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BROWNIAN MOTION: A CASE OF TEMPERATURE FLUCTUATIONS

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A diffusion process of a Brownian particle in a medium of temperature T is re-considered. We assume that temperature of the medium fluctuates around its mean value. The velocity probability distribution is obtained. It is shown that the stationary state is not a thermodynamic equilibrium state described by the Maxwell distribution. Instead a nonequilibrium state is produced by temperature fluctuations.

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1. Introduction

Generalized statistics or 'superstatistics' occur in non-equilibrium systems as a result of parameter (temperature, friction, energy dissipation, pressure, chemical potential, *etc.*) fluctuations [1]. An example of superstatistics is the Tsallis statistics in nonextensive statistical mechanics [2]. One of a dynamical realization of this statistics has been constructed by a Langevin equation for the Brownian particle [3,4] with the inverse temperature being a fluctuating parameter. In the paper we consider a more natural model with fluctuating temperature instead of its inverse. Fluctuations of temperature can play a significant role in many processes and phenomena. *E.g.*, in astrophysics, the spectrum of temperature fluctuations of the cosmic microwave background radiation can change our view on the universe at epochs from redshifts of the order of ten thousand to nearly the present and can provide important clues to inflationary models and the dark matter-energy problem [5]. In plasma physics, an experimental evidence of substantial temperature fluctuations has been found in mechanisms responsible for anomalous transport observed in tokamaks and stellarators [6]. The concept of temperature fluctuations is used in the theory of heavy ion collisions and multiparticle production [7]. In the Rayleigh–Benard convection, temperature fluctuations can be passively transported in the turbulence regimes [8].

Characteristics of temperature fluctuations in living tissue has been studied in [9]. Below, we study the influence of temperature fluctuations on motion of the Brownian particle. As mentioned, the similar problem has been studied previously, mainly in the context of the Tsallis statistics [3] with application to velocity fluctuations in a turbulent flow [4]. However, the studies have been limited to inverse temperature fluctuations and to the 'static' case when fluctuations are represented by a time-independent random variable. The problem is mathematically trivial in the sense that the probability density of velocity can be calculated for an arbitrary random variable modeling fluctuations. The more realistic model seems to be the 'dynamic' model based on time-dependent noise for which temperature fluctuations are represented by a stationary stochastic process. Then, as is shown below, the problem becomes non-trivial, even for the simplest model of temperature fluctuations.

Let us remind that in the classical theory of diffusion, a position $x = x(t)$ of a one-dimensional motion of a Brownian particle of mass m moving in an equilibrium homogeneous medium of temperature T is described by a Newton equation with a random force which, according to the fluctuation–dissipation theorem, has the form [10]

$$m\ddot{x} + \gamma\dot{x} = \sqrt{2\gamma kT}\,\xi(t)\,, \tag{1}$$

where γ is the friction coefficient (given by *e.g.* the Stokes formula), k is the Boltzmann constant and $\xi(t)$ is a random force modeled by the Gaussian white noise,

$$\langle\xi(t)\rangle = 0\,, \quad \langle\xi(t)\xi(s)\rangle = \delta(t-s)\,. \tag{2}$$

The velocity $v = \dot{x}$ is the Ornstein–Uhlenbeck stochastic process and its probability density $P(v,t)$ obeys the Fokker–Planck equation

$$\frac{\partial P(v,t)}{\partial t} = \frac{\gamma}{m}\frac{\partial P(v,t)}{\partial v} + \frac{\gamma kT}{m^2}\frac{\partial^2 P(v,t)}{\partial v^2}\,. \tag{3}$$

A general solution of this equation is given by the expression

$$P(v,t) = \int_{-\infty}^{\infty} p(v,t|v_0,0)P(v_0,0)\;dv_0\,, \tag{4}$$

where $P(v,0)$ is an initial distribution and the transition probability distribution

$$p(v,t|v_0,0) = \left[2\pi\sigma^2(t)\right]^{-1/2}\exp\left\{-\frac{\left[v - v_0\mathrm{e}^{-\gamma t/m}\right]^2}{2\sigma^2(t)}\right\}. \tag{5}$$

