal Fermi coupling $G_{\rm F}$ and the Cabibbo angle $\theta_{\rm C}$. The spinors of the neutron, proton, positron and antineutrino are denoted by u_n, u_p, v_e, and v_ν, respectively, with momenta specified in parentheses.

The vertex of the hadronic weak current in general depends on nucleon's momenta, thus written as $W_\lambda(p_1, p_2)$, but is well-approximated at low energies by

$$W_\lambda(p_1, p_2) = \gamma_\lambda(f_V - g_A\gamma^5)\,. \tag{4}$$

We retain the constant f_V to trace the vector current contributions. We do not consider the terms of $O(1/m_p)$.

The differential cross section is given in terms of the invariant amplitude $\mathcal{M}$,

$$\begin{aligned}\frac{d\sigma(\bar{\nu}_e + p \longrightarrow e^+ + n)}{d(\cos\theta)} &= \frac{1}{64\pi}\frac{E\beta}{m_p m_n E_\nu}\sum_{\rm spin}|\mathcal{M}|^2 \\ &= \frac{G_V^2}{2\pi}\,E^2\beta\left\{A(\beta) + B(\beta)\beta\cos\theta\right\}\,, \end{aligned}\tag{5}$$

where θ is the angle between incident antineutrino momentum $\boldsymbol{p}_\nu$ and positron's, $\boldsymbol{\ell}$. The spin summation is taken over all external fermions. The energy of the positron in the final state is denoted by $E = E_\nu + m_p - m_n$, E_ν being the incident antineutrino energy; $\beta = \sqrt{E^2 - m_e^2}/E$ is the velocity of the positron. The forward–backward asymmetry is given by

$$\langle\cos\theta\rangle = \frac{B(\beta)\beta}{3A(\beta)}\,. \tag{6}$$

Upon spin summation for the Born amplitude (3), we arrive at

$$\sum_{\rm spin}\left|\mathcal{M}^{(0)}\right|^2 = 32G_V^2 m_n m_p E E_\nu\left\{(f_V^2 + 3g_A^2) + (f_V^2 - g_A^2)\beta\cos\theta\right\}\,. \tag{7}$$

In the tree level,

$$A(\beta) = A_0 \equiv f_V^2 + 3g_A^2\,, \qquad B(\beta) = B_0 \equiv f_V^2 - g_A^2\,. \tag{8}$$

We will write $f_V^2 = f_V^2\langle 1\rangle^2$, and $3g_A^2 = g_A^2\langle\boldsymbol{\sigma}\rangle^2$ to make the Fermi and Gamow–Teller contributions explicit.

3. QED corrections

The QED radiative corrections in $A(\beta)$ were partly obtained by Vogel [14] and Fayans [15]. These authors, however, did not calculate the inner part correction. In this paper we compute the full corrections to both $A(\beta)$ and $B(\beta)$. Some of our formulae presented in this paper are already obtained in [14] and [15]. Nonetheless, since the derivation is not given in the above references, we give sketches of calculations so that interested readers can check them easily.

The diagrams we consider are depicted in figure 1, where (v) is the vertex correction, (s) is the self energy correction and (b) is bremsstrahlung. We consider the static limit for nucleons, $q^2 = (p_1 - p_2)^2 \ll m_p^2$. Our calculation is done in the Feynman gauge throughout.

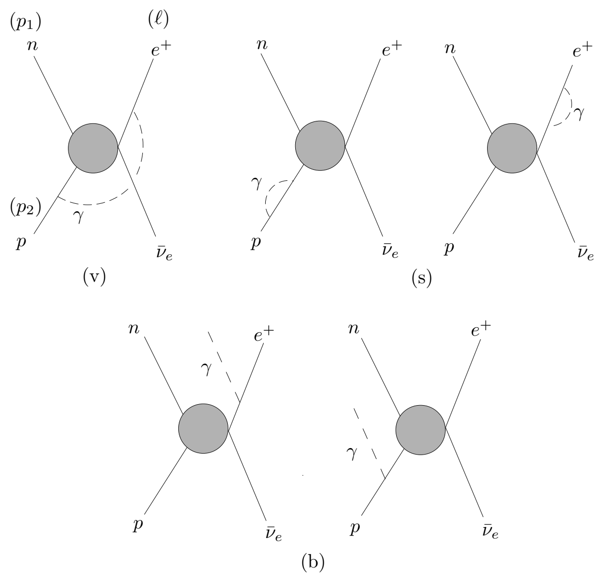

Fig. 1. QED corrections: (v) vertex correction, (s) self-energy correction, and (b) bremsstrahlung.

3.1. Vertex corrections

The vertex correction is given by

$$\begin{aligned}\mathcal{M}^{(v)} &= \frac{i}{\sqrt{2}} G_V e^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k-\ell)^2 - m_e^2} \frac{1}{(p_2-k)^2 - m_p^2} \frac{1}{k^2 - \lambda^2} \\ &\quad \times \bar{v}_\nu \gamma^\lambda (1-\gamma^5) \left\{ \gamma \cdot (k-\ell) + m_e \right\} \gamma^\mu v_e(\ell) \\ &\quad \times \bar{u}_n(p_1) W_\lambda(p_1, p_2 - k) \left\{ \gamma \cdot (p_2 - k) + m_p \right\} \gamma_\mu u_p(p_2) \,, \end{aligned} \tag{9}$$

where λ is the photon mass to regulate the infrared divergence.

Using the identities obtained with the use of the Dirac equation for the positron and proton,

$$\begin{aligned} \{\gamma\cdot(k-\ell)+m_e\}\,\gamma^\mu v_e(\ell) &= \{(k-2\ell)^\mu + i\sigma^{\mu\nu}k_\nu\}\,v_e(\ell)\,, &(10)\\ \{\gamma\cdot(p_2-k)+m_p\}\,\gamma_\mu u_p(p_2) &= \{(2p_2-k)_\mu - i\sigma_{\mu\nu}k^\nu\}\,u_p(p_2)\,, &(11)\end{aligned}$$

we decompose (9) into three parts (see [4]),

$$\mathcal{M}^{(v)} = \mathcal{M}^{(v1)} + \mathcal{M}^{(v2)} + \mathcal{M}^{(v3)}\,. \tag{12}$$

Here $\mathcal{M}^{(v1)}$ picks up the product of $(k-2\ell)^\mu$ in (10) and $(2p_2-k)_\mu$ in (11), and at the same time $W_\lambda(p_1,p_2-k)$ is replaced by $W_\lambda(p_1,p_2)$. It has apparently the same gamma matrix structure as the Born term (3), and is then written as

$$\begin{aligned} \mathcal{M}^{(v1)} &= e^2 I(\beta)\times\mathcal{M}^{(0)}\,, &(13)\\ I(\beta) &= i\int\frac{d^4k}{(2\pi)^4}\frac{(k-2\ell)\cdot(2p_2-k)}{\{(k-\ell)^2-m_e^2\}\{(p_2-k)^2-m_p^2\}\{k^2-\lambda^2\}}\,. &(14)\end{aligned}$$

The integral $I(\beta)$ in (14) is given in Appendix of [3]; the real part reads

$$\begin{aligned} I(\beta)+I(\beta)^* = \frac{1}{8\pi^2}\Bigg[&1+\log\left(\frac{M^2}{m_e^2}\right)-\frac{2}{\beta}\tanh^{-1}\beta\log\left(\frac{m_e^2}{\lambda^2}\right)\\ &+\frac{2}{\beta}L\left(\frac{2\beta}{1+\beta}\right)-\frac{2}{\beta}(\tanh^{-1}\beta)^2\Bigg]\,, \end{aligned}\tag{15}$$

where

$$L(z) = \int_0^z \frac{dt}{t}\log(1-t) \tag{16}$$

is the Spence function. The ultraviolet cutoff M has been brought about by the cut-off of the integral in region (i) of (2).

The term $\mathcal{M}^{(v2)}$ represents the combination of $i\sigma^{\mu\nu}k_\nu$ in (10) and $(2p_2-k)_\mu$ in (11), and $W_\lambda(p_1,p_2-k)$ is again replaced with $W_\lambda(p_1,p_2)$, *i.e.*,

$$\begin{aligned} \mathcal{M}^{(v2)} = \;& \frac{i}{\sqrt{2}}G_V e^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k-\ell)^2-m_e^2}\frac{1}{(p_2-k)^2-m_p^2}\frac{1}{k^2-\lambda^2}\\ &\times\bar{v}_\nu\gamma^\lambda(1-\gamma^5)i\sigma^{\mu\nu}k_\nu v_e(\ell)\\ &\times\bar{u}_n(p_1)W_\lambda(p_1,p_2)(2p_2-k)_\mu u_p(p_2)\,. \end{aligned}\tag{17}$$

The gamma matrix structure of the nucleon currents in (17) is the same as that in (3), although that of leptons is not. The contributions of $\mathcal{M}^{(v1)}$ and $\mathcal{M}^{(v2)}$ are thus evaluated without referring to details of the hadronic part of the currents (see Appendix A). The radiative correction to the cross section is given as the tree-one-loop interference amplitude, which takes the form

$$\sum_{\rm spin}\left\{\left(\mathcal{M}^{(v1)}+\mathcal{M}^{(v2)}\right)\mathcal{M}^{(0)*}+\left(\mathcal{M}^{(v1)*}+\mathcal{M}^{(v2)*}\right)\mathcal{M}^{(0)}\right\}$$
$$=32G_V^2 m_n m_p E E_\nu\left[\left\{e^2\left(I(\beta)+I(\beta)^*\right)+\frac{e^2}{4\pi^2}\beta\tanh^{-1}\beta\right\}\left(f_V^2+3g_A^2\right)\right.$$
$$\left.+\left\{e^2\left(I(\beta)+I(\beta)^*\right)+\frac{e^2}{4\pi^2}\frac{1}{\beta}\tanh^{-1}\beta\right\}\left(f_V^2-g_A^2\right)\beta\cos\theta\right]. \qquad (18)$$

The remaining terms are collected in $\mathcal{M}^{(v3)}$, which reads

$$\begin{aligned}\mathcal{M}^{(v3)} &= \frac{i}{\sqrt{2}}G_V e^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k-\ell)^2-m_e^2}\frac{1}{(p_2-k)^2-m_p^2}\frac{1}{k^2-\lambda^2}\\ &\times\bar{v}_\nu\gamma^\lambda\left(1-\gamma^5\right)\left\{(k-2\ell)^\mu+i\sigma^{\mu\nu}k_\nu\right\}v_e(\ell)\\ &\times\bar{u}_n(p_1)R_{\mu\lambda}(p_1,p_2,k)u_p(p_2)\,, \end{aligned}\qquad (19)$$

where

$$\begin{aligned}R_{\mu\lambda}(p_1,p_2,k) &\equiv \left\{W_\lambda(p_1,p_2-k)-W_\lambda(p_1,p_2)\right\}(2p_2-k)_\mu\\ &\quad -iW_\lambda(p_1,p_2-k)\sigma_{\mu\nu}k^\nu \qquad (20)\\ &\simeq -iW_\lambda(p_1,p_2-k)\sigma_{\mu\nu}k^\nu\,, \qquad (21)\end{aligned}$$

in the approximation of the point nucleons. It is only this $\mathcal{M}^{(v3)}$ term that depends on the details of the structure of the weak currents. It is clear from the powers of k that this term is infrared convergent. The explicit use of the four-Fermi interaction (3) gives an ultraviolet divergence. A straightforward calculation (see Appendix B) shows that

$$\sum_{\rm spin}\left\{\mathcal{M}^{(v3)}\mathcal{M}^{(0)*}+\mathcal{M}^{(v3)*}\mathcal{M}^{(0)}\right\}$$
$$=32G_V^2 m_n m_p E E_\nu\left[\Phi^{\rm F}(1+\beta\cos\theta)\langle 1\rangle^2+\Phi^{\rm GT}(3-\beta\cos\theta)\cdot\frac{1}{3}\langle\boldsymbol{\sigma}\rangle^2\right], \qquad (22)$$

where

$$\Phi^{\mathrm{F}} = \frac{e^2}{8\pi^2}\left[f_V^2\left\{\frac{3}{2}\log\left(\frac{M^2}{m_p^2}\right)+\frac{3}{4}\right\}+g_A f_V\left\{\frac{3}{2}\log\left(\frac{M^2}{m_p^2}\right)+\frac{9}{4}\right\}\right], \tag{23}$$

$$\Phi^{\mathrm{GT}} = \frac{e^2}{8\pi^2}\left[g_A^2\left\{\frac{3}{2}\log\left(\frac{M^2}{m_p^2}\right)+\frac{7}{4}\right\}+f_V g_A\left\{\frac{3}{2}\log\left(\frac{M^2}{m_p^2}\right)+\frac{5}{4}\right\}\right]. \tag{24}$$

The superscripts F and GT refer to the contribution to the Fermi and Gamow–Teller transitions. Note that the correction from $\mathcal{M}^{(v3)}$ is written as multiplicative factors on the coupling constants for both Fermi and Gamow–Teller parts, without disturbing the original angular dependence.

3.2. Self energy corrections

After subtracting the pole term by mass renormalization, we are left with the wave function renormalization. The wave function renormalization constant for the fermion of mass m is given by [17]

$$Z_2(m) = 1-\frac{e^2}{8\pi^2}\left\{\frac{1}{2}\log\left(\frac{M^2}{m^2}\right)+\frac{9}{4}-\log\left(\frac{m^2}{\lambda^2}\right)\right\} \tag{25}$$

for the on-shell renormalization. The self energy correction is given by

$$\mathcal{M}^{(s)} = \left\{\sqrt{Z_2(m_e)}-1+\sqrt{Z_2(m_p)}-1\right\}\mathcal{M}^{(0)}, \tag{26}$$

and the contribution to the cross section is

$$\begin{aligned}&\sum_{\text{spin}}\left(\mathcal{M}^{(s)}\mathcal{M}^{(0)*}+\mathcal{M}^{(s)*}\mathcal{M}^{(0)}\right)\\ &=2\left\{\sqrt{Z_2(m_e)}-1+\sqrt{Z_2(m_p)}-1\right\}\sum_{\text{spin}}\left|\mathcal{M}^{(0)}\right|^2.\end{aligned} \tag{27}$$

We see that the $\log M^2$ dependence in (18) is cancelled by that in (27), *i.e.*, the correction from $\mathcal{M}^{(v1)}+\mathcal{M}^{(v2)}+\mathcal{M}^{(s)}$ is ultraviolet finite. The infrared divergence is cancelled after the bremsstrahlung contribution $\mathcal{M}^{(b)}$ is added. The combination $\mathcal{M}^{(v1)}+\mathcal{M}^{(v2)}+\mathcal{M}^{(s)}+\mathcal{M}^{(b)}$ is identified as the outer part radiative correction of Sirlin (up to a numerical constant) for neutrino–nucleon quasi-elastic scattering. It is easy to show that this term is gauge independent. On the other hand, $\mathcal{M}^{(v3)}$ still contains ultraviolet divergence, and it also receives hadronic complications. This contribution is taken to be a part of the inner correction.

3.3. Bremsstrahlung

The cross section of single photon emission in figure 1(b) is given by

$$\sigma(\bar{\nu}_e + p \longrightarrow e^+ + n + \gamma) = \frac{1}{(2\pi)^5}\cdot\frac{1}{8m_p m_n E_\nu}\int\frac{d^3\boldsymbol{\ell}'}{2E'}\int\frac{d^3\boldsymbol{k}}{2\omega} \times\delta(E-E'-\omega)\times\frac{1}{2}\sum_{\rm spin}\left|\mathcal{M}^{(b)}\right|^2 , \quad (28)$$

where $E' = \sqrt{\boldsymbol{\ell}'^2 + m_e^2}$ and $\omega = \sqrt{\boldsymbol{k}^2 + \lambda^2}$. In the nucleon static limit, we find

$$\frac{1}{2}\sum_{\rm spin}\left|\mathcal{M}^{(b)}\right|^2 = \frac{64 m_p m_n G_V^2 e^2}{[(k+\ell')^2 - m_e^2]}\Bigg\{\left(f_V^2 + 3g_A^2\right)E_\nu\left[E\left(\boldsymbol{\ell}'^2 - \frac{(\boldsymbol{k}\cdot\boldsymbol{\ell}')^2}{\omega^2}\right) + (k\cdot\ell')\omega\right] +(f_V^2 - g_A^2)\Bigg[(\boldsymbol{\ell}'+\boldsymbol{k})\cdot\boldsymbol{p}_\nu\left(\boldsymbol{\ell}'^2 - \frac{(\boldsymbol{k}\cdot\boldsymbol{\ell}')^2}{\omega^2}\right) + (k\cdot\ell')(\boldsymbol{k}\cdot\boldsymbol{p}_\nu) +(k\cdot\ell)\left((\boldsymbol{\ell}'\cdot\boldsymbol{p}_\nu) - \frac{(\boldsymbol{k}\cdot\boldsymbol{\ell}')(\boldsymbol{k}\cdot\boldsymbol{p}_\nu)}{\omega^2}\right)\Bigg]\Bigg\}. \quad (29)$$

After integrating over photon phase space, the terms proportional to $(f_V^2 + 3g_A^2)$ in (29) contribute to $A(\beta)$, and those proportional to $(f_V^2 - g_A^2)$ to $B(\beta)$.

