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ENTROPY FOR COLOR SUPERCONDUCTIVITY IN QUARK MATTER

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We study a known model for color superconductivity with three colors and three massless quark flavors including pairing effects. By using the Hamiltonian in the color–flavor basis we can calculate the quantum entropy. From this calculation we are able to further investigate the phases of the color superconductor, for which we find a rather sharp transition to color superconductivity above a chemical potential around 290 MeV.

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1. Introduction

At high quark chemical potentials and low temperatures the internal structure of hadronic matter has been conjectured to dissolve into a degenerate system of quarks. Such material consisting of very cold dense quarks might exist in the interior of compact stellar objects. However, due to the difficulties of performing lattice simulations with high chemical potentials, it is still not possible to simulate the physics of these phases by using the usual lattice gauge computations. Nevertheless, a nonperturbative analysis at finite baryon density has been quite recently carried out on the lattice by using the Nambu–Jona-Lasinio model. The degenerate quarks near to the

Fermi surfaces are generally expected to interact according to quantum chromodynamics (QCD) so that they can build up Cooper pairs. This process may lead to superconducting quark matter [1].

Before we discuss the actual model for calculating the quantum entropy S in color superconducting quark matter, one might well ask: "Why should S be taken as a significant physical quantity, which is in no way connected to the properties of the phase transition to a superconducting state?" The simplest answer to this question is that S relates directly to the *order parameters*, which are for the superconductors simply the energy gaps. The quantum entropy puts all the information from all the gaps into a single physical quantity, which is then characteristic of the ordered phase. We shall later find a single parameter called σ, which is calculated directly from the momentum spread of S at a given quark chemical potential μ. We show that σ is effectively *zero* for μ below a critical value $\mu_{\rm c}$ and is finite and monotonically increasing above this value. This behavior clearly reflects the expected nature of an order parameter.

There are a number of physical systems, which we have previously discussed [2–4], for which we have reconsidered the meaning of the third law of thermodynamics when stated in its original form. There we have found that the entropy remains finite even at absolute zero. However, when we had taken this fact into consideration, there were some aspects of the usual thermodynamical formulation which had become more complicated [2, 4]. Our objective here in this work is to further interpret the reason for the finiteness of this entropy at absolute zero in quark matter in relation to the quantum correlations present in the superconducting ground state at large values of the quark chemical potential μ. How the presence of this finite quantum entropy term would affect the actual thermodynamics at finite temperatures will be discussed elsewhere. In particular, the effects of including S in the hadronic equation of state at low temperatures we have already studied in special cases [4].

To our knowledge, the works of Elze [6] have provided the start for addressing the question about the origin of the *entropy puzzle* in the high-energy collisions. These works established a theoretical framework for the discussion of how two hadronic scattering initial states undergo hard collisions in quantum mechanically pure initial states. This situation can result in a high-multiplicity event corresponding to a highly impure thermal density matrix on the partonic level before hadronization. According to these works [6] the entropy is an unambiguous characteristic property of the quantum nature of the system. The entropy production is clearly due to the *environmentally induced quantum decoherence* in the observable subsystem.

Therefore, we consider that there is no obvious theoretical reason to consider finite entropy for the interpretation of the particle multiplicity, while then explicitly setting its value to zero in other comparable cases.

In our previous works [2–4] we have also used the prescription of von Neumann for the entropy, which make use of the eigenvalues of the reduced density matrices. Thus we are able to give a first quantitative evaluation for the quarks' entropy inside the hadrons under the condition that in the singlet state of the hadronic groundstates the individual quarks' colors are maximally mixed. In a more recent work [5] we have also given a general evaluation of the entropy for the condensate in quark color superconductivity using a nonrelativistic model based on the BCS theory. In particular we studied the pair structure in the BCS model which relates to the $\mathrm{SU}(2)_c$ of the broken color symmetry [7]. In general for SU(2) symmetry [2] it is well known that one finds simply that the entropy is $\ln 2$. For a pair of such constituents there is the factor *two* so that the result is $2 \ln 2$. We shall later use this result for any given number N of paired states with this doubly degenerate groundstate as the factor $N \ln 4$.