The variance

$$\sigma^2(t) = \frac{kT}{m}\left(1 - \mathrm{e}^{-2\gamma t/m}\right) . \tag{6}$$

The stationary velocity distribution function $P_M(v)$ does not depend on the initial distribution $P(v,0)$ and is the Maxwell distribution,

$$P_M(v) = \sqrt{\frac{m}{2\pi kT}}\ \exp\left(-\frac{mv^2}{2kT}\right) . \tag{7}$$

It means that the stationary state in the velocity space is a thermodynamic equilibrium state.

Now, let us consider the situation when the fluid is in a nonequilibrium steady state. What is a stationary state of the Brownian particle in this case? We should use a theoretical framework which enables one to answer this question in a unified manner. For the moment, however, such a universal theory does not exist. First of all, we should specify a nonequilibrium state of the fluid: We assume that this state is not far from equilibrium and can be described similarly as an equilibrium state with the only exception that now temperature T is time-dependent, $T = T(t)$. We follow and extend the proposal of Beck [1,4] from adiabatic to non-adiabatic temperature changes and apply Eq. (1) to this case. The problem is whether and when we can use Eq. (1) if $T = T(t)$, *cf.* also polemics in [11]. Let us remember that when (1) is derived from the microscopic Hamiltonian model [12], it is assumed that the fluid (medium) is in the thermodynamic equilibrium state of temperature T. If, however, the fluid is in the nonequilibrium steady state and can be characterized by the time-dependent temperature $T = T(t)$, then (1) cannot be rigorously justified. We are optimistic (because of good agreement between theory and experimental data presented in [4]) and believe that it can be used as a first approximation to an exact (but non-existing) theory. In Section 2, we present a method how to treat such a system when temperature can fluctuate with $T(t)$ represented by a stochastic process and how to obtain the velocity distribution. In Section 3, we consider a curtailed characteristic functional for which an evolution equation is determined by the infinitesimal generator of the stochastic process representing temperature fluctuations. In Section 4, we consider a case of dichotomic temperature fluctuations and solve the evolution equation for the curtailed characteristic functional. In Section 5, we discuss properties of the velocity probability density. In Section 6, we analyze statistical moments of the velocity.

2. Temperature fluctuations: characteristic functional

Now, let us assume that temperature of the fluid fluctuates around its mean value T_0,

$$T = T(t) = T_0 + \eta(t)\,. \qquad (8)$$

The zero-mean stationary stochastic process $\eta(t)$ describes temperature fluctuations and is independent of the stochastic process $\xi(t)$ describing thermal noise (interaction with surroundings). The formal restriction on this process follows from the condition $T(t) > 0$ and its phase space Y is

$$\eta(t) \in Y = (-T_0, \infty)\,. \qquad (9)$$

The velocity probability distribution $P(v,t)$ can be obtained from the relation (4), in which the initial transition probability density $p(v,t|v_0,0)$ is expressed as

$$p(v,t|v_0,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \mathrm{e}^{-i\omega v} C_v(\omega,t;v_0)\,, \qquad (10)$$

where the conditional characteristic function $C_v(\omega,t;v_0)$ of the velocity is defined by

$$C_v(\omega,t;v_0) = \left\langle \mathrm{e}^{i\omega v(t)} \right\rangle . \qquad (11)$$