Putting (29) into (28) and integrating over photon momentum, we obtain

$$\frac{d\sigma(\bar{\nu}_e + p \longrightarrow e^+ + n + \gamma)}{d(\cos\theta)} = \frac{G_V^2}{2\pi}E^2\left(\frac{e^2}{8\pi^2}\right)\beta\left\{g^{(b1)}(\beta)A_0 + g^{(b2)}(\beta)B_0\beta\cos\theta\right\} , \quad (30)$$

where the function $g^{(b1)}(\beta)$, which was calculated by Vogel [14] (partly numerically) and by Fayans [15] using SCHOONSHIP, is written as

$$g^{(b1)}(\beta) = 4\log\left(\frac{2(E-m_e)}{\lambda}\right)\left(\frac{1}{\beta}\tanh^{-1}\beta - 1\right) +4\left(\frac{1}{\beta}\tanh^{-1}\beta - 1\right)\log\left(\frac{2(E+m_e)}{m_e}\right) + \frac{6}{\beta}L\left(\frac{2\beta}{1+\beta}\right) -\frac{6}{\beta}(\tanh^{-1}\beta)^2 + \frac{7-\beta^2}{2\beta}\tanh^{-1}\beta + \frac{17}{2} . \quad (31)$$

The other function $g^{(b2)}(\beta)$ is calculated to give

$$\begin{aligned} g^{(b2)}(\beta) &= \frac{4}{\beta^2} + \frac{7}{2} - \frac{4\sqrt{1-\beta^2}}{\beta^2} \\ &\quad - \left(4 + \frac{1}{\beta}\right)\tanh^{-1}\beta + \left(-\frac{1}{2\beta^2} - \frac{3}{2} + \frac{4}{\beta}\right)(\tanh^{-1}\beta)^2 \\ &\quad -\frac{2}{\beta}L\left(\frac{2\beta}{1+\beta}\right) + \frac{8}{\beta}L\left(1 - \sqrt{\frac{1-\beta}{1+\beta}}\right) \\ &\quad +4\left(1 - \frac{1}{\beta}\tanh^{-1}\beta\right)\log\left\{\frac{\lambda}{2m_e}\left(1+\frac{1}{\beta}\right)\frac{\sqrt{1+\beta}+\sqrt{1-\beta}}{\sqrt{1+\beta}-\sqrt{1-\beta}}\right\} . \end{aligned} \tag{32}$$

The treatment of the infrared divergence for $g^{(b2)}$ is somewhat subtle. Steps of the bremsstrahlung calculation are presented in Appendix C.

3.4. QED corrections: summary

We summarise the QED radiative corrections as

$$A(\beta) = \langle 1\rangle^2 f_V^2\left[1 + \delta_{\rm out}^{\rm F} + \delta_{\rm in}^{\rm F\,\prime}\right] + \langle\boldsymbol{\sigma}\rangle^2 g_A^2\left[1 + \delta_{\rm out}^{\rm GT} + \delta_{\rm in}^{\rm GT\prime}\right] , \tag{33}$$

$$B(\beta) = \langle 1\rangle^2 f_V^2\left[1 + \tilde{\delta}_{\rm out}^{\rm F} + \delta_{\rm in}^{\rm F\,\prime}\right] - \frac{1}{3}\langle\boldsymbol{\sigma}\rangle^2 g_A^2\left[1 + \tilde{\delta}_{\rm out}^{\rm GT} + \delta_{\rm in}^{\rm GT\prime}\right] . \tag{34}$$

Here

$$\begin{aligned} \delta_{\rm out}^{\rm F} = \delta_{\rm out}^{\rm GT} &= \delta_{\rm out}(E) \\ &= e^2\{I(\beta) + I(\beta)^*\} + \frac{e^2}{4\pi^2}\beta\tanh^{-1}\beta \\ &\quad +2\left\{\sqrt{Z_2(m_e)} - 1 + \sqrt{Z_2(m_p)} - 1\right\} \\ &\quad +\frac{e^2}{8\pi^2}\left\{g^{(b1)}(\beta) + \frac{3}{4}\right\} \qquad (35) \\ &= \frac{e^2}{8\pi^2}\left[\frac{23}{4} + \frac{3}{2}\log\left(\frac{m_p^2}{m_e^2}\right) + \frac{8}{\beta}L\left(\frac{2\beta}{1+\beta}\right)\right. \\ &\quad +4\log\left(\frac{4\beta^2}{1-\beta^2}\right)\left(\frac{1}{\beta}\tanh^{-1}\beta - 1\right) \\ &\quad \left. -\frac{8}{\beta}(\tanh^{-1}\beta)^2 + \left(\frac{7}{2\beta} + \frac{3\beta}{2}\right)\tanh^{-1}\beta\right] \qquad (36) \end{aligned}$$

is the so-called outer correction, which assembles the contributions from $\mathcal{M}^{(v1)}$, $\mathcal{M}^{(v2)}$, $\mathcal{M}^{(s)}$, and $\mathcal{M}^{(b)}$. This term does not contain infrared or ultraviolet divergence, and does not receive complications due to hadronic structure and strong interaction. All positron-velocity dependences are contained in this correction. We note that the last term 3/4 in (35) is added (then subtracted from $\delta_{\rm in}$) to make the definitions of $\delta_{\rm out}$ and $\delta_{\rm in}$ consistent with those of Sirlin for beta decay. The outer correction is common to the Fermi and Gamow–Teller transitions, and it can be factored out to the order of $O(e^2)$.

The inner correction $\delta_{\rm in}{}'$ (prime means the correction from QED only) arises from $\mathcal{M}^{(v3)}$. We find

$$\delta_{\rm in}^{\rm F}{}' = \frac{e^2}{8\pi^2}\left[\frac{3}{2}\log\left(\frac{M^2}{m_p^2}\right) + \frac{g_A}{f_V}\left\{\frac{3}{2}\log\left(\frac{M^2}{m_p^2}\right) + \frac{9}{4}\right\}\right], \quad (37)$$

$$\delta_{\rm in}^{\rm GT}{}' = \frac{e^2}{8\pi^2}\left[\frac{3}{2}\log\left(\frac{M^2}{m_p^2}\right) + 1 + \frac{f_V}{g_A}\left\{\frac{3}{2}\log\left(\frac{M^2}{m_p^2}\right) + \frac{5}{4}\right\}\right]. \quad (38)$$

Here the constant 3/4 is subtracted from $\Phi^{\rm F}$ and $\Phi^{\rm GT}$ to define $\delta_{\rm in}^{\rm F}{}'$ and $\delta_{\rm in}^{\rm GT}{}'$. The inner correction is infrared finite, but contains ultraviolet divergences, $\log M^2$, which are made finite only with electroweak theory. These terms do not contain positron-velocity dependent factors, and thus absorbed into the vector and axial-vector coupling constants. They also agree with the inner correction for beta decay. The inner correction for the Fermi transition has been identified in [4]. The first logarithmic divergent parts, both in $\delta_{\rm in}^{\rm F}{}'$ and $\delta_{\rm in}^{\rm GT}{}'$, do not receive corrections (see Sect. 5). However, the constant terms may, in principle, receive corrections from strong interactions. We shall argue that effects of the hadron structure are essential for terms proportional to g_A/f_V in (37) and f_V/g_A in (38), including the coefficients of logarithmic divergences.

For the correction of the angular dependent part, we have

$$\begin{aligned}\tilde{\delta}_{\rm out}^{\rm F} = \tilde{\delta}_{\rm out}^{\rm GT} &= \tilde{\delta}_{\rm out}(E) \\ &= e^2\{I(\beta) + I(\beta)^*\} + \frac{e^2}{4\pi^2}\frac{1}{\beta}\tanh^{-1}\beta \\ &\quad +2\left\{\sqrt{Z_2(m_e)} - 1 + \sqrt{Z_2(m_p)} - 1\right\} + \frac{e^2}{8\pi^2}\left\{g^{(b2)}(\beta) + \frac{3}{4}\right\} \\ &= \frac{e^2}{8\pi^2}\left[\frac{3}{4} + \frac{3}{2}\log\left(\frac{m_p^2}{m_e^2}\right)\right. \\ &\quad +\frac{8}{\beta}L\left(1 - \sqrt{\frac{1-\beta}{1+\beta}}\right) + \frac{4}{\beta^2} - \frac{4\sqrt{1-\beta^2}}{\beta^2}\end{aligned}$$

$$+4\left(1-\frac{1}{\beta}\tanh^{-1}\beta\right)\log\left(\frac{1}{2}\left(1+\frac{1}{\beta}\right)\frac{\sqrt{1+\beta}+\sqrt{1-\beta}}{\sqrt{1+\beta}-\sqrt{1-\beta}}\right)$$
$$+\left(\frac{1}{\beta}-4\right)\tanh^{-1}\beta+\left(\frac{2}{\beta}-\frac{3}{2}-\frac{1}{2\beta^2}\right)(\tanh^{-1}\beta)^2\Bigg]. \quad (39)$$

The inner correction $\delta_{\rm in}^{\rm F\prime}$ and $\delta_{\rm in}^{\rm GT\prime}$ are, as evident from (22), identical with those that appear in the angular independent correction (33); hence we have not attached tildes.

4. Electroweak corrections and the continuation to low energy

The short distance correction from the integration region (*ii*) in (2) is evaluated using electroweak theory [7]. For the application of electroweak theory, we must deal with quarks, so that the continuation of quark theory to low-energy effective theory for nucleons is essential. When we consider corrections relative to muon decay, we only need to consider the photon and Z exchange diagrams shown in figure 2.

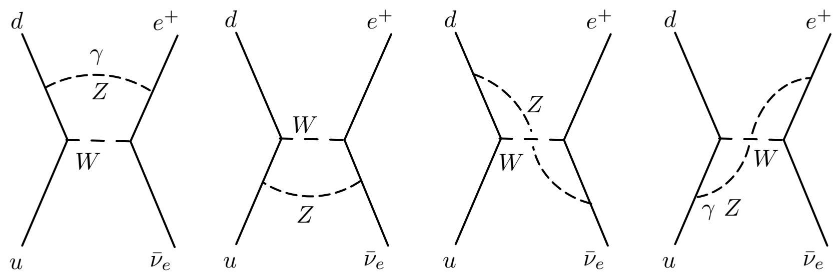

Fig. 2. Photon and Z-boson exchange diagrams.

For a short distance calculation, we can neglect mass and momenta of the external fermions. This calculation was done in [7]. A calculation for the photon exchange yields,

$$\hat{\mathcal{M}}^{(\gamma W)}=\frac{e^2}{16\pi^2}\left(\frac{3}{2}+3\bar{Q}\right)\log\left(\frac{m_W^2}{M^2}\right)\hat{\mathcal{M}}^{(0)}\,, \quad (40)$$

where $\hat{\mathcal{M}}^{(0)}$ is the tree amplitude for neutrino scattering off the assembly of free quarks, and $\bar{Q}=1/6$ is the mean charge of the up and down quarks. Note that the second term (proportional to $\bar{Q}$) originates from the product of two antisymmetric tensors $\varepsilon^{\lambda\mu\nu\rho}$ of leptonic and quark gamma matrices. Similarly for the Z-boson exchange, we obtain

$$\hat{\mathcal{M}}^{(ZW)}=\frac{e^2}{16\pi^2}\left(3\bar{Q}+\frac{5}{2\tan^4\theta_W}\right)\log\left(\frac{m_Z^2}{m_W^2}\right)\hat{\mathcal{M}}^{(0)}\,. \quad (41)$$

The first term is again due to the product of two antisymmetric tensors. The second term is all the rest.

In order to connect the quark-level amplitudes with hadronic ones, we assume that the ratio of the tree and loop amplitudes for neutrino quark scattering is the same as that for neutrino nucleon scattering [7]. This is justified at least for the logarithmic divergent part of the correction for the Fermi transition by current conservation. In the next section we justify this for the logarithmic divergent part of the correction for the Gamow–Teller transition in the presence of partial conservation of the axial current. With this prescription we write the radiative correction to neutrino nucleon scattering at the short distance as

$$\left\{\hat{\mathcal{M}}^{(0)}\left(\hat{\mathcal{M}}^{(\gamma W)*}+\hat{\mathcal{M}}^{(ZW)*}\right)+\hat{\mathcal{M}}^{(0)*}\left(\hat{\mathcal{M}}^{(\gamma W)}+\hat{\mathcal{M}}^{(ZW)}\right)\right\}\frac{\left|\mathcal{M}^{(0)}\right|^2}{\left|\hat{\mathcal{M}}^{(0)}\right|^2}$$
$$=\frac{e^2}{8\pi^2}\left\{\frac{3}{2}\log\left(\frac{m_W^2}{M^2}\right)+3\bar{Q}\log\left(\frac{m_Z^2}{M^2}\right)+\frac{5}{2\tan^4\theta_W}\log\left(\frac{m_Z^2}{m_W^2}\right)\right\}\left|\mathcal{M}^{(0)}\right|^2 . \tag{42}$$

We observe that the M dependence (upper cutoff) that appears in the long-distance radiative correction (37) is cancelled by the first term in the brackets in (42), which renders the ultraviolet divergence in the QED radiative correction finite [7]. Rigorously speaking, this universality applies only to the logarithmically divergent part, and the constant terms might receive extra contributions. Here we simply take the calculation with the point nucleon for the constant term.

Note that the weak coupling constant is determined by the rate of muon decay, where the same box-type Feynman integrals appears. The formula used to determine $G_{\rm F}$ from muon lifetime

$$\frac{1}{\tau_\mu}=\frac{G_{\rm F}^2 m_\mu^5}{192\pi^3}\left(1-\frac{8m_e^2}{m_\mu^2}\right)\left\{1+\frac{3m_\mu^2}{5m_W^2}+\frac{e^2}{8\pi^2}\left[\frac{25}{4}-\pi^2\right]\right\} \tag{43}$$

means that the effects of the box-type diagrams in muon decay is absorbed in $G_{\rm F}$. Thus, when we calculate radiative corrections in terms of $G_{\rm F}$, we should subtract those included in $G_{\rm F}$: the relevant contributions for muon decay are obtained by putting $\bar{Q}=-1/2$ in (42):

$$\frac{e^2}{8\pi^2}\left\{\frac{3}{2}\log\left(\frac{m_W^2}{M^2}\right)-\frac{3}{2}\log\left(\frac{m_Z^2}{M^2}\right)+\frac{5}{2\tan^4\theta_W}\log\left(\frac{m_Z^2}{m_W^2}\right)\right\}\left|\mathcal{M}^{(0)}\right|^2$$
$$=\frac{e^2}{8\pi^2}\left(-\frac{3}{2}+\frac{5}{2\tan^4\theta_W}\right)\log\left(\frac{m_Z^2}{m_W^2}\right)\left|\mathcal{M}^{(0)}\right|^2 . \tag{44}$$

Adding (42) to the QED corrections ${\delta_{\rm in}^{\rm F}}'$ and subtracting (44), the Fermi part inner correction of (37) is

$$\begin{aligned}\delta_{\rm in}^{\rm F} &\equiv {\delta_{\rm in}^{\rm F}}' + \frac{e^2}{8\pi^2}\left[\frac{3}{2}\log\left(\frac{m_W^2}{M^2}\right) + 3\bar{Q}\log\left(\frac{m_Z^2}{M^2}\right) + \frac{5}{2\tan^4\theta_W}\log\left(\frac{m_Z^2}{m_W^2}\right)\right]\\ &\quad -\frac{e^2}{8\pi^2}\left(-\frac{3}{2} + \frac{5}{2\tan^4\theta_W}\right)\log\left(\frac{m_Z^2}{m_W^2}\right)\\ &= \frac{e^2}{8\pi^2}\left[\frac{3}{2}\log\left(\frac{m_Z^2}{m_p^2}\right) + 3\bar{Q}\log\left(\frac{m_Z^2}{M^2}\right) + C^{\rm F}\right], \qquad (45)\end{aligned}$$

where the terms proportional to g_A/f_V are collected in $C^{\rm F}$,

$$C^{\rm F} = \frac{g_A}{f_V}\left\{\frac{3}{2}\log\left(\frac{M^2}{m_p^2}\right) + \frac{9}{4}\right\}. \qquad (46)$$

As noted above, the M-dependence in the first logarithm in (45) is cancelled by the term $\frac{3}{2}\log\left(M^2/m_p^2\right)$ in ${\delta_{\rm in}^{\rm F}}'$. The other M dependent term $(3g_A/2f_V)\log(M^2/m_p^2)$ in ${\delta_{\rm in}^{\rm F}}'$ that arises from the vector–axial-vector interference, however, is hadron structure dependent, which is manifested by the fact that it is not cancelled by the electroweak counterpart $3\bar{Q}\log(m_Z^2/M^2)$: the coefficients of the two logarithmic terms agree only for an unrealistic value $\bar{Q} = g_A/2f_V$. For the evaluation of these contributions, one separates long- and short-distance contributions [7,9], leaving the M dependence that appear in (45) as the lower cut-off of the short-distance integral, whereas estimating the long distance contribution $C^{\rm F}$ by rendering its logarithmic dependence milder with nucleon structure taken into account [9].