In the present work we start with an ultrarelativistic model Hamiltonian for quarks with three colors and flavors which was proposed a few years ago [8, 9] for color superconductivity. In this framework we shall apply our previous calculations for the quantum ground state entropy [2–4] to these quarks at large values of the quark chemical potential. As mentioned above, the quantum entropy is clearly that entropy which arises from the quantum correlations in the hadronic groundstates. Thereupon, these correlations relate directly to the presence of quantum fluctuations which are characteristic of a quantum phase transition [10]. Furthermore, we state that these entities differ from the usual thermal fluctuations of a statistical system in that they can also exist at zero temperature. Thus in the above mentioned model with three colors and flavors a total of nine mixed quark states are present. As we have explained above for the pair states which consist of two colored quarks, the quantum entropy can be expected to be temperature independent and equal to just $9 \ln 4$ in maximally mixed states [5].

2. Color superconductor model

We now look more explicitly at the effects on the ground state in our model with three colors and three flavors for the structure of the superconductivity. In this part we shall calculate the quantum entropy from the gap equations for the color superconducting state. For massless quarks the form for the Hamiltonian already has been written down for the three colors and flavors. This color superconductivity [8] arises from the quark-pairs at low temperatures and high quark chemical potentials. It takes the following form:

$$
\begin{aligned}
H \;=\; & \sum_{\rho,k>\mu} (k-\mu) a_\rho^\dagger(\boldsymbol{k}) a_\rho(\boldsymbol{k}) \\
& + \sum_{\rho,k<\mu} (\mu-k) a_\rho^\dagger(\boldsymbol{k}) a_\rho(\boldsymbol{k}) + \sum_{\rho,\boldsymbol{k}} (k+\mu) b_\rho^\dagger(\boldsymbol{k}) b_\rho(\boldsymbol{k}) \\
& + \frac{1}{2} \sum_{\rho,\boldsymbol{p}} F(p)^2 Q_\rho e^{-i\phi(\boldsymbol{p})} \Big(a_\rho(\boldsymbol{p}) a_\rho(-\boldsymbol{p}) + b_\rho^\dagger(\boldsymbol{p}) b_\rho^\dagger(-\boldsymbol{p}) \Big) \\
& + \frac{1}{2} \sum_{\rho,\boldsymbol{p}} F(p)^2 Q_\rho e^{i\phi(\boldsymbol{p})} \Big(a_\rho^\dagger(\boldsymbol{p}) a_\rho^\dagger(-\boldsymbol{p}) + b_\rho(\boldsymbol{p}) b_\rho(-\boldsymbol{p}) \Big) . \qquad (1)
\end{aligned}
$$

$F(p)^2$ is the form factor containing the cut-off Λ. Q_ρ stands for the diagonalized form for the gap parameters, for which $\rho = 1$ yields the color–flavor singlet gap-parameter Δ_1 and $\rho = 2, \cdots, 9$ result in the color–flavor gap $\pm\Delta_8$. The first two lines of the Hamiltonian represent only the non-interacting parts, while the third and fourth lines thereof are the complex conjugate terms of the interactions with opposite momenta. $a_\rho^\dagger$ and a_ρ together with $b_\rho^\dagger$ and b_ρ are the creation and annihilation operators of the particle and antiparticle states, respectively. The index ρ, as given above, stands for both the color and flavor degrees of freedom.

For our present purpose we can treat ρ in the same way as we had previously taken only the color degrees of freedom since the flavors now provide an exact symmetry in the limit of massless quarks. We take μ as the quark chemical potential, for which all the momenta up to $\mu = p_F$ have all the particle and antiparticle states completely occupied in the groundstate.