The velocity $v(t)$ is a solution of Eq. (1) with time-dependent temperature $T = T(t)$, namely,

$$v(t) = v_0 \exp\left(-\frac{\gamma t}{m}\right) + \sqrt{\frac{2k\gamma}{m^2}} \int_0^t ds \exp\left[-\frac{\gamma(t-s)}{m}\right] \sqrt{T(s)}\,\xi(s)\,, \qquad (12)$$

where v_0 is the initial velocity of the Brownian particle. Inserting the above equation into (11) yields

$$C_v(\omega,t;v_0) = \exp\left(i\omega v_0 \mathrm{e}^{-\gamma t/m}\right) C(\omega,t), \qquad (13)$$

where

$$C(\omega,t) = \left\langle \exp\left[i\omega \sqrt{\frac{2k\gamma}{m^2}} \int_0^t ds\, \mathrm{e}^{-\gamma(t-s)/m} \sqrt{T(s)}\,\xi(s)\right]\right\rangle_{\xi,\eta} . \qquad (14)$$

The subscripts ξ and η denote average over all realizations of thermal noise $\xi(t)$ and temperature fluctuations $\eta(t)$, respectively. The averaging over the Gaussian white noise $\xi(t)$ can be performed leading to the expression

$$C(\omega, t) = \exp\left[-\frac{kT_0}{2m}\,\omega^2\left(1 - \mathrm{e}^{-2\gamma t/m}\right)\right]\varPhi_\eta(\omega, t), \tag{15}$$

where

$$\varPhi_\eta(\omega, t) = \left\langle \exp\left[-\frac{\gamma k}{m^2}\,\omega^2 \mathrm{e}^{-2\gamma t/m}\int_0^t ds\,\mathrm{e}^{2\gamma s/m}\eta(s)\right]\right\rangle_\eta \tag{16}$$

is the characteristic functional of the stochastic process $\eta(t)$. In this approach, the velocity probability of the Brownian particle is determined by the characteristic functional of temperature fluctuations. The explicit form of this functional will be obtained by the method of the so-called 'curtailed' characteristic functional.

3. Curtailed characteristic functional

In order to calculate the functional (16) we proceed in the following way [13]. For fixed time $t = \tilde{t}$ we define [14]

$$\varOmega = \frac{\gamma k}{m^2}\,\omega^2 \mathrm{e}^{-2\gamma\tilde{t}/m}\,. \tag{17}$$

Let us introduce the auxiliary functional

$$\varPsi[\eta; \varOmega, \tilde{t}] = \left\langle \exp\left[-\varOmega\int_0^{\tilde{t}} ds\,\mathrm{e}^{2\gamma s/m}\eta(s)\right]\right\rangle_\eta\,. \tag{18}$$

Then the relation

$$\varPhi_\eta(\omega, t) = \varPsi[\eta; \varOmega, \tilde{t} = t] \tag{19}$$

holds.

The curtailed characteristic functional corresponding to (18) is defined as [15]

$$V(y, t) = \left\langle \delta(\eta(t), y)\exp\left[-\varOmega\int_0^t ds\,\mathrm{e}^{2\gamma s/m}\eta(s)\right]\right\rangle_\eta\,, \tag{20}$$

where $y \in Y$ takes values from the phase space of the stochastic process $\eta(t)$ and $\delta(\eta(t), y)$ is the Kronecker delta when $\eta(t)$ is a discrete process or the Dirac delta for the continuous stochastic processes $\eta(t)$. The relation between these two functions is the following

$$\Phi_\eta(\omega, t) = \int_Y V(y,t) dy \,, \tag{21}$$

where the integration for the continuous (or summation for discrete) process is over the phase space Y. We introduce curtailed characteristic functional because for it, in contrary to (16), an evolution equation is known. In the abbreviated notation, it has the form [15]

$$\frac{\partial}{\partial t} V(y,t) = \hat{L} V(y,t) - \Omega \mathrm{e}^{2\gamma t/m} \, y V(y,t) \,, \tag{22}$$

where $\hat{L}$ is an infinitesimal generator (a forward operator) of the stochastic process $\eta(t)$. If $\eta(t)$ is determined by an Ito stochastic equation, the infinitesimal generator is a differential operator which occurs in the *forward Kolmogorov equation* (*i.e.* in the Fokker–Planck equation). Now, the problem reduces to solving the evolution equation (22) which, in dependence of $\eta(t)$, can be a single or a set of differential or integro-differential equations. Below, we present an example which can be solved exactly.