We can proceed in a parallel manner for the Gamow–Teller transition. The basic observation is the universality of the logarithmic divergence that also applies to the Gamow–Teller transition for the axial–axial contribution, as shown in the next section. With this universality we can identify the QED correction given in terms of nucleon matrix elements in (38) with that of free quark theory, (40), and the match of the coefficients means the cancellation of the M dependences between the two expressions. We cannot prove the universality for the constant terms and those proportional to f_V/g_A. The corrections for the Gamow–Teller transition may have a more room of extra contributions from strong interaction than for the Fermi transition, unless chiral symmetry is exact. Again, we simply assume the calculation of the point nucleons as was done for the Fermi transition except for those proportional to f_V/g_A. We obtain for the Gamow–Teller transition,

$$\delta_{\rm in}^{\rm GT} = \frac{e^2}{8\pi^2}\left[\frac{3}{2}\log\left(\frac{m_Z^2}{m_p^2}\right) + 1 + 3\bar{Q}\log\left(\frac{m_Z^2}{M^2}\right) + C^{\rm GT}\right], \qquad (47)$$

where the terms proportional to f_V/g_A collected in $C^{\rm GT}$ are

$$C^{\rm GT} = \frac{f_V}{g_A}\left\{\frac{3}{2}\log\left(\frac{M^2}{m_p^2}\right) + \frac{5}{4}\right\} \tag{48}$$

for point nucleons.

For the vector–axial-vector interference term, we encounter exactly the same problem as for the Fermi transition. The two logarithmic terms, $3\bar{Q}\log(m_Z^2/M^2)$ in (47) and $(3f_V/2g_A)\log(M^2/m_p^2)$ in (48), match only when $\bar{Q} = f_V/2g_A$, another unrealistic value. So, we take the same prescription carried out for the Fermi transition by Marciano and Sirlin [9], by including the long-distance part correction introducing nucleon form factors. We present the calculation of $C^{\rm GT}$ in Sect. 6.

5. Universality of the logarithmic divergent part for the Gamow–Teller transition

Abers *et al.* [3] gave a proof that the logarithmic divergences $\frac{3}{2}\log(M^2/m_p^2)$ that appear in the Fermi part, *i.e.*, $\delta_{\rm in}^{\rm F'}$, is universal in that their coefficients are model-independent regarding the structure of hadrons: the coefficient of the logarithmic divergences does not depend on whether one deals with hadronic or quark current, in so far as these currents satisfy the generic commutation relations. Here, we argue that the same statement applies to the Gamow–Teller counterpart in $\delta_{\rm in}^{\rm GT'}$. The currents and the current commutation relations are given in Appendix D. We note that the proof described here is implicitly included in the work of Sirlin [16], which shows that the logarithmic divergences that appear in semi-leptonic processes are universal in the electroweak theory. We show explicitly the universality of the divergences between the four-Fermi theory and the electroweak theory for the axial-vector correction to the Gamow–Teller part.

We introduce two types of Green's functions:

$$T_{\lambda\mu}(k,p_1,p_2) = i\int d^4x e^{ik\cdot x}\langle p_1|T\left(t_\lambda(0)j_\mu^{\rm e.m.}(x)\right)|p_2\rangle\,, \tag{49}$$

$$T_{\lambda\mu\nu}(k,q,p_1,p_2) = \int d^4x e^{iq\cdot x}\int d^4y e^{ik\cdot y}\langle p_1|T\left(t_\lambda(x)j_\mu^{\rm e.m.}(y)j_\nu^{\rm e.m.}(0)\right)|p_2\rangle - \delta T_{\lambda\mu\nu}\,, \tag{50}$$

where $j_\mu^{\rm e.m.}(x)$ is the electromagnetic current and $t_\lambda(x)$ is the V–A weak current

$$t_\lambda(x) = V_\lambda(x) - A_\lambda(x). \tag{51}$$

The term $-\delta T_{\lambda\mu\nu}$ in (50) subtracts the pole term at $p_2^2 = m_p^2$ in $T_{\lambda\mu\nu}(k, p_1, p_2)$, which corresponds to mass renormalization. The considerations in what follows do not depend on whether the weak current (51) is expressed in terms of quark or nucleon fields, because our discussion is based solely on the current–current commutation relation between vector and axial-vector currents.

With conservation of the electromagnetic current and current algebra applied to (49) and (50), we obtain identities:

$$k^\mu T_{\lambda\mu}(k, p_1, p_2) = \langle p_1 | t_\lambda(0) | p_2 \rangle \,, \tag{52}$$

$$k^\lambda T_{\lambda\mu}(k, p_1, p_2) = \langle p_1 | t_\mu(0) | p_2 \rangle + M_\mu - (p_1 - p_2)^\lambda T_{\lambda\mu}(k, p_1, p_2) \,, \tag{53}$$

$$k^\mu T_{\lambda\mu\nu}(k, q, p_1, p_2) = -T_{\lambda\nu}(p_2 - p_1 - k - q, p_1, p_2) - k^\mu \delta T_{\lambda\mu\nu} \,, \tag{54}$$

$$q^\lambda T_{\lambda\mu\nu}(k, q, p_1, p_2) = T_{\mu\nu}(p_2 - p_1 - k - q, p_1, p_2) + T_{\nu\mu}(k, p_1, p_2) + M_{\mu\nu} - q^\lambda \delta T_{\lambda\mu\nu} \,, \tag{55}$$

where

$$M_\mu = -\int d^4x \mathrm{e}^{i(p_2 - p_1 - k)\cdot x} \langle p_1 | T(\partial \cdot A(x) j_\mu^{\mathrm{e.m.}}(0) | p_2 \rangle \,, \tag{56}$$

and

$$M_{\mu\nu} = -i \int d^4x \mathrm{e}^{iq\cdot x} \int d^4y \mathrm{e}^{ik\cdot y} \langle p_1 | T\left(\partial \cdot A(x) j_\mu^{\mathrm{e.m.}}(y) j_\nu^{\mathrm{e.m.}}(0)\right) | p_2 \rangle \tag{57}$$

arise in (53) and (55), as a result of non-conservation of the axial current. If we can set $p_2 - p_1 \approx 0$, then (53) is simplified as

$$k^\lambda T_{\lambda\mu}(k, p_1, p_2) = \langle p_1 | t_\mu(0) | p_2 \rangle + M_\mu \,. \tag{58}$$

By differentiating (55) with respect to q^λ and setting $q = p_2 - p_1$, we get the identity

$$T_{\lambda\mu\nu}(k, p_2 - p_1, p_1, p_2) + q^\rho \frac{\partial}{\partial q^\lambda} T_{\rho\mu\nu}(k, q, p_1, p_2)\Big|_{q = p_2 - p_1} = \frac{\partial}{\partial k^\lambda} T_{\mu\nu}(-k, p_1, p_2) + \frac{\partial}{\partial q^\lambda}\left(M_{\mu\nu} - q^\rho \delta T_{\rho\mu\nu}\right)\Big|_{q = p_2 - p_1} \,. \tag{59}$$

Neglecting again $q = p_2 - p_1$, we write

$$T_{\lambda\mu\nu}(k, p_2 - p_1, p_1, p_2) = \frac{\partial}{\partial k^\lambda} T_{\mu\nu}(-k, p_1, p_2) + \frac{\partial}{\partial q^\lambda}(M_{\mu\nu} - q^\rho \delta T_{\rho\mu\nu})\Big|_{q = p_2 - p_1} \,. \tag{60}$$

We redefine $\mathcal{M}^{(v)}$ of figure 1, including the effect of strong interaction in terms of the Green's functions (49) and (50):

$$\begin{aligned}\mathcal{M}^{(v)} &= -\frac{G_V}{\sqrt{2}}\mathrm{e}^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\lambda^2}\bar{v}_\nu\gamma^\lambda(1-\gamma^5)S_{\mathrm{F}}(k-\ell)\gamma^\mu v_e(\ell)T_{\lambda\mu}(k,p_1,p_2)\\ &= -\frac{G_V}{\sqrt{2}}e^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\lambda^2}\frac{i}{(k-\ell)^2-m_e^2}T_{\lambda\mu}(k,p_1,p_2)\\ &\quad\times\bar{v}_\nu\left(k^\lambda\gamma^\mu+k^\mu\gamma^\lambda-g^{\lambda\mu}\gamma\cdot k-i\varepsilon^{\lambda\rho\mu\sigma}\gamma_\sigma\gamma^5k_\rho-2\ell^\mu\gamma^\lambda\right)(1-\gamma^5)v_e(\ell)\end{aligned} \tag{61}$$

after some manipulation of gamma matrices in the leptonic part. The first and second terms in the brackets of (61) can be expressed by using (52) and (58) in terms of the single-current matrix element.

The third term requires a manipulation of $g^{\lambda\mu}T_{\lambda\mu}(k,p_1,p_2)$. It is known [12, 13] that the large energy limit of the Green's function (49) may be expressed in terms of the equal-time (ET) current commutator

$$\lim_{k^0\to+\infty}T_{\lambda\mu}(k,p_1,p_2)=\frac{1}{k^0}\int d^3\boldsymbol{x}\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\langle p_1|\left[t_\lambda(0),j_\mu^{\mathrm{e.m.}}(x)\right]_{\mathrm{ET}}|p_2\rangle\,. \tag{62}$$

The current algebra relation, used in (62),

$$g^{\lambda\mu}\left[t_\lambda(0),j_\mu^{\mathrm{e.m.}}(x)\right]_{\mathrm{ET}}=-2t_0(x)\delta^3(\boldsymbol{x}) \tag{63}$$

determines the high energy behaviour of $g^{\lambda\mu}T_{\lambda\mu}(k,p_1,p_2)$, which reads

$$\lim_{k^0\to+\infty}T_{\lambda\mu}(k,p_1,p_2)g^{\lambda\mu}=-\frac{2}{k^0}\langle p_1|t_0(0)|p_2\rangle\,. \tag{64}$$

This is non-covariant looking. If we are interested only in the leading term in k^2, we can follow the trick of [3] to rewrite it in a covariant form:

$$\lim_{k^2\to+\infty}T_{\lambda\mu}(k,p_1,p_2)g^{\lambda\mu}\longrightarrow-\frac{2k^\mu}{k^2-2k\cdot p_2}\langle p_1|t_\mu(0)|p_2\rangle\,. \tag{65}$$

The evaluation of the fourth term in (61) requires knowledge beyond generic current commutation relations. Writing explicitly the current in terms of nucleons or quarks (see Appendix D), we obtain

$$-i\varepsilon^{\lambda\rho\mu\sigma}k_\rho\left[t_\lambda(0),j_\mu^{\mathrm{e.m.}}(x)\right]_{\mathrm{ET}}=4\bar{Q}\left\{k_0t^\sigma(0)-g_0^\sigma k\cdot t(0)\right\}\delta^3(\boldsymbol{x})\,, \tag{66}$$

where $\bar{Q}$ is the mean electric charge of the fundamental doublet, *i.e.*, $\bar{Q}=1/2$ for proton and neutron, or $+1/6$ for quarks. The presence of $\bar{Q}$ means that this current commutator is not model-independent. Note also that the commutators used to derive this expression are those beyond ordinary current algebra; they are determined only with the aid of models of hadrons (see Appendix D). We see that the antisymmetric tensor in the l.h.s. of (66) converts the vector (axial-vector) currents into the axial-vector (vector) currents. Therefore, the g_A dependence appears for the vector current contributions upon taking the hadronic matrix element. This is in contrast to the situation for the first three terms in (61), which are determined by current algebra and conservation laws; the effect of strong interaction for the axial-vector current is properly absorbed in g_A after taking the matrix element with nucleons.

We are interested only in the leading high energy behavior and again write (66) in a covariant form:

$$\begin{aligned} &-i \lim_{k^2\to+\infty} \varepsilon^{\lambda\rho\mu\sigma} k_\rho T_{\lambda\mu}(k,p_1,p_2) \\ &\longrightarrow 4\bar{Q}\left(g^{\kappa\sigma}-\frac{k^\kappa k^\sigma}{k^2}\right)\langle p_1|t_\kappa(0)|p_2\rangle \frac{k^2}{k^2-2k\cdot p_2}\,. \end{aligned} \tag{67}$$

The fifth term in the brackets of (61) does not contribute to its divergent part. Thus, the divergent part of (61) is evaluated as

$$\begin{aligned} \mathcal{M}^{(v)}\Big|_{\text{divergent}} &= -\frac{G_V}{\sqrt{2}}e^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\lambda^2}\frac{i}{(k-\ell)^2-m_e^2}\bar{v}_\nu\gamma^\lambda(1-\gamma^5)v_e(\ell) \\ &\quad\times\left\{2\langle p_1|t_\lambda(0)|p_2\rangle + M_\lambda + \frac{2k^\mu k_\lambda}{k^2-2k\cdot p_2}\langle p_1|t_\mu(0)|p_2\rangle\right. \\ &\quad\left. +4\bar{Q}\left(g_\lambda^\kappa-\frac{k^\kappa k_\lambda}{k^2}\right)\langle p_1|t_\kappa(0)|p_2\rangle\frac{k^2}{k^2-2k\cdot p_2}\right\} \\ &\sim -\frac{G_V}{\sqrt{2}}e^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\lambda^2}\frac{i}{(k-\ell)^2-m_e^2}\bar{v}_\nu\gamma^\lambda(1-\gamma^5)v_e(\ell) \\ &\quad\times\left\{\left(2+\frac{1}{2}\right)\langle p_1|t_\lambda(0)|p_2\rangle + M_\lambda\right. \\ &\quad\left. +3\bar{Q}\langle p_1|t_\lambda(0)|p_2\rangle\frac{k^2}{k^2-2k\cdot p_2}\right\} \end{aligned} \tag{68}$$

$$\sim \frac{e^2}{16\pi^2}\log M^2\left(\frac{5}{2}\mathcal{M}^{(0)}+3\bar{Q}\mathcal{M}^{(0)}\right), \tag{69}$$

where

$$\mathcal{M}^{(0)} = \frac{G_V}{\sqrt{2}}\bar{v}_\nu\gamma^\lambda(1-\gamma^5)v_e(\ell)\langle p_1|t_\lambda(0)|p_2\rangle \tag{70}$$

is a generalisation of (3) including the strong interaction effect. We have implicitly assumed above that M_λ in (68) does not contribute to the divergent part, because axial current conservation is broken only by soft operators.

The external nucleon-line renormalization also receives strong interaction effects. Using the same notation $\mathcal{M}^{(s)}$ as in (26), while including strong interaction effects, $\mathcal{M}^{(s)}$ is expressed in terms of the Green's function (50),

$$\begin{aligned}\mathcal{M}^{(s)} &= \frac{i}{2\sqrt{2}}G_V e^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\lambda^2}\bar{v}_\nu \gamma^\lambda(1-\gamma^5)v_e(\ell) \\ &\quad \times g^{\mu\nu}T_{\lambda\mu\nu}(k, p_2-p_1, p_1, p_2) \\ &\quad + \left\{\sqrt{Z_2(m_e)}-1\right\}\mathcal{M}^{(0)}. \end{aligned} \tag{71}$$

The last term is an external line renormalization of the lepton.