A proper parameterization for the annihilation and creation operators, respectively, had been already suggested [8] as follows:

$$
\begin{aligned}
y_\rho(\boldsymbol{k}) &= \cos[\theta_\rho^y(\boldsymbol{k})] a_\rho(\boldsymbol{k}) + \sin[\theta_\rho^y(\boldsymbol{k})] e^{i\xi_\rho^y(\boldsymbol{k})} a_\rho^\dagger(-\boldsymbol{k}) , & (2) \\
z_\rho(\boldsymbol{k}) &= \cos[\theta_\rho^z(\boldsymbol{k})] b_\rho(\boldsymbol{k}) + \sin[\theta_\rho^z(\boldsymbol{k})] e^{i\xi_\rho^z(\boldsymbol{k})} b_\rho^\dagger(-\boldsymbol{k}) , & (3) \\
y_\rho^\dagger(\boldsymbol{k}) &= \cos[\theta_\rho^y(\boldsymbol{k})] a_\rho^\dagger(\boldsymbol{k}) + \sin[\theta_\rho^y(\boldsymbol{k})] e^{-i\xi_\rho^y(\boldsymbol{k})} a_\rho(-\boldsymbol{k}) , & (4) \\
z_\rho^\dagger(\boldsymbol{k}) &= \cos[\theta_\rho^z(\boldsymbol{k})] b_\rho^\dagger(\boldsymbol{k}) + \sin[\theta_\rho^z(\boldsymbol{k})] e^{-i\xi_\rho^z(\boldsymbol{k})} b_\rho(-\boldsymbol{k}) . & (5)
\end{aligned}
$$

Therefrom the following definitions are given:

$$
\begin{aligned}
\theta_\rho^y(\boldsymbol{k}) &= \frac{1}{2} \arccos\left(\frac{|k-\mu|}{\sqrt{(k-\mu)^2 + F(k)^4 Q_\rho^2}} \right) , & (6) \\
\xi_\rho^y(\boldsymbol{k}) &= \phi(\boldsymbol{k}) + \pi , & (7)
\end{aligned}
$$

$$\theta_\rho^z(\boldsymbol{k}) = \frac{1}{2}\arccos\left(\frac{|k+\mu|}{\sqrt{(k+\mu)^2+F(k)^4Q_\rho^2}}\right), \tag{8}$$

$$\xi_\rho^z(\boldsymbol{k}) = -\phi(\boldsymbol{k})\,. \tag{9}$$

We may compare these complex expressions with the usual forms for the Bogoliubov transformations, from which we can explicitly write down the parameters as follows:

$$u_\rho(\boldsymbol{k}) \equiv \cos[\theta_\rho(\boldsymbol{k})]\,, \tag{10}$$

$$v_\rho(\boldsymbol{k}) \equiv \sin[\theta_\rho(\boldsymbol{k})]e^{i\xi_\rho(\boldsymbol{k})}\,. \tag{11}$$

Obviously, we get the necessary relationship between these quantities

$$u_\rho^*(\boldsymbol{k})u_\rho(\boldsymbol{k}) + v_\rho^*(\boldsymbol{k})v_\rho(\boldsymbol{k}) = 1 \tag{12}$$

which shows the canonical nature of these transformations.

After we have carried out these canonical transformations, the form of the Hamiltonian for noninteracting quasiquarks takes on the quadratic canonical structure:

$$H = \sum_{\boldsymbol{k},\rho}\Big[\left((k-\mu)^2+F(k)^4Q_\rho^2\right)^{1/2} y_\rho^\dagger(\boldsymbol{k})y_\rho(\boldsymbol{k}) + \left((k+\mu)^2+F(k)^4Q_\rho^2\right)^{1/2} z_\rho^\dagger(\boldsymbol{k})z_\rho(\boldsymbol{k})\Big]\,. \tag{13}$$

We now write out the results using the above given definitions for $u_\rho(\boldsymbol{k})$ and $v_\rho(\boldsymbol{k})$. These quantities are easily derived using elementary trigonometrical identities and the previously given definitions of the angles. After a small amount of algebra we find that two new quantities can be simply defined,

$$\Upsilon_\rho^y(\boldsymbol{k}) \equiv \frac{u_\rho^{y*}(\boldsymbol{k})u_\rho^y(\boldsymbol{k})}{v_\rho^{y*}(\boldsymbol{k})v_\rho^y(\boldsymbol{k})} = \frac{\sqrt{(k-\mu)^2+F(k)^4Q_\rho^2}+|k-\mu|}{\sqrt{(k-\mu)^2+F(k)^4Q_\rho^2}-|k-\mu|}, \tag{14}$$

$$\Upsilon_\rho^z(\boldsymbol{k}) \equiv \frac{u_\rho^{z*}(\boldsymbol{k})u_\rho^z(\boldsymbol{k})}{v_\rho^{z*}(\boldsymbol{k})v_\rho^z(\boldsymbol{k})} = \frac{\sqrt{(k+\mu)^2+F(k)^4Q_\rho^2}+|k+\mu|}{\sqrt{(k+\mu)^2+F(k)^4Q_\rho^2}-|k+\mu|}. \tag{15}$$