4. Dichotomic fluctuations

Here, we consider a caricature of temperature fluctuations, *i.e.* a discrete, two-state model [1]. An extension to a many-state or continuous model of fluctuations is in principle possible [16]. However, physics should be similar but mathematics would be much more complicated because of difficulties in solving the evolution equation (22). So, we represent temperature fluctuations by dichotomic noise [17]

$$\eta(t) = \{-a, b\} \,, \quad 0 < a < T_0 \,, \quad b > 0 \,. \tag{23}$$

Transition probabilities per unit time from one state to the other are given by the relations

$$\begin{aligned} Pr(-a \to b) \; &= \; \mu = \frac{1}{\tau_a} \,, \\ Pr(b \to -a) \; &= \; \lambda = \frac{1}{\tau_b} \,, \end{aligned} \tag{24}$$

where τ_a and τ_b are mean waiting times in states $-a$ and b, respectively. We assume that

$$b\mu = a\lambda\,. \tag{25}$$

Then the process is stationary and the probabilities

$$\begin{aligned} Pr(\eta(t)=-a) &= \frac{\lambda}{\mu+\lambda} = \frac{b}{a+b}\,,\\ Pr(\eta(t)=b) &= \frac{\mu}{\mu+\lambda} = \frac{a}{a+b}\,. \end{aligned} \tag{26}$$

The first two moments read

$$\langle\eta(t)\rangle = 0\,, \quad \langle\eta(t)\eta(s)\rangle = ab\exp\left(-|t-s|/\tau_{\mathrm{c}}\right), \tag{27}$$

where the correlation time τ_{c} is given by the formula $1/\tau_{\mathrm{c}} = \mu+\lambda$.

The relation (21) takes the form

$$\Phi_\eta(\omega,t) = V(-a,t) + V(b,t) \tag{28}$$

and the explicit form of (22) reads

$$\frac{\partial}{\partial t}V(-a,t) = -\mu V(-a,t) + \lambda V(b,t) + \Omega \mathrm{e}^{2\gamma t/m} aV(-a,t)\,, \tag{29}$$

$$\frac{\partial}{\partial t}V(b,t) = \mu V(-a,t) - \lambda V(b,t) - \Omega \mathrm{e}^{2\gamma t/m} bV(b,t)\,. \tag{30}$$

The initial conditions follow from (20) and read (*cf.* (26))

$$V(-a,0) = \langle\delta(\eta(t),-a)\rangle = \frac{\lambda}{\mu+\lambda}\,, \tag{31}$$

$$V(b,0) = \langle\delta(\eta(t),b)\rangle = \frac{\mu}{\mu+\lambda}\,. \tag{32}$$

Let us define a new time variable

$$\tau = \tau(t) = \Omega \mathrm{e}^{2\gamma t/m} \tag{33}$$

and two new functions $\tilde V(-a,\tau)$ and $\tilde V(b,\tau)$ via the relations

$$V(-a,t) = \tilde V(-a,\tau(t))\,, \tag{34}$$

$$V(b,t) = \tilde V(b,\tau(t))\,. \tag{35}$$

Then Eqs (30) and (29) can be transformed to the form

$$\frac{2\gamma\tau}{m}\frac{\partial}{\partial\tau}\tilde V(-a,\tau) = -\mu\tilde V(-a,\tau) + \lambda\tilde V(b,\tau) + \tau a\tilde V(-a,\tau)\,, \tag{36}$$

$$\frac{2\gamma\tau}{m}\frac{\partial}{\partial\tau}\tilde V(b,\tau) = \mu\tilde V(-a,\tau) - \lambda\tilde V(b,\tau) - \tau b\tilde V(b,\tau) \tag{37}$$