The integral on the r.h.s. is expressed in terms of $T_{\mu\nu}$ by using (60) and further $\langle p_1|t_\mu(0)|p_2\rangle$. Retaining only the divergent part, and using (65), we find

$$\begin{aligned} &\frac{i}{2\sqrt{2}}G_V e^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\lambda^2}\bar{v}_\nu \gamma^\lambda(1-\gamma^5)v_e(\ell) \\ &\quad \times g^{\mu\nu}\left\{\frac{\partial}{\partial k^\lambda}T_{\mu\nu}(-k,p_1,p_2) + \frac{\partial}{\partial q^\lambda}(M_{\mu\nu} - q^\rho \delta T_{\rho\mu\nu})\Big|_{q=p_2-p_1}\right\} \\ &\sim \frac{i}{2\sqrt{2}}G_V e^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-\lambda^2}\bar{v}_\nu \gamma^\lambda(1-\gamma^5)v_e(\ell) \\ &\quad \times\left\{\frac{\partial}{\partial k^\lambda}\left(\frac{2k^\mu}{k^2+2k\cdot p_2}\right)\langle p_1|t_\mu(0)|p_2\rangle \right. \\ &\quad \left. + g^{\mu\nu}\frac{\partial}{\partial q^\lambda}(M_{\mu\nu} - q^\rho \delta T_{\rho\mu\nu})\Big|_{q=p_2-p_1}\right\}. \end{aligned} \tag{72}$$

Again, axial-current conservation is broken only by soft operators and the $M_{\mu\nu}$ term does not contribute to the divergent part of $\mathcal{M}^{(s)}$. Thus we can drop $M_{\mu\nu} - q^\rho \delta T_{\rho\mu\nu}$. We, therefore, retain only the first term of (72) to evaluate the divergent part of $\mathcal{M}^{(s)}$:

$$\begin{aligned}\mathcal{M}^{(s)}\Big|_{\text{divergent}} &= \frac{e^2}{16\pi^2}\times\left(-\frac{1}{2}\right)\log M^2 \mathcal{M}^{(0)} + \left\{\sqrt{Z_2(m_e)}-1\right\}\mathcal{M}^{(0)} \\ &\sim -\frac{e^2}{16\pi^2}\log M^2 \mathcal{M}^{(0)}. \end{aligned} \tag{73}$$

Adding (69) to (73), the divergent part of $\mathcal{M}^{(v)} + \mathcal{M}^{(s)}$ becomes

$$\mathcal{M}^{(v)} + \mathcal{M}^{(s)}\bigg|_{\text{divergent}} = \frac{e^2}{16\pi^2}\log M^2 \left(\frac{3}{2}\mathcal{M}^{(0)} + 3\bar{Q}\mathcal{M}^{(0)}\right) . \qquad (74)$$

The first part in the brackets of (74) gives divergent terms that are proportional to f_V^2 and g_A^2 in the amplitude squared, as determined by current algebra, conservation of the vector current, and softly-broken axial current conservation. Hence, we can conclude that the coefficient of the logarithmic divergence $\frac{3}{2}\log(M^2/m_p^2)$ in (37) and that in (38) are both independent of the model of strong interactions, and smoothly continue to the electroweak calculation. On the other hand, the second term of logarithmic divergence which arises from the interference of the vector and axial-vector currents are model-dependent; the appearance of $\bar{Q}$ is a manifestation of the model dependence.

6. Long-distance contributions to the vector–axial-vector interference terms

To evaluate the long-distance contribution to the axial-current correction to the Fermi transition, Marciano and Sirlin [9] proposed the prescription to make the logarithmic 'divergence' milder by introducing nucleon form factors in the evaluation of vertex corrections. The result does not contain the cut-off mass M any longer. The calculation for the Gamow–Teller transition is carried out parallel with that for the Fermi transition.

Marciano and Sirlin [9] considered only the vector and axial-vector current contributions, implicitly assuming that the contribution from weak magnetism is suppressed by the proton mass $O(1/m_p)$. We find that, since the loop momentum becomes of the order of the form factor mass, typically m_ρ (ρ meson mass), an $O(m_\rho/m_p)$ contribution may be non-negligible. This contribution turns out to be particularly large for the Gamow–Teller transition.

We work with the effective electromagnetic vertex,

$$\gamma^\mu F_1^{(p)}(k^2) - \frac{i}{2m_p}\sigma^{\mu\nu}k_\nu F_2^{(p)}(k^2)\,, \qquad (75)$$

for the proton, where $F_1^{(p)}(k^2)$ and $F_2^{(p)}(k^2)$ are the form factors, and similarly for the neutron with the form factors, $F_1^{(n)}(k^2)$ and $F_2^{(n)}(k^2)$. The form factors are also included at the weak vertex of the nucleons, $F_V((p_2 - k - p_1)^2) \approx F_V(k^2)$ for the vector vertex and $F_A((p_2-k-p_1)^2) \approx F_A(k^2)$ for the axial-vector vertex. We also retain in the weak vertex the weak magnetism term with the form factor $F_W(k^2)$ via

$$-\frac{i}{2m_N}\sigma_{\lambda\rho}(p_2-k-p_1)^\rho F_W\left((p_2-k-p_1)^2\right) \approx \frac{i}{2m_N}\sigma_{\lambda\rho}k^\rho F_W(k^2)\,, \quad (76)$$

where $m_N \equiv (m_p+m_n)/2$. The effective nucleon vertex $R_{\mu\lambda}(p_1,p_2,k)$ defined in (20) that contributes to $\mathcal{M}^{(v3)}$ then reads,

$$\begin{aligned} R_{\mu\lambda}(p_1,p_2,k) &= \frac{i}{2m_N}(\sigma_{\lambda\rho}k^\rho)(2p_2-k)_\mu F_1^{(p)}(k^2)F_W(k^2) \\ &+\Big\{ -i\gamma_\lambda\left(f_V F_V(k^2) - g_A\gamma^5 F_A(k^2)\right) \\ &+\frac{1}{2m_N}(\sigma_{\lambda\rho}k^\rho)F_W(k^2)\Big\} \\ &\times(\sigma_{\mu\nu}k^\nu)\Big\{F_1^{(p)}(k^2)+F_2^{(p)}(k^2)\Big\}\,. \end{aligned} \quad (77)$$

We now decompose the inner part $\mathcal{M}^{(v3)}$ as,

$$\mathcal{M}^{(v3)} = \mathcal{M}_p^{(v3,VA)} + \mathcal{M}_p^{(v3,wm)} + \mathcal{M}_n^{(v3,VA)} + \mathcal{M}_n^{(v3,wm)}, \quad (78)$$

where

$$\begin{aligned} \mathcal{M}_p^{(v3,VA)} &= \frac{1}{\sqrt{2}}G_V e^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k-\ell)^2-m_e^2}\frac{1}{(p_2-k)^2-m_p^2}\frac{1}{k^2-\lambda^2} \\ &\times\bar{v}_\nu\gamma^\lambda(1-\gamma^5)\left\{(k-2\ell)^\mu + i\sigma^{\mu\nu}k_\nu\right\}v_e(\ell) \\ &\times\bar{u}_n(p_1)\gamma_\lambda\left\{f_V F_V(k^2) - g_A\gamma^5 F_A(k^2)\right\}(\sigma_{\mu\rho}k^\rho)u_p(p_2) \\ &\times\{F_1^{(p)}(k^2)+F_2^{(p)}(k^2)\}\,, \end{aligned} \quad (79)$$

$$\begin{aligned} \mathcal{M}_p^{(v3,wm)} &= \frac{i}{\sqrt{2}}G_V e^2\left(\frac{i}{2m_N}\right)\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k-\ell)^2-m_e^2}\frac{1}{(p_2-k)^2-m_p^2}\frac{1}{k^2-\lambda^2} \\ &\times\bar{v}_\nu\gamma^\lambda(1-\gamma^5)\left\{(k-2\ell)^\mu + i\sigma^{\mu\nu}k_\nu\right\}v_e(\ell) \\ &\times\bar{u}_n(p_1)\Big[(\sigma_{\lambda\rho}k^\rho)(2p_2-k)_\mu F_1^{(p)}(k^2) \\ &-i(\sigma_{\lambda\rho}k^\rho)(\sigma_{\mu\nu}k^\nu)\Big\{F_1^{(p)}(k^2)+F_2^{(p)}(k^2)\Big\}\Big]u_p(p_2)F_W(k^2) \end{aligned} \quad (80)$$

and likewise for the neutron, $\mathcal{M}_n^{(v3,VA)}$ and $\mathcal{M}_n^{(v3,wm)}$. Eq. (79) is the term due to the V and A current interaction and (80) is due to the weak magnetism. The calculation of the four terms in $\mathcal{M}^{(v3)}$ of (78) is given in Appendix E.

Results for $\mathcal{M}_p^{(v3,VA)}$ *and* $\mathcal{M}_n^{(v3,VA)}$

We are interested only in the $f_V g_A$-terms and we subtract the pure vector and pure axial-vector parts that are proportional f_V^2 and g_A^2. We find

$$\sum_{\text{spin}} \Big\{ \Big(\mathcal{M}_p^{(v3,VA)} + \mathcal{M}_n^{(v3,VA)}\Big)\mathcal{M}^{(0)*} + \Big(\mathcal{M}_p^{(v3,VA)*} + \mathcal{M}_n^{(v3,VA)*}\Big)\mathcal{M}^{(0)} \Big\} \Big|_{f_V g_A}$$
$$= 32 G_V^2 m_n m_p E E_\nu \left(\frac{e^2}{8\pi^2}\right) f_V g_A \Big[6\left(\mathcal{C}_\sigma^{(p,A)} + \mathcal{C}_\sigma^{(n,A)}\right)(1+\beta\cos\theta)\langle 1\rangle^2$$
$$+ \left(6\mathcal{C}_\sigma^{(p,V)} + 6\mathcal{C}_\sigma^{(n,V)} + 2\mathcal{C}_\tau^{(p,V)} + 2\mathcal{C}_\tau^{(n,V)}\right)(3-\beta\cos\theta)\frac{1}{3}\langle \boldsymbol{\sigma}\rangle^2 \Big], \tag{81}$$

where $\mathcal{C}_\sigma$ and $\mathcal{C}_\tau$ are defined by the integrals

$$\int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2)^2\{(p_2-k)^2 - m_p^2\}} \left\{F_1^{(p)}(k^2) + F_2^{(p)}(k^2)\right\} F_V(k^2)$$
$$\equiv \frac{i}{16\pi^2}\left\{ g_{\mu\nu}\mathcal{C}_\sigma^{(p,V)} + \frac{1}{m_p^2}(p_2)_\mu (p_2)_\nu \mathcal{C}_\tau^{(p,V)} \right\}, \tag{82}$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{(k^2)^2\{(p_2-k)^2 - m_p^2\}} \left\{F_1^{(p)}(k^2) + F_2^{(p)}(k^2)\right\} F_A(k^2)$$
$$\equiv \frac{i}{16\pi^2}\left\{ g_{\mu\nu}\mathcal{C}_\sigma^{(p,A)} + \frac{1}{m_p^2}(p_2)_\mu (p_2)_\nu \mathcal{C}_\tau^{(p,A)} \right\}, \tag{83}$$

and are evaluated numerically using the empirical dipole nucleon form factors (see Appendix F) as

$$\mathcal{C}_\sigma^{(p,V)} = (1+\mu_p) \times 0.240 = 0.671\,, \tag{84}$$
$$\mathcal{C}_\tau^{(p,V)} = (1+\mu_p) \times (-0.198) = -0.554\,, \tag{85}$$
$$\mathcal{C}_\sigma^{(p,A)} = (1+\mu_p) \times 0.261 = 0.729\,, \tag{86}$$
$$\mathcal{C}_\tau^{(p,A)} = (1+\mu_p) \times (-0.212) = -0.592\,, \tag{87}$$

where $\mu_p = 1.793$ and $\mu_n = -1.913$ are the anomalous magnetic moments of the proton and the neutron.

The evaluation of the neutron's counterpart $\mathcal{C}_\sigma^{n,V}$, $\mathcal{C}_\tau^{n,V}$, $\mathcal{C}_\sigma^{n,A}$ and $\mathcal{C}_\tau^{n,A}$ goes similarly, with the result,

$$\mathcal{C}_\sigma^{(n,V)} = \mu_n \times 0.240 = -0.459\,, \tag{88}$$
$$\mathcal{C}_\tau^{(n,V)} = \mu_n \times (-0.198) = 0.379\,, \tag{89}$$
$$\mathcal{C}_\sigma^{(n,A)} = \mu_n \times 0.261 = -0.499\,, \tag{90}$$
$$\mathcal{C}_\tau^{(n,A)} = \mu_n \times (-0.212) = 0.405\,. \tag{91}$$

The first terms in the square brackets of (81) are the axial-vector current contribution to the Fermi transition, and the second terms are the vector current contribution to the Gamow–Teller transitions. The computation of the former terms (*i.e.*, those proportional to $\langle 1 \rangle^2$) was already done by the authors of [9] and [10], and our evaluation

$$f_V g_A \times 6 \left(\mathcal{C}_\sigma^{(p,A)} + \mathcal{C}_\sigma^{(n,A)} \right) = 2 \times 0.875 = 1.75 \tag{92}$$

agrees with their results allowing for the difference that arises from the different values of the input data[3].

Let us remark that (84)–(87) and (88)–(91) reduce to

$$\mathcal{C}_\sigma^{(p,V)}, \mathcal{C}_\sigma^{(p,A)} \longrightarrow \frac{1}{4} \log \left(\frac{M^2}{m_p^2} \right) + \frac{3}{8} \,, \tag{93}$$

$$\mathcal{C}_\sigma^{(n,V)}, \mathcal{C}_\sigma^{(n,A)} \longrightarrow 0 \,, \tag{94}$$

$$\mathcal{C}_\tau^{(p,V)}, \mathcal{C}_\tau^{(p,A)} \longrightarrow -\frac{1}{2} \,, \tag{95}$$

$$\mathcal{C}_\tau^{(n,V)}, \mathcal{C}_\tau^{(n,A)} \longrightarrow 0 \,, \tag{96}$$

for the point-like nucleon with a vanishing anomalous magnetic moment, and (81) reduces to the $f_V g_A$-terms in (22).

Results for $\mathcal{M}_p^{(v3,wm)}$ *and* $\mathcal{M}_n^{(v3,wm)}$

The calculation is also straightforward for the effects of weak magnetism, though it is rather lengthy. We refer the readers who are interested in the calculation to Appendix E; here we only record the final result:

$$\begin{aligned} &\sum_{\text{spin}} \left\{ \left(\mathcal{M}_p^{(v3,wm)} + \mathcal{M}_n^{(v3,wm)} \right) \mathcal{M}^{(0)*} + \left(\mathcal{M}_p^{(v3,wm)*} + \mathcal{M}_n^{(v3,wm)*} \right) \mathcal{M}^{(0)} \right\} \\ &= 32 G_V^2 m_n m_p E E_\nu \left(\frac{e^2}{8\pi^2} \right) \frac{1}{m_N} \left[2 g_A \left(m_p \mathcal{D}_\sigma^{(p)} + m_n \mathcal{D}_\sigma^{(n)} \right) (3 - \beta \cos\theta) \frac{1}{3} \langle \boldsymbol{\sigma} \rangle^2 \right. \\ &\quad + g_A \left(m_p \mathcal{E}^{(p)} - m_n \mathcal{E}^{(n)} \right) (3 - \beta \cos\theta) \frac{1}{3} \langle \boldsymbol{\sigma} \rangle^2 \\ &\quad \left. + \frac{3}{2} f_V \left(m_p \mathcal{E}^{(p)} + m_n \mathcal{E}^{(n)} \right) (1 + \beta \cos\theta) \langle 1 \rangle^2 \right] , \end{aligned} \tag{97}$$

[3] Our value 0.875 in (92) should be compared with 0.885 of Marciano and Sirlin [9] and with 0.881 of Towner [10]. The slight difference is due to different choices of g_A and m_A in the form factor.

where the constants $\mathcal{D}_\sigma^{(p)}$ and $\mathcal{E}^{(p)}$ are defined by

$$\int \frac{d^4k}{(2\pi)^4}\frac{k^\mu k^\nu}{(k^2)^2}\frac{1}{(p_2-k)^2-m_p^2}F_W(k^2)F_1^{(p)}(k^2)$$
$$\equiv \frac{i}{16\pi^2}\left\{g^{\mu\nu}\mathcal{D}_\sigma^{(p)}+\frac{1}{m_p^2}p_2^\mu p_2^\nu\mathcal{D}_\tau^{(p)}\right\}, \tag{98}$$
$$\int \frac{d^4k}{(2\pi)^4}\frac{k^\mu}{k^2}\frac{1}{(p_2-k)^2-m_p^2}F_W(k^2)\left\{F_1^{(p)}(k^2)+F_2^{(p)}(k^2)\right\}$$
$$\equiv \frac{i}{16\pi^2}p_2^\mu\mathcal{E}^{(p)}, \tag{99}$$

and are evaluated, by using the phenomenological form factors of the proton in Appendix F, as

$$\mathcal{D}_\sigma^{(p)} = (\mu_p-\mu_n)\times 0.240 = 0.890\,, \tag{100}$$
$$\mathcal{D}_\tau^{(p)} = (\mu_p-\mu_n)\times(-0.198) = -0.735\,, \tag{101}$$
$$\mathcal{E}^{(p)} = (\mu_p-\mu_n)(1+\mu_p)\times 0.0836 = 0.866\,. \tag{102}$$