The quantum entropy of entanglement for an ideal Bose gas at $T=0$ vanishes, which means that the occupied state with zero momentum does not contribute. For finite interactions between constituents there appears an entropy, which requires a momentum exchange between the particles [11].

In the same context in the BCS model without a finite momentum arising in the interaction there would be no Cooper pairs with oppositely directed momenta. Thus for $k \neq 0$ we can write down the quantum entropy of entanglement for this color superconducting model following the method we have previously used for the BCS model [5] for finding $S_{\rm BCS}$. The main difference from the simple BCS model is that in the color superconducting model we have nine states with two different contributions from the gaps. Now we write down $S_{\rm CSM}$ with the degeneracy factor $g = 2$ for the spins.

$$S_{\rm CSM} = g \sum_{\rho=1}^{9} \left\{ \frac{\ln \varUpsilon_\rho^y(\boldsymbol{k})}{\varUpsilon_\rho^y(\boldsymbol{k})+1} + \ln\left(1+\frac{1}{\varUpsilon_\rho^y(\boldsymbol{k})}\right) + \frac{\ln \varUpsilon_\rho^z(\boldsymbol{k})}{\varUpsilon_\rho^z(\boldsymbol{k})+1} + \ln\left(1+\frac{1}{\varUpsilon_\rho^z(\boldsymbol{k})}\right) \right\}. \tag{16}$$

In order to actually evaluate the above contributions to $S_{\rm CSM}$, we must first set up the equations for the gaps $\varDelta_1$ and $\varDelta_8$, which have been previously studied by Alford, Rajagopal and Wilczek [8]. Before we write down the complete gap equations, we briefly discuss the two color–flavor superconducting model known as the 2SC Model. In structure it is very similar to the simple nonrelativistic BCS model with the addition of the antiparticle contribution. A not too great generalization of this type of model leads to a two flavor and three color model [8], which is the prior step to the above model. The 2SC phase is such that the diquarks condense while the chiral symmetry is restored. It has a simple equation for the gap $\varDelta$ similar to the above BCS Model. We write the gap equation in the form [9]

$$1 = \frac{2G}{V} \sum_p \left\{ \frac{1}{\sqrt{(p-\mu)^2+\varDelta^2}} + \frac{1}{\sqrt{(p+\mu)^2+\varDelta^2}} \right\}. \tag{17}$$

We can convert the sum over all the momenta p into an integral $\mathcal{I}[\varDelta]$ so that

$$\mathcal{I}[\varDelta] = \frac{G}{\pi^2} \int_0^{\varLambda} k^2 dk \left(\frac{1}{\sqrt{(k-\mu)^2+\varDelta^2}} + \frac{1}{\sqrt{(k+\mu)^2+\varDelta^2}} \right). \tag{18}$$

This form of the gap equation

$$1 = \mathcal{I}[\varDelta] \tag{19}$$

can be evaluated and substituted in the above equation for the entropy. However, it does not properly reflect the fully extended color–flavor symmetry of our above model Hamiltonian. Nevertheless, we can use this simpler equation with the constituent quark mass M replacing the gap parameter at vanishing chemical potential in order to determine the coupling G. The coupled integral equations for the gaps represent the singlet and octet decomposition of this extended symmetry [8]. In our evaluation of the equations for the gaps Δ_1 and Δ_8 we use the step-function cut-off with $\Lambda = 800$ MeV. Thus we write the two gap equations [9] in the following form:

$$\Delta_1 = -2\Delta_8 \mathcal{I}[\Delta_8]\,, \tag{20}$$

$$\Delta_8 = -\frac{\Delta_1\,(1+\mathcal{I}[\Delta_1])}{4}\,. \tag{21}$$

We use these gap equations for Δ_1 and Δ_8 in order to obtain $\Upsilon^y_\rho(\boldsymbol{k})$ and $\Upsilon^z_\rho(\boldsymbol{k})$ with $F(k) = 1$ for $k < \Lambda$ and zero above. These quantities are substituted into the above equation for the quantum entropy S_{CSM}. The results for $9\ln 4 - S_{\mathrm{CSM}}$ are shown in Fig. 1 for different values of the quark chemical potential μ. We can see the effects of the gaps Δ_1 and Δ_8 on the quantum entropy. Below a critical value of the chemical potential around 290 MeV the gaps vanish so that the spread of S_{CSM} also vanishes. We see this in a very narrow line that extends downwards to zero at values of μ under 290 MeV.

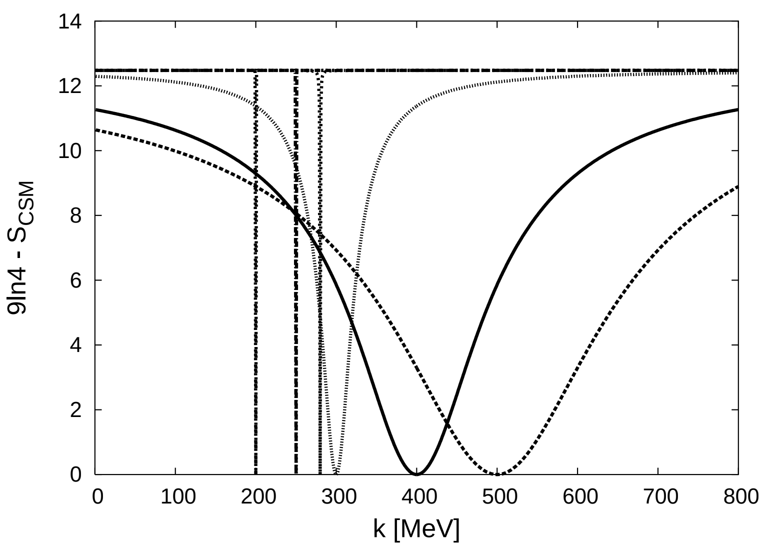

Fig. 1. The momentum dependence of the difference between the maximum entropy in the ground state $9\ln 4$ and the entropy from the excitation of color superconductors is shown for different values of the chemical potential μ indicated by the zero-point on the graph.

3. Results and discussion

Now we discuss the results from the computation of the quantum entropy $S_{\rm CSM}$ derived in the equation 16. Figure 1 shows the difference between the entropy from the quark pairing in color superconductors Eq. 16 and the maximum value of the groundstate entropy for nine states given by $9\ln 4$. The dependence of $9\ln 4 - S_{\rm CSM}$ upon the momenta k is plotted for different values of the chemical potential μ. This difference has a zero-point when the value of the momentum $k = \mu$. In this figure we notice that for the zero-point values above a critical value $\mu_{\rm c} = 290$ MeV there is always a finite spread in the curve around the zero-point. From this fact we can extract at the halfheight value the fullwidth Γ. By means of a direct comparison with the Gaussian distribution we take the dispersion to be the standard deviation σ. In this case we have simply the fullwidth $\Gamma = 2\sqrt{2\ln 2}\,\sigma$ at half maximum. Then we can compute the dispersion σ from the distributions given in the next figure 2 as a function of the corresponding μ value. We can make a fit of these points as a function of μ as is shown in the second figure with

$$\sigma(\mu) = A + B(\mu - \mu_{\rm c})^{\beta}. \tag{22}$$

In this fit the best values are found to be $A = -3.095$ and $B = 3.294$. The best value of the exponent is $\beta = 0.506$, which is notably very near to the classical value of one-half. If we were to set the proper units for the