with the initial conditions at $\tau(t=0)=\Omega$,

$$\tilde{V}(-a,\Omega) = \frac{\lambda}{\mu+\lambda}\,, \qquad (38)$$

$$\tilde{V}(b,\Omega) = \frac{\mu}{\mu+\lambda}\,. \qquad (39)$$

We define two new functions in the following way

$$F(\tau) = \tilde{V}(-a,\tau)+\tilde{V}(b,\tau)\,, \qquad (40)$$

$$G(\tau) = b\tilde{V}(b,\tau)-a\tilde{V}(-a,\tau)\,. \qquad (41)$$

Then from Eqs (36) and (37) one gets

$$\frac{2\gamma}{m}\dot{F}(\tau) = -G(\tau)\,, \qquad (42)$$

$$\frac{2\gamma\tau}{m}\dot{G}(\tau)+(\mu+\lambda-\tau(a-b))G(\tau)+\tau abF(\tau)=0\,, \qquad (43)$$

where the dot denotes a derivative with respect to the argument. The initial conditions follow from (40)–(43) and take the form

$$F(\Omega)=1\,, \qquad G(\Omega)=0\,. \qquad (44)$$

The function $F(\tau)$ is crucial because the characteristic functional (16) is related to it in a simple way. Indeed,

$$\Phi_\eta(\omega,t)=F(\tau) \quad \text{for} \quad \tau=\Omega \mathrm{e}^{2\gamma t/m} \quad \text{and} \quad \Omega=\frac{\gamma k}{m^2}\,\omega^2\mathrm{e}^{-2\gamma t/m}\,. \qquad (45)$$

From the above system of two coupled differential equations (42) and (43), we obtain a closed differential equation for the function $F(\tau)$ only. It has the form

$$\tau\ddot{F}(\tau)+\frac{m}{2\gamma}\left[\mu+\lambda+\tau(b-a)\right]\dot{F}(\tau)-\frac{m^2ab\tau}{4\gamma^2}F(\tau)=0 \qquad (46)$$

with the initial conditions

$$F(\Omega)=1\,, \qquad \dot{F}(\Omega)=0\,. \qquad (47)$$

It belongs to a class of hypergeometric equations. Its solution is the function [18]

$$F(\tau)=\mathrm{e}^{-mb\tau/2\gamma}\left\{C_1(\Omega)\ \Phi[\alpha,\beta,\chi(\tau)]+C_2(\Omega)\ \Psi[\alpha,\beta,\chi(\tau)]\right\}, \qquad (48)$$

where Φ and Ψ stand for the confluent hypergeometric Kummer and Tricomi functions, respectively [19]. The parameters

$$\alpha = \frac{mb}{2\gamma}\frac{\mu+\lambda}{a+b} = \frac{mb}{2\gamma\tau_c(a+b)}\,, \tag{49}$$

$$\beta = \frac{m(\mu+\lambda)}{2\gamma} = \frac{m}{2\gamma\tau_c} \tag{50}$$

and

$$\chi(\tau) = \frac{m(a+b)}{2\gamma}\,\tau\,. \tag{51}$$

The constants $C_1(\Omega)$ and $C_2(\Omega)$ are determined from the conditions (47) and read

$$C_1(\Omega) = \frac{b\Gamma(\alpha)}{(a+b)\Gamma(\beta)}\,\chi(\Omega)^{\beta}\,\mathrm{e}^{-ma\Omega/2\gamma} \times\left(\Psi[\alpha,\beta,\chi(\Omega)] + \frac{m(\mu+\lambda)}{2\gamma}\,\Psi[\alpha+1,\beta+1,\chi(\Omega)]\right) \tag{52}$$

and

$$C_2(\Omega) = \frac{b\Gamma(\alpha)}{(a+b)\Gamma(\beta)}\,\chi(\Omega)^{\beta}\,\mathrm{e}^{-ma\Omega/2\gamma} \times\left(\Phi[\alpha+1,\beta+1,\chi(\Omega)] - \Phi[\alpha,\beta,\chi(\Omega)]\right)\,, \tag{53}$$

where $\Gamma(z)$ is the Euler Gamma function. In this way we found the function $F(\tau)$ and via the expressions in (45) we can find the characteristic functional $\Phi_\eta(\omega,t)$.