[Note that $\mathcal{D}_\tau^{(p)}$ does not appear in (97).] We also define the corresponding quantities for the neutron and obtain an expression similar to (97), where

$$\mathcal{D}_\sigma^{(n)} = 0\,, \tag{103}$$
$$\mathcal{D}_\tau^{(n)} = 0\,, \tag{104}$$
$$\mathcal{E}^{(n)} = (\mu_p-\mu_n)\mu_n\times 0.0835 = -0.592\,. \tag{105}$$

Summary of long-distance contributions

The long-distance contributions of $\mathcal{M}^{(v3)}$ defined by (78) is summarized as

$$\sum_{\text{spin}}\left(\mathcal{M}^{(v3)}\mathcal{M}^{(0)*}+\mathcal{M}^{(v3)*}\mathcal{M}^{(0)}\right)$$
$$=32G_V^2m_nm_pEE_\nu\cdot\frac{e^2}{8\pi^2}\left[C^{\mathrm{F}}(1+\beta\cos\theta)f_V^2\langle 1\rangle^2\right.$$
$$\left.+C^{\mathrm{GT}}(3-\beta\cos\theta)\frac{1}{3}g_A^2\langle\boldsymbol{\sigma}\rangle^2\right]. \tag{106}$$

Here the constants C^{F} and C^{GT}, which were introduced in (45) and (47), are obtained from (81) and (97):

$$f_V^2C^{\mathrm{F}} = 6f_Vg_A\left(\mathcal{C}_\sigma^{(p,A)}+\mathcal{C}_\sigma^{(n,A)}\right)+\frac{3f_V}{2m_N}\left(m_p\mathcal{E}^{(p)}+m_n\mathcal{E}^{(n)}\right), \tag{107}$$

$$\begin{aligned} g_A^2 C^{\rm GT} &= f_V g_A \left(6\mathcal{C}_\sigma^{(p,V)} + 6\mathcal{C}_\sigma^{(n,V)} + 2\mathcal{C}_\tau^{(p,V)} + 2\mathcal{C}_\tau^{(n,V)}\right) \\ &\quad + \frac{g_A}{m_N}\left\{2\left(m_p\mathcal{D}_\sigma^{(p)} + m_n\mathcal{D}_\sigma^{(n)}\right) + \left(m_p\mathcal{E}^{(p)} - m_n\mathcal{E}^{(n)}\right)\right\}, \end{aligned} \tag{108}$$

which are evaluated numerically as

$$\begin{aligned} C^{\rm F} &= 1.751 + 0.409 = 2.160\,, \\ C^{\rm GT} &= 0.727 + 2.554 = 3.281\,, \end{aligned} \tag{109}$$

where the two parts represent contributions from the (V, A) interaction and weak magnetism. Note that the weak magnetism contribution is non-negligible for $C^{\rm F}$, and is even larger than the (V, A) contributions in $C^{\rm GT}$.

7. Summary

Full one-loop radiative corrections are calculated for neutrino–nucleon quasi-elastic scattering for both Fermi and Gamow–Teller transitions. We separate the corrections into the outer and inner parts à la Sirlin. The outer part is infrared and ultraviolet finite, and involves the positron velocity. This part takes different forms for angular independent and dependent parts of the differential cross section. The calculation of the inner part requires a scrutiny regarding the continuation of the long-distance hadronic calculation to the short-distance quark treatment and the dependence on the model of hadron structure. We have shown that the logarithmically divergent parts do not depend on the structure of hadrons not only for the Fermi part, but also for the Gamow–Teller part. This observation has enabled us to deal with the inner part for the Gamow–Teller transition in a way parallel to that for the Fermi transition. The inner corrections contribute to the angular-independent and dependent parts of the cross section in the same manner, so that they are written as universal multiplicative factors on the coupling constants, f_V and g_A [4].

The resulting radiative correction to the differential cross section (5) takes the factorised form:

$$A(\beta) = \{1 + \delta_{\rm out}(E)\}\left(\bar{f}_V^2\langle 1\rangle^2 + \bar{g}_A^2\langle\boldsymbol{\sigma}\rangle^2\right), \tag{110}$$

$$B(\beta) = \left\{1 + \tilde{\delta}_{\rm out}(E)\right\}\left(\bar{f}_V^2\langle 1\rangle^2 - \frac{1}{3}\bar{g}_A^2\langle\boldsymbol{\sigma}\rangle^2\right), \tag{111}$$

where $\delta_{\rm out}(E)$ and $\tilde{\delta}_{\rm out}(E)$ are given by (36) and (39), respectively, and the inner corrections are absorbed into the modification of the coupling constants,

$$\begin{aligned} \bar{f}_V^2 &= f_V^2\left(1 + \delta_{\rm in}^{\rm F}\right), \\ \bar{g}_A^2 &= g_A^2\left(1 + \delta_{\rm in}^{\rm GT}\right), \end{aligned} \tag{112}$$

with $\delta_{\rm in}^{\rm F}$ and $\delta_{\rm in}^{\rm GT}$ given by

$$\delta_{\rm in}^{\rm F} = \frac{\alpha}{2\pi}\left\{4\log\left(\frac{m_Z}{m_p}\right) + \log\left(\frac{m_p}{M}\right) + C^{\rm F}\right\}, \tag{113}$$

$$\delta_{\rm in}^{\rm GT} = \frac{\alpha}{2\pi}\left\{4\log\left(\frac{m_Z}{m_p}\right) + \log\left(\frac{m_p}{M}\right) + 1 + C^{\rm GT}\right\}, \tag{114}$$

and the constants $C^{\rm F,GT}$ by (109) [α is the fine structure constant, $\alpha = e^2/(4\pi)$]. If we set the mass scale, M, at which the short-distance asymptotic behaviour onsets, to be 1 GeV, we obtain

$$\begin{aligned} \delta_{\rm in}^{\rm F} &= 0.02370 - 1.16\times 10^{-3}\times\log\left(\frac{M}{1\text{GeV}}\right) \\ &= 0.02370 \pm 0.0008\,, \\ \delta_{\rm in}^{\rm GT} &= 0.02616 - 1.16\times 10^{-3}\times\log\left(\frac{M}{1\text{GeV}}\right) \\ &= 0.02616 \pm 0.0008\,, \end{aligned} \tag{115}$$

the error corresponding to a multiplicative factor of 2 in the scale M.

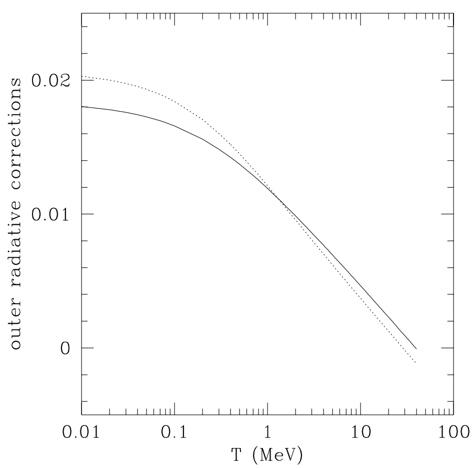

Fig. 3. The outer corrections, $\delta_{\rm out}(E)$ (solid line) and $\tilde{\delta}_{\rm out}(E)$ (dotted line) as a function of the kinetic energy $T = E - m_e$ of the recoil positron.

We present numerical values of $\delta_{\rm out}(E)$ and $\tilde{\delta}_{\rm out}(E)$ in Table I and figure 3, where $T = E - m_e$ is the kinetic energy of the positron. In particular, at the threshold $E = m_e$, they are given by

$$\delta_{\rm out}(m_e) = \frac{e^2}{8\pi^2}\left\{-\frac{27}{4} + \frac{3}{2}\log\left(\frac{m_p^2}{m_e^2}\right)\right\} = 0.01835\,, \tag{116}$$

$$\tilde{\delta}_{\rm out}(m_e) \; = \; \frac{e^2}{8\pi^2}\left\{-\frac{19}{4}+\frac{3}{2}\log\left(\frac{m_p^2}{m_e^2}\right)\right\} = 0.02067\,, \tag{117}$$

and for a large positron energy, they behave as

$$\delta_{\rm out}(E) \sim \frac{e^2}{8\pi^2}\left\{\frac{23}{4}-\frac{4\pi^2}{3}+3\log\left(\frac{m_p}{2E}\right)\right\}, \tag{118}$$

$$\tilde{\delta}_{\rm out}(E) \sim \frac{e^2}{8\pi^2}\left\{\frac{19}{4}-\frac{4\pi^2}{3}+3\log\left(\frac{m_p}{2E}\right)\right\}. \tag{119}$$

$\delta_{\rm out}(E)$ and $\tilde{\delta}_{\rm out}(E)$ differ only by $\approx 10\%$, and $\delta_{\rm in}^{\rm F}$ and $\delta_{\rm in}^{\rm GT}$ also differ by 10%; hence the radiative correction to the angular dependence is at a 0.1% level.

TABLE I

Numerical values of the outer corrections, $\delta_{\rm out}(E)$ and $\tilde{\delta}_{\rm out}(E)$ as a function of the kinetic energy $T = E - m_e$ of the recoil positron.

T (MeV)	β	$\delta_{\rm out}(E)$	$\tilde{\delta}_{\rm out}(E)$
0.01	0.1950	0.0180	0.0203
0.05	0.4127	0.0173	0.0193
0.1	0.5482	0.0166	0.0184
0.2	0.6953	0.0156	0.0171
0.3	0.7765	0.0149	0.0160
0.4	0.8279	0.0142	0.0152
0.5	0.8629	0.0137	0.0145
0.6	0.8879	0.0132	0.0139
0.7	0.9066	0.0129	0.0134
0.8	0.9209	0.0125	0.0129
0.9	0.9321	0.0122	0.0125
1.0	0.9411	0.0119	0.0121
2.0	0.9791	0.0099	0.0096
3.0	0.9894	0.0086	0.0081
4.0	0.9936	0.0077	0.0070
5.0	0.9957	0.0070	0.0062
6.0	0.9969	0.0064	0.0055
7.0	0.9977	0.0058	0.0500
8.0	0.9982	0.0054	0.0045
9.0	0.9986	0.0050	0.0041
10.0	0.9988	0.0047	0.0037
20.0	0.9997	0.0023	0.0013
30.0	0.9999	0.0009	−0.0002
40.0	0.9999	−0.0001	−0.0012

The dominant part of the inner correction arises from $4\log(m_Z/m_p)$, which amounts to 18.18. The uncertainty in the scale $M = 1$ GeV in $\log(m_p/M)$ by a factor of 2 leads to an error of ± 0.69. The model-dependent, long-distance contribution from axial-vector-vector interference terms contributes by 2.160 for the Fermi transition, and 3.281 for the Gamow–Teller transition (there is an additional contribution of unity for the Gamow–Teller transition). We find a sizable contribution to the axial-vector–vector interference terms from the weak magnetism, which has been ignored in [9] and [10]. This contribution is even larger than that of the (V, A) interaction for the correction to the Gamow–Teller transition. If we would use the point nucleon calculation for the Gamow–Teller transition as an extreme case, the long-distance contribution to C^{GT} will be 1.1. We may, therefore, conclude that the uncertainty from the model dependence is conservatively no more than 10% for δ_{in}.

The formulae we derived are applicable not only to neutrino–nucleon quasi elastic scattering, but also to other neutrino reactions off nuclei.

The radiative correction to nuclear beta decay is also described by (110) with δ_{out} replaced by the well-known g function of [1, 4]

$$\begin{aligned}\hat{\delta}_{\mathrm{out}} &= \frac{\alpha}{2\pi} g(E, E_0) \\ &= \frac{\alpha}{2\pi}\Bigg[3\,\ln\left(\frac{m_p}{m_e}\right) - \frac{3}{4} + \frac{4}{\beta}L\left(\frac{2\beta}{1+\beta}\right) \\ &\quad +4\left(\frac{1}{\beta}\tanh^{-1}\beta - 1\right)\left[\frac{E_0 - E}{3E} - \frac{3}{2} + \ln\frac{2(E_0 - E)}{m_e}\right] \\ &\quad +\frac{1}{\beta}\tanh^{-1}\beta\left\{2(1+\beta^2) + \frac{(E_0 - E)^2}{6E^2} - 4\tanh^{-1}\beta\right\}\Bigg], \end{aligned} \tag{120}$$

where E_0 is the end point energy of the electron. The difference from the case of $0^+ \to 0^+$ decay is that we have now somewhat larger inner part correction for the coupling constant that is multiplied on the Gamow–Teller matrix element. For example, the neutron decay width is given by

$$\Gamma = \frac{G_V^2}{2\pi^3} m_e^5 f(E_0)\left[\bar{f}_V^2 \langle 1 \rangle^2 + \bar{g}_A^2 \langle \sigma \rangle^2\right], \tag{121}$$

where $f(E_0) = 1.71483$ [18] includes the outer radiative correction. The value in the square bracket is 5.966, which is 0.20% larger than 5.954 that would be obtained by assuming $\delta_{\mathrm{in}}^{\mathrm{GT}}$ would equal $\delta_{\mathrm{in}}^{\mathrm{F}}$ [19], where $g_A = 1.2670$ [20] is used as a provisional value. Unfortunately, the values of g_A reported in the literature or in the table of Particle Data Group [20] are neither the bare value nor the value including the outer radiative corrections defined through the prescription discussed in this paper. Therefore, the current accuracy of g_A does not warrant the use of our formulae, *e.g.*, to determine precisely $|U_{ud}| = \cos\theta_C$[4], and we should postpone a detailed numerical analysis until an accurate estimate of g_A will become available.

Our final remark is that one may not necessarily need the formulae we derived for the purpose to obtain a precise estimate of neutrino nucleon scattering cross section or of $|U_{ud}|$, if g_A would be properly obtained by using the outer-part radiative correction formula in which the inner and outer corrections are accurately separated and the inner corrections are included in g_A. The true power of our formulae emerges when one would deal radiative corrections for processes involving the neutral current.

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Appendix A

Calculation of $\mathcal{M}^{(v2)}$ in (17)

We sketch the calculation of $\mathcal{M}^{(v2)}$ in (17). The integration formula necessary for (17) is given in Appendix of Ref. [3],

$$\begin{aligned} i\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k-\ell)^2-m_e^2}\frac{1}{(k-p_2)^2-m_p^2}\frac{1}{k^2-\lambda^2}k_\mu \\ = \frac{1}{16\pi^2}\frac{1}{m_p^2}\left[\left\{-\frac{1}{\beta}\tanh^{-1}\beta+\frac{1}{2}\log\left(\frac{m_p^2}{m_e^2}\right)\right\}(p_2)_\mu\right. \\ \left.+\left\{\frac{1}{E\beta}(m_p+E)\tanh^{-1}\beta-\frac{1}{2}\log\left(\frac{m_p^2}{m_e^2}\right)\right\}\ell_\mu\right]. \qquad \text{(A.1)}\end{aligned}$$

Putting (A.1) into (17) and taking the nucleon static limit, we get

$$\begin{aligned}\mathcal{M}^{(v2)} &= \frac{G_V}{\sqrt{2}}\left(\frac{e^2}{8\pi^2}\right)\frac{1}{E\beta}\tanh^{-1}\beta \\ &\times\left[\bar{v}_\nu\gamma^\lambda(1-\gamma^5)i\sigma^{0\nu}v_e(\ell)\right]\ell_\nu\left[\bar{u}_n(p_1)W_\lambda(p_1,p_2)u_p(p_2)\right], \qquad \text{(A.2)}\end{aligned}$$

[4] Particle Data Group [20] gives a value $g_A = 1.2670 \pm 0.0030$. This value is obtained by naively averaging over values reported in the literature, one of which is obtained including the outer radiative correction (up to an unkown constant; the inner correction is ignored), and some others are results which do not include radiative corrections at all. Unfortunately, the value with radiative correction reported by [22] differs from others that ignore radiative corrections [21] by an amount much larger than the difference that would be accounted for with radiative corrections, and hence much of this difference is probably ascribed to other systematic errors of experiments. For this reason we give up to apply corrections to find a bare g_A from the available data.

where only the term proportional to m_p in (A.1) is retained. The spin summation of the tree-one-loop interference amplitude is expressed as

$$\sum_{\text{spin}} \left(\mathcal{M}^{(v2)}\mathcal{M}^{(0)*} + \mathcal{M}^{(v2)*}\mathcal{M}^{(0)}\right) = \left(\frac{G_V}{\sqrt{2}}\right)^2 \left(\frac{e^2}{4\pi^2}\right) \frac{1}{E\beta}\tanh^{-1}\beta \times Q^{\lambda\rho} K_{\lambda\rho}\,, \tag{A.3}$$

where

$$\begin{aligned} K_{\lambda\rho} &= \sum_{\text{spin}} \left[\bar{u}_n(p_1)\gamma_\lambda \left(f_V - g_A\gamma^5\right) u_p(p_2)\right] \left[\bar{u}_n(p_1)\gamma_\rho(f_V - g_A\gamma^5)u_p(p_2)\right]^* \,, \\ Q^{\lambda\rho} &= \sum_{\text{spin}} \left[\bar{v}_\nu \gamma^\lambda (1-\gamma^5) i\sigma^{0\nu} v_e(\ell)\right] \ell_\nu \left[\bar{v}_\nu\gamma^\rho(1-\gamma^5)v_e(\ell)\right]^* \,, \end{aligned} \tag{A.4}$$

are evaluated as

$$K_{00} = 8f_V^2 m_n m_p\,, \qquad K_{ij} = 8g_A^2 m_n m_p \delta_{ij}\,, \tag{A.5}$$

$$Q^{00} = 8E^2E_\nu\beta(\beta+\cos\theta)\,, \quad Q^{ij}\delta_{ij} = 8E^2E_\nu\beta(3\beta-\cos\theta)\,. \tag{A.6}$$

We arrive at

$$\sum_{\text{spin}} \left(\mathcal{M}^{(v2)}\mathcal{M}^{(0)*} + \mathcal{M}^{(v2)*}\mathcal{M}^{(0)}\right) = 32G_V^2 m_n m_p E E_\nu \left(\frac{e^2}{4\pi^2}\right)\tanh^{-1}\beta \left\{f_V^2(\beta+\cos\theta) + g_A^2(3\beta-\cos\theta)\right\}\,, \tag{A.7}$$

which is the $\mathcal{M}^{(v2)}$ part of (18).