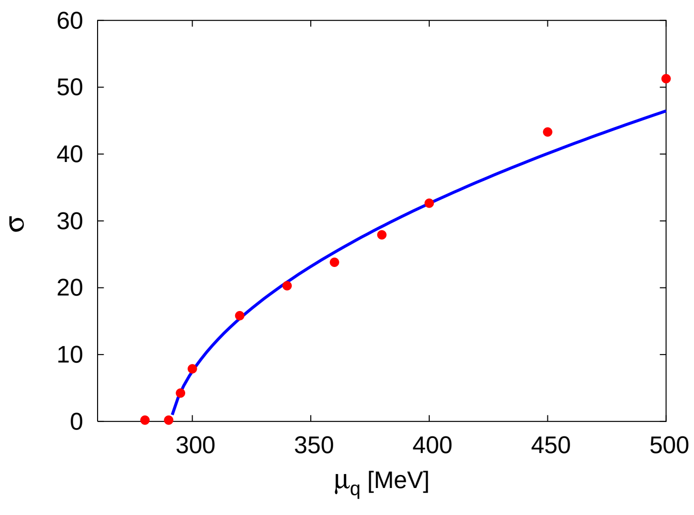

Fig. 2. The dispersion σ is taken from the distributions given in Fig. 1 plotted against the corresponding quark chemical potential. Finite values for the quark chemical potential first appear above the critical value $\mu > 290$ MeV.

scaling variable from the forefactor B, which we shall call r. Then we may rewrite σ as

$$\sigma(r) \;=\; A \;+\; B'(r)^{\beta}\,, \tag{23}$$

whereby $r \;=\; (\mu - \mu_{\rm c})/\mu_{\rm c}$ and $B' \;=\; B{\mu_{\rm c}}^{\beta}$. Thus the dispersion $\sigma(r)$ with the proper scaling relations serves as an order parameter for the critical region of the chemical potential near to $\mu_{\rm c}$. These results then show approximately the usual mean field behavior at the critical point with $\beta \;=\; 1/2$.

4. Conclusion

Finally we can conclude that the quantum entropy is a good indicator of the phase transition arising from the presence of the two gaps Δ_1 and Δ_8. As we have seen in figure 2, the dispersion σ becomes finite at a chemical potential of about 290 MeV, which is very near the previously calculated value of around 300 MeV for the gaps appearing in the CFL phase [8, 9]. Further discussion of these models and their properties relating to the phenomena of color superconductivity has appeared quite recently [13]. In this model with three massless flavors taken together with the three colors from the usual $\mathrm{SU}(3)_c$ we are able to analyze the dependence of the gaps Δ_1 and Δ_8 on the quark chemical potential μ using the physical quantity $S_{\rm CSM}$.

In our present work we have shown that by using the quantum entropy $S_{\rm CSM}$ we have a quantity in addition to the gaps which can be computed to show the transition to color superconductivity. Although the gaps in general relate to the quantum structure of the quark pairs, the entropy $S_{\rm CSM}$ is a thermodynamical quantity. Nevertheless, its deviation comes from the pairing structure in this three color and flavor model, which gives rise to the two gaps Δ_1 and Δ_8. Furthermore, we have derived the dispersion σ to relate directly to the fullwidth of the peak in $S_{\rm CSM}$. This fullwidth is directly related to the quark pair structure in momentum space, which we have shown in the first figure as a dip around the Fermi momentum, which is just μ in this massless quark model. The gap is well known to be characteristic of the correlations between the fermions with oppositely directed momenta and spins as in the BCS model. In this case these correlations are between the quarks within the pairs with different colors and flavors. That these flavors and colors form given states together is quite characteristic of the phenomena known as color–flavor locking [8, 9].

This pairing structure is actually quite similar to the quark–quark correlations oftentimes referred to as the diquark structure. This phenomenon has been noted in connection with various results in experimental high-energy physics. The diquark differs from the usual quark pairs used in superconductivity in that the full $3 \otimes 3$ is taken into consideration. Nevertheless, the ideas relating to the quantum entropy could possibly lead to further future investigations relating to the expected correlations between the diquarks as well as, perhaps, find future applications in the theoretical studies of the recently experimentally discovered pentaquark states [14]. These states have been interpreted [15] as a bound state of four quarks and an antiquark, which consist of two correlated diquarks.

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