5. Probability distribution

The probability distribution is obtained from Eqs (10)–(16) and the relations (45). It has the form

$$p(v,t|v_0,0) = \frac{1}{2\pi}\int\limits_{\infty}^{\infty} d\omega\,\mathrm{e}^{i\omega v} \times\exp\left[i\omega v_0\mathrm{e}^{-\gamma t/m} - \frac{kT_0}{2m}\omega^2\left(1-\mathrm{e}^{-2\gamma t/m}\right)\right]\Phi_\eta(\omega,t). \tag{54}$$

The explicit form of the characteristic functional is

$$\begin{aligned}\Phi_\eta(\omega,t) \;=\; & \frac{b\Gamma(\alpha)}{(a+b)\Gamma(\beta)}\,\mathrm{e}^{-bk\omega^2/2m}\left(A\omega^2\mathrm{e}^{-2\gamma t/m}\right)^{\beta}\exp\left[-\frac{ak\omega^2}{m}\mathrm{e}^{-2\gamma\, t/m}\right] \\ & \times\Bigg\{\Phi\left[\alpha,\beta,A\omega^2\right]\Bigg(\Psi\left[\alpha,\beta,A\omega^2\mathrm{e}^{-2\gamma t/m}\right] \\ & +\frac{m}{2\gamma\tau_\mathrm{c}}\,\Psi\left[\alpha+1,\beta+1,A\omega^2\mathrm{e}^{-2\gamma t/m}\right]\Bigg) \\ & +\Psi\left[\alpha,\beta,A\omega^2\right]\Bigg(\Phi\left[\alpha+1,\beta+1,A\omega^2\mathrm{e}^{-2\gamma t/m}\right] \\ & -\Phi\left[\alpha,\beta,A\omega^2\mathrm{e}^{-2\gamma t/m}\right]\Bigg)\Bigg\}, \end{aligned} \tag{55}$$

where the constant $A = k(a+b)/2m$. By use of (4), the distribution $p(v,t|v_0,0)$ allows to determine evolution of the one-dimensional velocity probability density $P(v,t)$ for an arbitrary initial state defined by the distribution $P(v,0)$ and analyze relaxation of the system to the stationary state.

5.1. Stationary distribution

The stationary velocity distribution function $P_{\mathrm{st}}(v)$ does not depend on the initial distribution $P(v,0)$. It is obtained from (4) and (54) performing the long time limit, $t\to\infty$. We use the relations [19]

$$\lim_{z\to 0}\ \Phi[\alpha,\beta,z] = 1 \tag{56}$$

and

$$\Psi[\alpha,\beta,z] = \frac{\Gamma(1-\beta)}{\Gamma(\alpha-\beta+1)} + \frac{\Gamma(\beta-1)}{\Gamma(\alpha)}\, z^{1-\beta}, \tag{57}$$

which represents the leading terms for small $z = A\omega^2\mathrm{e}^{-2\gamma t/m} \ll 1$ when $t\to\infty$. Then the stationary distribution takes the form

$$P_{\mathrm{st}}(v) = \frac{1}{\pi}\int_0^\infty d\omega\cos(\omega v)\mathrm{e}^{-(T_0+b)k\omega^2/2m}\,\Phi\left[\frac{b\,m}{2\gamma\tau_c(a+b)},\frac{m}{2\gamma\tau_c},\frac{a+b}{2m}k\omega^2\right]. \tag{58}$$

It is not a Maxwell distribution and we can conclude that the stationary state is not an equilibrium state: It is a nonequilibrium state. In the case of absence of temperature fluctuations, *i.e.* when $a = 0$ and next $b = 0$, then

$$\Phi\left[\frac{m}{2\,\tau_c\gamma}, \frac{m}{2\gamma\tau_c}, 0\right] = 1 \tag{59}$$

and the Maxwell distribution (7) is obtained for $T = T_0$.