Appendix B

Calculation of $\mathcal{M}^{(v3)}$ in (19)

Neglecting λ and m_e, $\mathcal{M}^{(v3)}$ is written as

$$\begin{aligned} \mathcal{M}^{(v3)} &= \frac{G_V}{\sqrt{2}}e^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2)^2}\frac{1}{(p_2-k)^2-m_p^2}\bar{v}_\nu\gamma^\lambda(1-\gamma^5)i\sigma^{\mu\nu}k_\nu v_e(\ell) \\ &\quad \times \bar{u}_n(p_1)\gamma_\lambda(f_V - g_A\gamma^5)\sigma_{\mu\rho}k^\rho u_p(p_2) \\ &= -\frac{G_V}{\sqrt{2}}\left(\frac{e^2}{16\pi^2}\right) I_\nu{}^\rho X^\nu{}_\rho\,, \end{aligned} \tag{B.1}$$

where

$$\frac{i}{16\pi^2} I_\nu{}^\rho \equiv \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2} \frac{1}{(p_2-k)^2 - m_p^2} k_\nu k^\rho$$
$$= \frac{i}{16\pi^2} \left\{ g_\nu^\rho \left[\frac{1}{4} \log\left(\frac{M^2}{m_p^2}\right) + \frac{3}{8} \right] + \frac{1}{m_p^2} (p_2)_\nu (p_2)^\rho \times \left(-\frac{1}{2}\right) \right\} \tag{B.2}$$

and

$$X^\nu{}_\rho = \left[\bar{v}_\nu \gamma^\lambda (1-\gamma^5) \sigma^{\mu\nu} v_e(\ell)\right] \left[\bar{u}_n(p_1)\gamma_\lambda (f_V - g_A\gamma^5) \sigma_{\mu\rho} u_p(p_2)\right] . \tag{B.3}$$

We shuffle $\gamma_\lambda \sigma_{\mu\rho}$ in (B.3) using

$$\gamma^\lambda \gamma^\mu \gamma^\nu = g^{\lambda\mu}\gamma^\nu + g^{\mu\nu}\gamma^\lambda - g^{\lambda\nu}\gamma^\mu + i\varepsilon^{\lambda\mu\nu\rho}\gamma_\rho\gamma^5 \tag{B.4}$$

in such a way that a single gamma matrix is sandwiched between spinors; we find in the nucleon static limit

$$g^{\nu\rho} X_{\nu\rho} = -6(f_V + g_A) \left[\bar{v}_\nu \gamma^0 (1-\gamma^5) v_e(\ell)\right] \left[\bar{u}_n(p_1)\gamma_0 u_p(p_2)\right]$$
$$+6(f_V + g_A) \left[\bar{v}_\nu \gamma^i (1-\gamma^5) v_e(\ell)\right] \left[\bar{u}_n(p_1)\gamma_i \gamma^5 u_p(p_2)\right] , \tag{B.5}$$

$$\frac{1}{m_p^2}(p_2)^\nu (p_2)^\rho X_{\nu\rho} = -3 f_V \left[\bar{v}_\nu \gamma^0 (1-\gamma^5) v_e(\ell)\right] \left[\bar{u}_n(p_1)\gamma_0 u_p(p_2)\right]$$
$$+(2f_V + g_A) \left[\bar{v}_\nu \gamma^i (1-\gamma^5) v_e(\ell)\right] \left[\bar{u}_n(p_1)\gamma_i \gamma^5 u_p(p_2)\right] . \tag{B.6}$$

The first and second terms correspond to the Fermi and Gamow–Teller parts, respectively, in (B.5) and (B.6). The angular distribution is determined by the leptonic parts,

$$\sum_{\text{spin}} \left[\bar{v}_\nu \gamma^0 (1-\gamma^5) v_e(\ell)\right] \left[\bar{v}_\nu \gamma_0 (1-\gamma^5) v_e(\ell)\right] = 8EE_\nu (1 + \beta\cos\beta) , \tag{B.7}$$

$$\sum_{\text{spin}} \left[\bar{v}_\nu \gamma^i (1-\gamma^5) v_e(\ell)\right] \left[\bar{v}_\nu \gamma^j (1-\gamma^5) v_e(\ell)\right] \delta_{ij} = 8EE_\nu (3 - \beta\cos\beta) . \tag{B.8}$$

Combining with the nucleon parts,

$$\sum_{\text{spin}} \left[\bar{u}_n(p_1)\gamma^0 u_p(p_2)\right] \left[\bar{u}_n(p_1)\gamma_0 u_p(p_2)\right]^* = 8 m_n m_p , \tag{B.9}$$

$$\sum_{\text{spin}} \left[\bar{u}_n(p_1)\gamma_i \gamma^5 u_p(p_2)\right] \left[\bar{u}_n(p_1)\gamma_j \gamma^5 u_p(p_2)\right]^* = -8 m_n m_p g_{ij} , \tag{B.10}$$

we end up with the $\mathcal{M}^{(v3)}$ contribution to the differential cross section given by

$$\sum_{\text{spin}}\left(\mathcal{M}^{(v3)}\mathcal{M}^{(0)*}+\mathcal{M}^{(v3)*}\mathcal{M}^{(0)}\right)=32G_V^2 m_n m_p E E_\nu\left(\frac{e^2}{8\pi^2}\right)(1+\beta\cos\beta)$$
$$\times\left[6f_V\,(f_V+g_A)\left\{\frac{1}{4}\log\left(\frac{M^2}{m_p^2}\right)+\frac{3}{8}\right\}+3f_V^2\cdot\left(-\frac{1}{2}\right)\right]$$
$$+32G_V^2 m_n m_p E E_\nu\left(\frac{e^2}{8\pi^2}\right)(3-\beta\cos\beta)$$
$$\times\left[6g_A\,(f_V+g_A)\left\{\frac{1}{4}\log\left(\frac{M^2}{m_p^2}\right)+\frac{3}{8}\right\}+g_A(2f_V+g_A)\cdot\left(-\frac{1}{2}\right)\right]. \quad \text{(B.11)}$$

This is (22). The first and second terms correspond to the Fermi and Gamow–Teller contributions, respectively [the matrix elements are explicitly denoted in (22)].

Appendix C

Calculation of bremsstrahlung

The bremsstrahlung diagrams contain infrared divergences and the Feynman integrations require careful treatments, in particular, to calculate the angular dependent part. The terms in the first line of (29) do not contain $\boldsymbol{p}_\nu$ and are angular-independent; hence, the angular integration of $\boldsymbol{k}$ is easy. Those in the second line of (29) are linear in $\boldsymbol{p}_\nu$ and are proportional to $\cos\theta$. After some manipulations, (28) becomes

$$\frac{d\sigma(\bar{\nu}_e+p\longrightarrow e^+ + n+\gamma)}{d(\cos\theta)}$$
$$=\frac{G_V^2}{\pi}\left(\frac{e^2}{4\pi^2}\right)\left[(f_V^2+3g_A^2)\left\{-E^2\tanh^{-1}\beta\cdot\log\left(\frac{\lambda}{E-m_e}\right)+\sum_{k=1}^{6}\mathcal{I}_k\right\}\right.$$
$$\left.+(f_V^2-g_A^2)\sum_{k=1}^{5}\mathcal{J}_k\right]. \quad \text{(C.1)}$$

Here the integrals $\mathcal{I}_k$, which do not contain $\cos\theta$, are given by

$$\mathcal{I}_1 \equiv E^2\int_{m_e}^{E-\lambda}dE'\frac{1}{E-E'}\log\left(\frac{E'+|\boldsymbol{\ell}'|}{E(1+\beta)}\right)$$
$$= E^2\left\{L\left(\frac{2\beta}{1+\beta}\right)+\tanh^{-1}\beta\cdot\log\left(\frac{2(E+m_e)}{m_e}\right)-(\tanh^{-1}\beta)^2\right\}, \quad \text{(C.2)}$$

$$\begin{aligned}
\mathcal{I}_2 &\equiv -E \int_{m_e}^{E-\lambda} dE' \log\left(\frac{E'+|\boldsymbol{\ell}'|}{m_e}\right) \\
&= E^2\left(-\tanh^{-1}\beta+\beta\right) , \qquad \text{(C.3)}\\
\mathcal{I}_3 &\equiv \frac{1}{2}\int_{m_e}^{E-\lambda} dE'(E-E')\log\left(\frac{E'+|\boldsymbol{\ell}'|}{m_e}\right)\\
&= E^2\left(\frac{3-\beta^2}{8}\tanh^{-1}\beta-\frac{3\beta}{8}\right) , \qquad \text{(C.4)}\\
\mathcal{I}_4 &\equiv -\frac{E}{2}\int_{m_e}^{E-\lambda} dE'\frac{1}{(E-E')^2}|\boldsymbol{k}||\boldsymbol{\ell}'|\\
&= E^2\left\{\frac{\beta}{2}\log\left(\frac{\lambda}{m_e}\frac{1-\beta^2}{4\beta^2}\right)+\frac{1}{2}\tanh^{-1}\beta+\beta\right\} , \qquad \text{(C.5)}\\
\mathcal{I}_5 &\equiv \frac{1}{2}m_e^2E\int_{m_e}^{E-\lambda} dE'\left\{\frac{1}{2E'(E-E')+2|\boldsymbol{k}||\boldsymbol{\ell}|+\lambda^2}\right.\\
&\qquad \left.-\frac{1}{2E'(E-E')-2|\boldsymbol{k}||\boldsymbol{\ell}|+\lambda^2}\right\}\\
&= E^2\left\{\frac{\beta}{2}+\frac{\beta}{2}\log\left(\frac{1-\beta^2}{4\beta^2}\right)+\tanh^{-1}\beta+\frac{\beta}{2}\log\left(\frac{\lambda}{m_e}\right)\right\} , \qquad \text{(C.6)}\\
\mathcal{I}_6 &\equiv \frac{E^2}{2}\int_{m_e}^{E-\lambda} dE'\frac{1}{E-E'}\log\left(\frac{2E'(E-E')+2|\boldsymbol{k}||\boldsymbol{\ell}|+\lambda^2}{2E'(E-E')-2|\boldsymbol{k}||\boldsymbol{\ell}|+\lambda^2}\cdot\frac{E'-|\boldsymbol{\ell}'|}{E'+|\boldsymbol{\ell}'|}\right)\\
&= E^2\left\{\frac{1}{2}L\left(\frac{2\beta}{1+\beta}\right)-\frac{1}{2}(\tanh^{-1}\beta)^2+\tanh^{-1}\beta\cdot\log 2\right\} . \qquad \text{(C.7)}
\end{aligned}$$

Collecting (C.2) to (C.7) together, we obtain the angular independent part (31).

The integrals in (29) that are proportional to $\cos\theta$ are given by

$$\begin{aligned}
\mathcal{J}_1 &\equiv \frac{1}{2\pi E_\nu}\int_{m_e}^{E-\lambda} dE'|\boldsymbol{\ell}'|\int\frac{d^3\boldsymbol{k}}{\omega}\delta(E-E'-\omega)\frac{(\boldsymbol{\ell}'\cdot\boldsymbol{p}_\nu)}{(2k\cdot\ell'+\lambda^2)^2}\left\{\boldsymbol{\ell}'^2-\frac{(\boldsymbol{k}\cdot\boldsymbol{\ell}')^2}{\omega^2}\right\}\\
&= E^2\beta\cos\theta\left\{\frac{2}{\beta}+\frac{5\beta}{4}-\frac{2\sqrt{1-\beta^2}}{\beta}-(1+\beta)\tanh^{-1}\beta\right.
\end{aligned}$$

$$+\left(-\frac{1}{4\beta}-\frac{\beta}{4}+1\right)(\tanh^{-1}\beta)^2-\frac{1}{2}L\left(\frac{2\beta}{1+\beta}\right)+2L\left(1-\sqrt{\frac{1-\beta}{1+\beta}}\right)$$
$$+(\beta-\tanh^{-1}\beta)\log\left(\frac{\lambda}{2m_e}\left(1+\frac{1}{\beta}\right)\frac{\sqrt{1+\beta}+\sqrt{1-\beta}}{\sqrt{1+\beta}-\sqrt{1-\beta}}\right)\Bigg\}, \qquad \text{(C.8)}$$

$$\mathcal{J}_2 \equiv \frac{1}{2\pi E_\nu}\int_{m_e}^{E-\lambda} dE'|\boldsymbol{\ell}'|\int\frac{d^3\boldsymbol{k}}{\omega}\delta(E-E'-\omega)\frac{(\boldsymbol{k}\cdot\boldsymbol{p}_\nu)}{(2k\cdot\ell'+\lambda^2)^2}\left\{\boldsymbol{\ell}'^2-\frac{(\boldsymbol{k}\cdot\boldsymbol{\ell}')^2}{\omega^2}\right\}$$
$$= E^2\beta\cos\theta\left\{\frac{1}{2}\tanh^{-1}\beta-\beta+\frac{1-\beta^2}{2\beta}(\tanh^{-1}\beta)^2\right\}, \qquad \text{(C.9)}$$

$$\mathcal{J}_3 \equiv \frac{1}{2\pi E_\nu}\int_{m_e}^{E-\lambda} dE'|\boldsymbol{\ell}'|\int\frac{d^3\boldsymbol{k}}{\omega}\delta(E-E'-\omega)\frac{(k\cdot\ell')(\boldsymbol{k}\cdot\boldsymbol{p}_\nu)}{(2k\cdot\ell'+\lambda^2)^2}$$
$$= E^2\beta\cos\theta\left\{\frac{1}{4}\tanh^{-1}\beta-\frac{1-\beta^2}{8\beta}(\tanh^{-1}\beta)^2-\frac{1}{\beta}+\frac{3\beta}{8}+\frac{\sqrt{1-\beta^2}}{\beta}\right\}, \qquad \text{(C.10)}$$

$$\mathcal{J}_4 \equiv \frac{1}{2\pi E_\nu}\int_{m_e}^{E-\lambda} dE'|\boldsymbol{\ell}'|\int\frac{d^3\boldsymbol{k}}{\omega}\delta(E-E'-\omega)\frac{(k\cdot\ell')(\boldsymbol{\ell}'\cdot\boldsymbol{p}_\nu)}{(2k\cdot\ell'+\lambda^2)^2}$$
$$= E^2\beta\cos\theta\left\{\frac{1}{4}\tanh^{-1}\beta-\frac{1-\beta^2}{8\beta}(\tanh^{-1}\beta)^2-\frac{\beta}{8}\right\}, \qquad \text{(C.11)}$$

$$\mathcal{J}_5 \equiv -\frac{1}{2\pi E_\nu}\int_{m_e}^{E-\lambda} dE'|\boldsymbol{\ell}'|\int\frac{d^3\boldsymbol{k}}{\omega}\delta(E-E'-\omega)\frac{(k\cdot\ell')}{(2k\cdot\ell'+\lambda^2)^2}\frac{(\boldsymbol{k}\cdot\boldsymbol{\ell}')(\boldsymbol{k}\cdot\boldsymbol{p}_\nu)}{\omega^2}$$
$$= E^2\beta\cos\theta\left\{-\frac{1}{4}\tanh^{-1}\beta-\frac{1-\beta^2}{8\beta}(\tanh^{-1}\beta)^2+\frac{3\beta}{8}\right\}. \qquad \text{(C.12)}$$

Collecting these terms, we get (32).