5.2. Limiting cases of short and long correlation time

The correlation time τ_c of temperature fluctuations is defined below Eq. (27). For very fast fluctuations when the correlation time is short,

$$\lim_{\tau_c \to 0} \Phi\left[\frac{b\,m}{2\gamma\tau_c(a+b)}, \frac{m}{2\gamma\tau_c}, \frac{a+b}{2\,m}\,k\,\omega^2\right] = \mathrm{e}^{bk\,\omega^2/2m} \tag{60}$$

and (58) reduces to the Maxwell distribution (7) with temperature $T = T_0$. The short correlation time limit can be achieved when (*cf.* (25)):

(i) $\mu \to \infty$ and $b \to 0$ but $b\mu = a\lambda = \text{const.}$

(ii) $\lambda \to \infty$ and $a \to 0$ but $a\lambda = b\mu = \text{const.}$

(iii) $b \to \infty$ and $\lambda \to \infty$ but $b/\lambda = \text{const.}$ (the latter corresponds to the Poisson white shot noise)

(iv) $\mu \to \infty$ and $\lambda \to \infty$ but $\mu/\lambda = \text{const.}$

In these limits, the system is not able to react to very fast fluctuations and effectively it feels the mean temperature $T = T_0$.

The opposite limit is the adiabatic limit when fluctuations are slow and the correlation time is very long, $\tau_c \to \infty$. The Kummer function takes the form

$$\lim_{\tau_c \to \infty} \Phi\left[\frac{b\,m}{2\gamma\tau_c(a+b)}, \frac{m}{2\gamma\tau_c}, \frac{a+b}{2\,m}\,k\,\omega^2\right] = \frac{a}{a+b} + \frac{b}{a+b}\,\mathrm{e}^{(a+b)k\omega^2/2m} \tag{61}$$

and Eq. (58) reduces to the function

$$\begin{aligned} P_{\mathrm{st}}(v) \;=\; & \frac{b}{a+b}\sqrt{\frac{m}{2\pi k(T_0-a)}}\;\exp\left[-\frac{mv^2}{2k(T_0-a)}\right] \\ & + \frac{a}{a+b}\sqrt{\frac{m}{2\pi k(T_0+b)}}\;\exp\left[-\frac{mv^2}{2k(T_0+b)}\right]. \end{aligned} \tag{62}$$

It is a linear combinations of two Maxwellian distributions for two temperatures $T_0 - a$ and $T_0 + b$ and with the weights given by Eqs (26). The long correlation time limit can be achieved when μ, $\lambda \to 0$ and the mean residence times in the states $-a$ and b tend to infinity, τ_a, $\tau_b \to \infty$. The adiabatic limit for other models of inverse temperature fluctuations has been considered in [1].

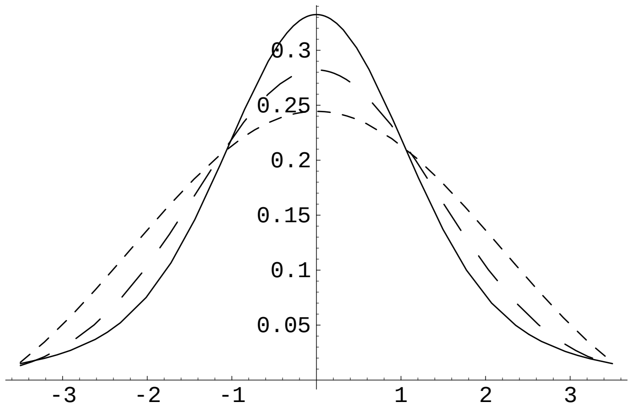

Fig. 1. The stationary velocity distribution $P_{\rm st}(v)$ for three values of the correlation time $\tau_{\rm c}$ of temperature fluctuations: $\tau_{\rm c} = 0$ (dotted line, the Maxwell distribution), $\tau_{\rm c} = 5$ (dashed line) and $\tau_{\rm c} \to \infty$ (solid line).

6. Discussion

Applying a method of the curtailed characteristic functional we obtained a time-dependent probability distribution of velocity of the Brownian particle moving in medium in which temperature fluctuates. We considered temperature fluctuations to be as simple as possible, *i.e.*, a two-state stationary Markovian process $\eta(t)$. Nevertheless, the problem is formulated for an arbitrary Markovian stochastic process $\eta(t)$ because what we need is the infinitesimal generator $\hat{L}$ of the process $\eta(t)$, see Eq. (22). We note that the correlation function of the force $F(t) = \sqrt{2\gamma k T(t)}\, \xi(t)$ in Eq. (1) has the form

$$\langle F(t)F(s)\rangle = 2\gamma k T_0 \delta(t-s), \tag{63}$$

independently of statistics of fluctuations $\eta(t)$ and has the same form as in the case without temperature fluctuations. It resembles the dissipation-fluctuation relation. However, we showed that the stationary state is a nonequilibrium state. We can ask how far the system is from equilibrium.

To this aim, let us analyze statistical moments of the velocity. From Eq. (58) it follows that the stationary characteristic function $C_v(\omega)$ of the velocity is

$$C_v(\omega) = \mathrm{e}^{-(T_0+b)k\omega^2/2m} \ \Phi\left[\frac{b\, m}{2\gamma\tau_c(a+b)}, \frac{m}{2\gamma\tau_c}, \frac{a+b}{2\, m}\ k\, \omega^2\right]. \tag{64}$$

The statistical moments $\langle v^n \rangle$, $(n = 1, 2, 3, \ldots)$ can be obtained from the relation $\langle v^n \rangle = i^n d^n C_v(\omega)/d\omega^n|_{\omega=0}$. The odd order moments are equal to zero. The second moment

$$\langle v^2 \rangle = \frac{kT_0}{2}\,. \tag{65}$$

It does not depend on the statistics of temperature fluctuations and is the same as for the Maxwell distribution! The forth order moment measure a deviation from the Maxwell distribution. We use it to calculate the kurtosis,

$$\mathrm{Kurt}\,(v) = \frac{\langle v^4 \rangle}{3\langle v^2 \rangle^2} - 1 = \frac{2ab}{T_0^2(2+\tau_{\mathrm{d}}/\tau_{\mathrm{c}})}\,, \tag{66}$$

where $\tau_{\mathrm{d}} = m/\gamma$ is the relaxation time of the velocity in the deterministic case (*cf.* Eq. (12) when $\xi(t) = 0$) and τ_{c} is the correlation time of fluctuations (see Eq. (27)). For the equilibrium state, *i.e.* for the Maxwell distribution, the kurtosis is zero. In the case considered, the kurtosis is always positive and it means that $P_{\mathrm{st}}(v)$ is more peaked than the Maxwell distribution. It is an increasing function of the variance $\langle \eta^2(t) \rangle = ab$ and the correlation time τ_{c} of temperature fluctuations. As a function of the correlation time, it grows from zero for $\tau_{\mathrm{c}} = 0$ to the maximal value ab/T_0^2 when $\tau_{\mathrm{c}} \to \infty$. Generally, all even order moments are greater than for the Maxwell distribution.

One can determine the moments for the position of the Brownian particle. *E.g.*, for long times, $t \gg \tau_{\mathrm{d}}$, the mean squared displacement

$$\langle x^2(t) \rangle \sim 2Dt\,, \tag{67}$$

where the diffusion coefficient $D = kT_0/\gamma$ is the same as in the case without temperature fluctuations. It means that for long time, the process in the position space is the standard normal diffusion with the same diffusion constant. We can conclude that the first two moments of position and of velocity are the same in both cases: without and with temperature fluctuations. So, when we measure only first two moments, we cannot distinguish these two states. We emphasize that it does not depend on the model of temperature fluctuations.

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