Let us now explain the treatment of infrared divergences that appear in the integral of $\mathcal{J}_k$, taking $\mathcal{J}_1$ as an example. After angular integration of the $\boldsymbol{k}$ variable, the integral is

$$\mathcal{J}_1 = \cos\theta\int_{m_e}^{E-\lambda} dE'|\boldsymbol{\ell}'|\Bigg[-\frac{m_e^2}{2}\left\{\frac{1}{2E'(E-E')-2|\boldsymbol{k}||\boldsymbol{\ell}'|+\lambda^2}\right.$$
$$\left.-\frac{1}{2E'(E-E')+2|\boldsymbol{k}||\boldsymbol{\ell}'|+\lambda^2}\right\}-\frac{|\boldsymbol{k}||\boldsymbol{\ell}'|}{2(E-E')^2}$$

$$-\frac{E'}{2(E-E')}\log\left(\frac{2E'(E-E')-2|\boldsymbol{k}||\boldsymbol{\ell}'|+\lambda^2}{2E'(E-E')+2|\boldsymbol{k}||\boldsymbol{\ell}'|+\lambda^2}\right)\Bigg]. \tag{C.13}$$

A special care is needed because there are two sources of the photon mass (λ) dependence. One comes from the the integrand and the other from the integration region. To handle this problem, we split the integral into two terms, as $\mathcal{J}_1 = \mathcal{J}_1' + (\mathcal{J}_1 - \mathcal{J}_1')$, where

$$\mathcal{J}_1' \equiv \cos\theta \int_{m_e}^{E-\lambda} dE'|\boldsymbol{\ell}'|\left[-\frac{|\boldsymbol{\ell}'|}{E-E'} - \frac{E'}{2(E-E')}\log\left(\frac{E'-|\boldsymbol{\ell}'|}{E'+|\boldsymbol{\ell}'|}\right)\right], \tag{C.14}$$

which is defined by putting $\lambda = 0$ in the integrand of (C.13), while retaining λ that appears in the integration region of E'. (Recall the relation $|\boldsymbol{\ell}'| = \sqrt{E'^2 - m_e^2}$ and $|\boldsymbol{k}| = \sqrt{(E-E')^2-\lambda^2}$) .

Integral (C.14) is readily handled by changing the integration variable as

$$E' \equiv \frac{m_e}{2}\left(\xi + \frac{1}{\xi}\right), \tag{C.15}$$

with the result

$$\begin{aligned}
\mathcal{J}_1' &= E^2\beta\cos\theta\Bigg[\frac{3}{4}\beta + \frac{2}{\beta} - \frac{2\sqrt{1-\beta^2}}{\beta} \\
&\quad -\left(\frac{3}{2}+\beta\right)\tanh^{-1}\beta + \left(-\frac{1}{4\beta} - \frac{\beta}{4} + \frac{3}{2}\right)(\tanh^{-1}\beta)^2 \\
&\quad -L\left(\frac{2\beta}{1+\beta}\right) + 2L\left(1-\sqrt{\frac{1-\beta}{1+\beta}}\right) \\
&\quad +\left(\beta - \tanh^{-1}\beta\right)\log\left(\frac{\lambda}{m_e}\left(1+\frac{1}{\beta}\right)\frac{\sqrt{1+\beta}+\sqrt{1-\beta}}{\sqrt{1+\beta}-\sqrt{1-\beta}}\right)\Bigg].
\end{aligned} \tag{C.16}$$

The evaluation of $\mathcal{J}_1 - \mathcal{J}_1'$ requires a great care, since its integrand is non-vanishing only in the vicinity of the edge point $E' = E - \lambda$. To perform this integration, we choose the variable

$$\eta = \frac{E - E' + \sqrt{(E-E')^2-\lambda^2}}{\lambda}. \tag{C.17}$$

In the limit $\lambda \longrightarrow 0$

$$\begin{aligned}\mathcal{J}_1 - \mathcal{J}_1' &= E^2\beta\cos\theta\Bigg[-\frac{1}{2}(\tanh^{-1}\beta)^2 + \frac{\beta}{2} + \frac{1}{2}\tanh^{-1}\beta \\ &\quad - \left(\beta - \tanh^{-1}\beta\right)\log 2 + \frac{1}{2}L\left(\frac{2\beta}{1+\beta}\right)\Bigg], \end{aligned} \tag{C.18}$$

which is infrared finite. By adding (C.18) and (C.16), the integration of $\mathcal{J}_1$ is completed to give (C.8).

Appendix D

Currents and their commutation relations

We present explicitly the currents and the current commutation relations that appear in Sect. 5. Consider a fermion field ψ, which belongs to some representation of the internal symmetry group and satisfies the equal time (ET) canonical commutation relations $\left\{\psi_\alpha(x), \psi_\beta^\dagger(y)\right\}_{\rm ET} = \delta_{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y})$. The vector and axial-vector currents are constructed as,

$$j_\mu^a(x) = \bar\psi(x)\gamma_\mu T^a\psi(x), \quad j_\mu^{5a}(x) = \bar\psi(x)\gamma_\mu\gamma^5 T^a\psi(x)\,, \tag{D.1}$$

where T^a are the generators of the internal symmetry group.

By applying the canonical commutation relations, we obtain

$$\begin{aligned}\left[j_\mu^a(x), j_\nu^b(y)\right]_{\rm ET} &= \bar\psi(x)\gamma_\mu\gamma^0\gamma_\nu T^aT^b\psi(y)\delta^3(\boldsymbol{x}-\boldsymbol{y}) \\ &\quad -\bar\psi(y)\gamma_\nu\gamma^0\gamma_\mu T^bT^a\psi(x)\delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \end{aligned} \tag{D.2}$$

$$\begin{aligned}\left[j_\mu^{5a}(x), j_\nu^b(y)\right]_{\rm ET} &= \bar\psi(x)\gamma_\mu\gamma^5\gamma^0\gamma_\nu T^aT^b\psi(y)\delta^3(\boldsymbol{x}-\boldsymbol{y}) \\ &\quad -\bar\psi(y)\gamma_\nu\gamma^0\gamma_\mu\gamma^5 T^bT^a\psi(x)\delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \end{aligned} \tag{D.3}$$

$$\begin{aligned}\left[j_\mu^{5a}(x), j_\nu^{5b}(y)\right]_{\rm ET} &= \bar\psi(x)\gamma_\mu\gamma^0\gamma_\nu T^aT^b\psi(y)\delta^3(\boldsymbol{x}-\boldsymbol{y}) \\ &\quad -\bar\psi(y)\gamma_\nu\gamma^0\gamma_\mu T^bT^a\psi(x)\delta^3(\boldsymbol{x}-\boldsymbol{y})\,. \end{aligned} \tag{D.4}$$

From (D.2)–(D.4), conventional current algebra follows for $\nu = 0$.

The commutation relations we use are derived from (D.2)–(D.4). One class of the commutation relations is

$$g^{\mu\nu}\left[j_\mu^a(x), j_\nu^b(y)\right]_{\rm ET} = -2\bar\psi(x)\gamma_0\left[T^a, T^b\right]\psi(x)\delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \tag{D.5}$$

$$g^{\mu\nu}\left[j_\mu^{5a}(x), j_\nu^b(y)\right]_{\rm ET} = -2\bar\psi(x)\gamma_0\gamma^5\left[T^a, T^b\right]\psi(x)\delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \tag{D.6}$$

$$g^{\mu\nu}\left[j_\mu^{5a}(x), j_\nu^{5b}(y)\right]_{\rm ET} = -2\bar\psi(x)\gamma_0\left[T^a, T^b\right]\psi(x)\delta^3(\boldsymbol{x}-\boldsymbol{y})\,. \tag{D.7}$$

For the other class of the commutation relations, by noting

$$\varepsilon^{\lambda\mu\nu\rho}\gamma_\mu\gamma_0\gamma_\nu = 2i\left(g_0^\lambda g_\sigma^\rho - g_\sigma^\lambda g_0^\rho\right)\gamma^\sigma\gamma^5 \tag{D.8}$$

we obtain

$$\begin{aligned}\varepsilon^{\lambda\mu\nu\rho}&\left[j_\mu^a(x), j_\nu^b(y)\right]_{\mathrm{ET}} \\ &= 2i\left(g_0^\lambda g_\sigma^\rho - g_\sigma^\lambda g_0^\rho\right)\bar\psi(x)\gamma^\sigma\gamma^5\left\{T^a, T^b\right\}\psi(x)\delta^3(\boldsymbol{x}-\boldsymbol{y}),\end{aligned} \tag{D.9}$$

$$\begin{aligned}\varepsilon^{\lambda\mu\nu\rho}&\left[j_\mu^{5a}(x), j_\nu^b(y)\right]_{\mathrm{ET}} \\ &= 2i\left(g_0^\lambda g_\sigma^\rho - g_\sigma^\lambda g_0^\rho\right)\bar\psi(x)\gamma^\sigma\left\{T^a, T^b\right\}\psi(x)\delta^3(\boldsymbol{x}-\boldsymbol{y}),\end{aligned} \tag{D.10}$$

$$\begin{aligned}\varepsilon^{\lambda\mu\nu\rho}&\left[j_\mu^{5a}(x), j_\nu^{5b}(y)\right]_{\mathrm{ET}} \\ &= 2i\left(g_0^\lambda g_\sigma^\rho - g_\sigma^\lambda g_0^\rho\right)\bar\psi(x)\gamma^\sigma\gamma^5\left\{T^a, T^b\right\}\psi(x)\delta^3(\boldsymbol{x}-\boldsymbol{y}).\end{aligned} \tag{D.11}$$

Let us consider the case of SU(2) and suppose that ψ belongs to the doublet representation. The electromagnetic and weak currents are

$$j_\mu^{\mathrm{em}} = \bar\psi\gamma_\mu T^Q\psi, \qquad t_\lambda = \bar\psi\gamma_\lambda T^-\psi - \bar\psi\gamma_\lambda\gamma^5 T^-\psi, \tag{D.12}$$

where the charge matrix T^Q and T^- are defined by

$$T^Q = \begin{pmatrix} Q_+ & 0 \\ 0 & Q_- \end{pmatrix}, \qquad T^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \tag{D.13}$$

the electric charges of upper and lower components of the doublet are denoted by Q_+ and Q_-.

The commutation and anticommutation relations between T^Q and T^- are

$$\left[T^-, T^Q\right] = (Q_+ - Q_-)T^- = T^-, \tag{D.14}$$

$$\left\{T^-, T^Q\right\} = (Q_+ + Q_-)T^- \equiv 2\bar Q T^-. \tag{D.15}$$

The relations are translated as

$$g^{\lambda\mu}\left[t_\lambda(x), j_\mu^{\mathrm{em}}(y)\right]_{\mathrm{ET}} = -2t_0(x)\delta^3(\boldsymbol{x}-\boldsymbol{y}), \tag{D.16}$$

$$\varepsilon^{\lambda\mu\nu\rho}\left[t_\mu(x), j_\nu^{\mathrm{em}}(y)\right]_{\mathrm{ET}} = -4i\bar Q\left(g_0^\lambda g_\sigma^\rho - g_\sigma^\lambda g_0^\rho\right)t^\sigma(x)\delta^3(\boldsymbol{x}-\boldsymbol{y}). \tag{D.17}$$

The relation $Q_+ - Q_- = 1$ holds independently of the model of the constituents of hadrons; hence (D.16) is model-independent. On the contrary, the mean charge $\bar{Q}$ depends on the constituents, *e.g.*, whether ψ denotes the nucleon- or quark-doublet ($\bar{Q} = 1/2$ or $1/6$), so (D.17) is model-dependent.

Some commutators of the local currents may in principle receive model-dependent Schwinger terms. Within the standard model, however, the Schwinger term is c-numbers, and it does not contribute to beta decay of nucleons, as argued by Sirlin [8].

Appendix E

Calculation of $\mathcal{M}^{(v3,VA)}$ *and* $\mathcal{M}^{(v3,wm)}$

To shuffle the gamma matrices in (79), we use

$$\begin{aligned}&\frac{1}{2}\left\{\gamma_\lambda\left(f_V F_V(k^2) - g_A F_A(k^2)\gamma^5\right), \sigma_{\mu\rho}\right\}\\ &= -\varepsilon_{\lambda\mu\rho\sigma}\gamma^\sigma\gamma^5\left(f_V F_V(k^2) - g_A F_A(k^2)\gamma^5\right)\,,\end{aligned} \tag{E.1}$$

$$\begin{aligned}&\frac{1}{2}\left[\gamma_\lambda\left(f_V F_V(k^2) - g_A F_A(k^2)\gamma^5\right), \sigma_{\mu\rho}\right]\\ &= i\left(g_{\lambda\mu}\gamma_\rho - g_{\lambda\rho}\gamma_\mu\right)\left(f_V F_V(k^2) - g_A F_A(k^2)\gamma^5\right)\,,\end{aligned} \tag{E.2}$$

and simplify nucleon's gamma matrices in (79) using

$$\begin{aligned}&\gamma_\lambda\left(f_V F_V(k^2) - g_A F_A(k^2)\gamma^5\right)\sigma_{\mu\rho}\\ &= \frac{1}{2}\left\{\gamma_\lambda\left(f_V F_V(k^2) - g_A F_A(k^2)\gamma^5\right), \sigma_{\mu\rho}\right\}\\ &\quad+\frac{1}{2}\left[\gamma_\lambda\left(f_V F_V(k^2) - g_A F_A(k^2)\gamma^5\right), \sigma_{\mu\rho}\right]\,.\end{aligned} \tag{E.3}$$

Applying these formulae and using the definition of $\mathcal{C}_{\sigma,\tau}$ in (82) and (83), (79) is re-expressed as

$$\begin{aligned}\mathcal{M}_p^{(v3,VA)} &= \frac{G_V}{\sqrt{2}}\frac{e^2}{16\pi^2}\bar{v}_\nu\gamma^\lambda(1-\gamma^5)\sigma^{\mu\nu}v_e(\ell)\\ &\times\Bigg[\varepsilon_{\lambda\mu\nu\sigma}\bar{u}_n(p_1)\gamma^\sigma\gamma^5\left(f_V\mathcal{C}_\sigma^{(p,V)} - g_A\gamma^5\mathcal{C}_\sigma^{(p,A)}\right)u_p(p_2)\\ &+\frac{1}{m_p^2}\varepsilon_{\lambda\mu\rho\sigma}(p_2)_\nu(p_2)^\rho\bar{u}_n(p_1)\gamma^\sigma\gamma^5\left(f_V\mathcal{C}_\tau^{(p,V)} - g_A\gamma^5\mathcal{C}_\tau^{(p,A)}\right)u_p(p_2)\\ &-i\bar{u}_n(p_1)\left(g_{\lambda\mu}\gamma_\nu - g_{\lambda\nu}\gamma_\mu\right)\left(f_V\mathcal{C}_\sigma^{(p,V)} - g_A\gamma^5\mathcal{C}_\sigma^{(p,A)}\right)u_p(p_2)\end{aligned}$$

$$
-\frac{i}{m_p^2}(p_2)^\nu(p_2)^\rho\bar{u}_n(p_1)\left(g_{\lambda\mu}\gamma_\rho - g_{\lambda\rho}\gamma_\mu\right)\left(f_V\mathcal{C}_\tau^{(p,V)} - g_A\gamma^5\mathcal{C}_\tau^{(p,A)}\right)u_p(p_2)\Bigg]. \tag{E.4}
$$

We are interested only in those terms that are proportional to $f_V g_A$ in the amplitude squared. By inspecting the gamma matrices in (E.4) in the nucleon static limit, we see that only the first and second terms in the brackets of (E.4) are potential sources of the $f_V g_A$ terms; the third and fourth terms give those proportional to f_V^2 or g_A^2. The spin summation relevant to the first and second terms are

$$
\begin{aligned}
&\sum_{\text{spin}}\varepsilon_{\lambda\mu\nu\sigma}[\bar{u}_n(p_1)\gamma^\sigma\gamma^5\left(f_V\mathcal{C}_\sigma^{(p,V)} - g_A\gamma^5\mathcal{C}_\sigma^{(p,A)}\right)u_p(p_2)] \\
&\quad\times[\bar{u}_n(p_1)\gamma_\rho\left(f_V - g_A\gamma^5\right)u_p(p_2)]^* \\
&\quad\times\sum_{\text{spin}}[\bar{v}_\nu\gamma^\lambda\sigma^{\mu\nu}(1-\gamma^5)][\bar{v}_\nu\gamma^\rho(1-\gamma^5)v_e(\ell)]^* \\
&= 384 m_n m_p E E_\nu f_V g_A\left\{\mathcal{C}_\sigma^{(p,A)}(1+\beta\cos\theta) + \mathcal{C}_\sigma^{(p,V)}(3-\beta\cos\theta)\right\},
\end{aligned} \tag{E.5}
$$

$$
\begin{aligned}
&\sum_{\text{spin}}\frac{1}{m_p^2}\varepsilon_{\lambda\mu\tau\sigma}(p_2)_\nu(p_2)^\tau[\bar{u}_n(p_1)\gamma^\sigma\gamma^5\left(f_V\mathcal{C}_\tau^{(p,V)} - g_A\gamma^5\mathcal{C}_\tau^{(p,A)}\right)u_p(p_2)] \\
&\quad\times[\bar{u}_n(p_1)\gamma_\rho\left(f_V - g_A\gamma^5\right)u_p(p_2)]^* \\
&\quad\times\sum_{\text{spin}}[\bar{v}_\nu\gamma^\lambda\sigma^{\mu\nu}(1-\gamma^5)][\bar{v}_\nu\gamma^\rho(1-\gamma^5)v_e(\ell)]^* \\
&= 128 m_n m_p E E_\nu f_V g_A\mathcal{C}_\tau^{(p,V)}(3-\beta\cos\theta).
\end{aligned} \tag{E.6}
$$

It is obvious that $(1+\beta\cos\theta)$ and $(3-\beta\cos\theta)$ terms correspond to the Fermi and Gamow–Teller parts, respectively.

The interference with $\mathcal{M}^{(0)}$ turns out to be

$$
\begin{aligned}
&\sum_{\text{spin}}\left(\mathcal{M}_p^{(v3,VA)}\mathcal{M}^{(0)*} + \mathcal{M}_p^{(v3,VA)*}\mathcal{M}^{(0)}\right)\Bigg|_{f_V g_A} \\
&= 32 G_V^2 m_n m_p E E_\nu\left(\frac{e^2}{8\pi^2}\right)f_V g_A\Bigg[6\mathcal{C}_\sigma^{(p,A)}(1+\beta\cos\theta)\langle 1\rangle^2 \\
&\quad + (6\mathcal{C}_\sigma^{(p,V)} + 2\mathcal{C}_\tau^{(p,V)})(3-\beta\cos\theta)\frac{1}{3}\langle\boldsymbol{\sigma}\rangle^2\Bigg],
\end{aligned} \tag{E.7}
$$

where only the $f_V g_A$ terms are retained. The same procedure applies to the neutron amplitude, and we arrive at (81).

For the weak magnetism terms, the gamma matrices in the last line of (80) are simplified by applying

$$(\sigma_{\lambda\rho}k^\rho)(\sigma_{\mu\nu}k^\nu) = \frac{1}{2}\{\sigma_{\lambda\rho}k^\rho, \sigma_{\mu\nu}k^\nu\} + \frac{1}{2}[\sigma_{\lambda\rho}k^\rho, \sigma_{\mu\nu}k^\nu] \,, \tag{E.8}$$

$$[\sigma_{\lambda\rho}k^\rho, \sigma_{\mu\nu}k^\nu] = -ik_\lambda\sigma_{\mu\rho}k^\rho + ik_\mu\sigma_{\lambda\rho}k^\rho - ik^2\sigma_{\mu\nu} + i\varepsilon_{\lambda\mu\nu\kappa}k^\nu(\gamma\cdot k)\gamma^\kappa\gamma^5 \,, \tag{E.9}$$

$$\{\sigma_{\lambda\rho}k^\rho, \sigma_{\mu\nu}k^\nu\} = -2\left(k_\lambda k_\mu - k^2 g_{\mu\lambda}\right) . \tag{E.10}$$

Note that (E.9) and (E.10) are antisymmetric and symmetric under the interchange of the indices λ and μ. We also simplify the gamma matrices in the leptonic sector in (80) by using

$$\begin{aligned} i\gamma^\lambda(1-\gamma^5)\sigma^{\mu\nu}k_\nu = &\left\{\frac{1}{2}\left(k^\lambda\gamma^\mu + k^\mu\gamma^\lambda\right) - g^{\lambda\mu}(\gamma\cdot k)\right\}(1-\gamma^5) \\ &+\left\{\frac{1}{2}\left(k^\lambda\gamma^\mu - k^\mu\gamma^\lambda\right) - i\varepsilon^{\lambda\mu\nu\rho}\gamma_\rho\gamma^5 k_\nu\right\}(1-\gamma^5) . \end{aligned} \tag{E.11}$$

The first (second) line is symmetric (antisymmetric) under the interchange of λ and μ. Neglecting m_e, ℓ and λ, and using (E.9), (E.10) and (E.11), we simplify (80) as

$$\begin{aligned} &\mathcal{M}_p^{(v3,wm)} \\ &= \frac{i}{\sqrt{2}}G_V e^2\left(\frac{i}{2m_N}\right)\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2)^2}\frac{1}{(p_2-k)^2-m_p^2}F_1^{(p)}(k^2)F_W(k^2) \\ &\quad\times\Big\{-[\bar{v}_\nu\gamma^\lambda(1-\gamma^5)v_e(\ell)][\bar{u}_n(p_1)\sigma_{\lambda\rho}k^\rho u_p(p_2)](k^2-2k\cdot p_2) \\ &\quad+2[\bar{v}_\nu\gamma^\lambda(1-\gamma^5)\left(k\cdot p_2 + i\sigma^{\mu\nu}k_\nu(p_2)_\mu\right)v_e(\ell)][\bar{u}_n(p_1)\sigma_{\lambda\rho}k^\rho u_p(p_2)]\Big\} \\ &\quad+\frac{i}{\sqrt{2}}G_V e^2\left(\frac{i}{2m_N}\right)\left(\frac{-i}{2}\right)\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2)^2}\frac{1}{(p_2-k)^2-m_p^2} \\ &\quad\times\left\{F_1^{(p)}(k^2)+F_2^{(p)}(k^2)\right\}F_W(k^2) \\ &\quad\times\Big\{-6[\bar{v}_\nu\gamma\cdot k(1-\gamma^5)v_e(\ell)][\bar{u}_n(p_1)u_p(p_2)] \\ &\quad+2\varepsilon^{\lambda\mu\nu\rho}k_\nu[\bar{v}_\nu\gamma_\rho(1-\gamma^5)v_e(\ell)][\bar{u}_n(p_1)\sigma_{\lambda\mu}u_p(p_2)]\Big\} . \end{aligned} \tag{E.12}$$

In (E.12) the term containing $[\bar{u}_n(p_1)\sigma_{\lambda\rho}k^\rho u_p(p_2)](k^2-2k\cdot p_2)$ does not survive the symmetric integration over k. In the static nucleon limit, we retain only those terms containing $\bar{u}_n(p_1)u_p(p_2)$ for the Fermi part and $\bar{u}_n(p_1)\sigma_{ij}u_p(p_2)$ for the Gamow–Teller part. Using the definitions in (98) and (99), (E.12) is rewritten

$$
\begin{aligned}
&\mathcal{M}_p^{(v3,wm)} \\
&= \frac{G_V}{\sqrt{2}}\frac{e^2}{16\pi^2}\left(\frac{m_p}{m_N}\right)\mathcal{D}_\sigma^{(p)}\sum_{i,j=1}^{3}[\bar{v}_\nu\gamma^i(1-\gamma^5)\sigma^{0j}v_e(\ell)][\bar{u}_n(p_1)\sigma_{ij}u_p(p_2)] \\
&\quad+\frac{G_V}{\sqrt{2}}\frac{e^2}{16\pi^2}\left(\frac{m_p}{m_N}\right)\mathcal{E}^{(p)}\left\{\frac{3}{2}[\bar{v}_\nu\gamma^0(1-\gamma^5)v_e(\ell)][\bar{u}_n(p_1)u_p(p_2)]\right. \\
&\quad\left.-\frac{1}{2}\sum_{i,j,k=1}^{3}\varepsilon^{0ijk}[\bar{v}_\nu\gamma_k(1-\gamma^5)v_e(\ell)][\bar{u}_n(p_1)\sigma_{ij}u_p(p_2)]\right\}. \qquad \text{(E.13)}
\end{aligned}
$$

The following spin summations are employed to calculate the interference with $\mathcal{M}^{(0)}$:

$$
\begin{aligned}
&\sum_{\text{spin}}[\bar{v}_\nu\gamma^i(1-\gamma^5)\sigma^{0j}v_e(\ell)][\bar{v}_\nu\gamma_\rho(1-\gamma^5)v_e(\ell)]^* \\
&\quad\times\sum_{\text{spin}}[\bar{u}_n(p_1)\sigma_{ij}u_p(p_2)][\bar{u}_n(p_1)\gamma^\rho(f_V-g_A\gamma^5)u_p(p_2)]^* \\
&= 128 g_A m_n m_p E E_\nu(3-\beta\cos\theta)\,, \qquad \text{(E.14)}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{\text{spin}}[\bar{v}_\nu\gamma^0(1-\gamma^5)v_e(\ell)][\bar{v}_\nu\gamma_\rho(1-\gamma^5)v_e(\ell)]^* \\
&\quad\times\sum_{\text{spin}}[\bar{u}_n(p_1)u_p(p_2)][\bar{u}_n(p_1)\gamma^\rho(f_V-g_A\gamma^5)u_p(p_2)]^* \\
&= 64 f_V m_n m_p E E_\nu(1+\beta\cos\theta)\,, \qquad \text{(E.15)}
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon^{0ijk}\sum_{\text{spin}}[\bar{v}_\nu\gamma_k(1-\gamma^5)v_e(\ell)][\bar{v}_\nu\gamma_\rho(1-\gamma^5)v_e(\ell)]^* \\
&\quad\times\sum_{\text{spin}}[\bar{u}_n(p_1)\sigma_{ij}u_p(p_2)][\bar{u}_n(p_1)\gamma^\rho(f_V-g_A\gamma^5)u_p(p_2)]^* \\
&= -128 g_A m_n m_p E E_\nu(3-\beta\cos\theta). \qquad \text{(E.16)}
\end{aligned}
$$

We obtain for $\mathcal{M}_p^{(v3,wm)}$

$$
\sum_{\text{spin}}\left\{\mathcal{M}_p^{(v3,wm)}\mathcal{M}^{(0)*}+\mathcal{M}_p^{(v3,wm)*}\mathcal{M}^{(0)}\right\}
$$

$$= 32G_V^2 m_n m_p E E_\nu \left(\frac{e^2}{8\pi^2}\right) \frac{1}{m_N} \Big[2g_A m_p \mathcal{D}_\sigma^{(p)} (3 - \beta\cos\theta) + g_A m_p \mathcal{E}^{(p)} (3 - \beta\cos\theta) + \frac{3}{2} f_V m_p \mathcal{E}^{(p)} (1 + \beta\cos\theta) \Big] . \tag{E.17}$$

Similar calculations are made for $\mathcal{M}_n^{(v3,wm)}$ and we finally obtain (97).

Appendix F

Nucleon form factors

To calculate numerical constants that appear in (82) and (83), we use the dipole form factors ($\mu_p = 1.793$, $\mu_n = -1.913$):

$$F_1^{(p)}(k^2) + F_1^{(n)}(k^2) = \frac{1}{\left(1 - k^2/m_V^2\right)^2} , \tag{F.1}$$

$$F_1^{(p)}(k^2) - F_1^{(n)}(k^2) = \frac{1}{\left(1 - k^2/m_V^2\right)^2} , \tag{F.2}$$

$$F_2^{(p)}(k^2) + F_2^{(n)}(k^2) = \frac{\mu_p + \mu_n}{\left(1 - k^2/m_V^2\right)^2} , \tag{F.3}$$

$$F_2^{(p)}(k^2) - F_2^{(n)}(k^2) = \frac{\mu_p - \mu_n}{\left(1 - k^2/m_V^2\right)^2} , \tag{F.4}$$

where we adopt $m_V = 0.84$ GeV. We also adopt the dipole form factors

$$F_V(k^2) = \frac{1}{(1 - k^2/m_V^2)^2} , \qquad F_A(k^2) = \frac{1}{(1 - k^2/m_A^2)^2} \tag{F.5}$$

for the vector and axial-vector vertices; we take $m_A = 1.05$ GeV. Weak magnetism should also be endowed with the form factor

$$F_W(k^2) = \frac{\mu_p - \mu_n}{(1 - k^2/m_V^2)^2} . \tag{F.6}$$

The integrals of (82) are inverted as

$$\frac{i}{16\pi^2} \mathcal{C}_\sigma^{(p,V)} = \frac{1}{3} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2} \frac{1}{(p_2 - k)^2 - m_p^2} \left\{ k^2 - \frac{1}{m_p^2} (k \cdot p_2)^2 \right\} \times \left\{ F_1^{(p)}(k^2) + F_2^{(p)}(k^2) \right\} F_V(k^2) , \tag{F.7}$$

$$\frac{i}{16\pi^2}\mathcal{C}_\tau^{(p,V)} = -\frac{1}{3}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2)^2}\frac{1}{(p_2-k)^2-m_p^2}\left\{k^2-\frac{4}{m_p^2}(k\cdot p_2)^2\right\} \times\left\{F_1^{(p)}(k^2)+F_2^{(p)}(k^2)\right\}F_V(k^2)\,. \tag{F.8}$$

We use integral representations employing the Feynman trick to compute numerically the constants in (82) and (83),

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2)^2}\frac{1}{(p_2-k)^2-m_p^2}\frac{1}{(1-k^2/\Lambda_0^2)^2}\frac{1}{(1-k^2/\Lambda_1^2)^2} \times\left\{k^2-\frac{2\xi_0}{m_p^2}(k\cdot p)^2\right\} = \Lambda_0^2\Lambda_1^2 f(\Lambda_0,\Lambda_1)+\frac{\Lambda_0^2\Lambda_1^2}{\Lambda_1^2-\Lambda_0^2}\left\{\Lambda_0^2 f(\Lambda_0,0)-\Lambda_1^2 f(0,\Lambda_1)\right\}\,, \tag{F.9}$$

where

$$f(\Lambda_0,\Lambda_1) = \frac{i}{16\pi^2}\int_0^1 dx\int_0^{1-x} dy x(1-x-y) \times\left[\frac{(2-\xi_0)}{\left\{y^2m_p^2+x\Lambda_0^2+(1-x-y)\Lambda_1^2\right\}^2} + \frac{(4\xi_0-2)y^2m_p^2}{\left\{y^2m_p^2+x\Lambda_0^2+(1-x-y)\Lambda_1^2\right\}^3}\right], \tag{F.10}$$

and ξ_0 is either $1/2$ or 2. A similar expression is used to evaluate (98).

The evaluation of (99) goes similarly, starting from

$$\frac{i}{16\pi^2}\mathcal{E}^{(p)} = \frac{1}{m_p^2}\int \frac{d^4k}{(2\pi)^4}\frac{(k\cdot p_2)}{k^2}\frac{1}{(p_2-k)^2-m_p^2} \times F_W(k^2)\left\{F_1^{(p)}(k^2)+F_2^{(p)}(k^2)\right\}\,. \tag{F.11}$$

Our numerical computation employs the Feynman integral

$$\int \frac{d^4k}{(2\pi)^4}\frac{(k\cdot p_2)}{k^2}\frac{1}{(p_2-k)^2-m_p^2}\frac{1}{(1-k^2/\Lambda_0^2)^2(1-k^2/\Lambda_1^2)^2} = \frac{\Lambda_0^4\Lambda_1^4}{2}\left\{h(1,\Lambda_0,\Lambda_1)-h(0,\Lambda_0,\Lambda_1)\right\}\,, \tag{F.12}$$

where

$$h(\xi,\Lambda_0,\Lambda_1) = \frac{i}{16\pi^2}\int\limits_0^1 dx \int\limits_0^{1-x} dy x(1-x-y) \times \frac{-2}{\left\{y^2\xi^2 m_p^2 + x\Lambda_0^2 + (1-x-y)\Lambda_1^2\right\}^3}\,. \tag{F.13}$$

Likewise, $\mathcal{E}^{(n)}$ is computed via the same type of parameter integrals.

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