A NEW PROOF OF EXISTENCE OF A BOUND STATE IN THE QUANTUM COULOMB FIELD

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(Received June 9, 2004)

Let $S(x)$ be a massless scalar quantum field which lives on the three-dimensional hyperboloid $xx=(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2=-1$. The classical action is assumed to be $(\hbar=1=c)(8\pi e^2)^{-1}\int dx g^{ik}\partial_i S\partial_k S$, where e^2 is the coupling constant, dx is the invariant measure on the de Sitter hyperboloid $xx=-1$ and $g_{ik}, i,k=1,2,3$, is the internal metric on this hyperboloid. Let u be a fixed four-velocity *i.e.* a fixed unit time-like vector. The field $S(u)=(1/4\pi)\int dx\delta(ux)S(x)$is smooth enough to be exponentiated, being an average of the operator valued distribution $S(x)$ over the entire Cauchy surface $ux=0$. We prove that if $0<e^2<\pi$, then the state $|u\rangle=\exp(-iS(u))\mid 0\rangle$, where $\mid 0\rangle$ is the Lorentz invariant vacuum state, contains a normalizable eigenstate of the Casimir operator $C_1=-(1/2)M_{\mu\nu}M^{\mu\nu}$; $M_{\mu\nu}$ are generators of the proper orthochronous Lorentz group. The eigenvalue is $(e^2/\pi)(2-(e^2/\pi))$. This theorem was first proven by the Author in 1992 in his contribution to the Czyż Festschrift, see Erratum *Acta Phys. Pol. B* **23**, 959 (1992). In this paper a completely different proof is given: we derive the partial, differential equation satisfied by the matrix element $\langle u\mid\exp(-\sigma C_1)\mid u\rangle, \sigma>0$, and show that the function $\exp(z)\cdot(1-z)\cdot\exp[-\sigma z(2-z)], z=e^2/\pi$, is an exact solution of this differential equation, recovering thus both the eigenvalue and the probability of occurrence of the bound state. A beautiful integral is calculated as a byproduct.

PACS numbers: 12.20.Ds, 11.10.Jj

1. Introduction

We use mechanical units such that $\hbar=1=c$. We use electric units such that the fine structure constant is equal to $1/e^2$, where e is the electron's charge. We use space-time metric such that $g(x,x)=xx=(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2$ is the square of the length of the vector x.

In Ref. [1] we were led to consider the following theoretical structure. Let $x^\mu, \mu=0,1,2,3$, denote space-time Cartesian coordinates in an inertial reference frame. The equation $xx=-1$ defines a subspace which is locally a three-dimensional space-time and is maximally symmetric, admitting six Killing vectors. Thus it is a three-dimensional analogue of de Sitter space-time and will be called simply three-dimensional de Sitter space-time. A scalar massless quantum field is assumed to live on the de Sitter space-time $xx=-1$. Its classical action is assumed to be

$$\frac{1}{8\pi e^2}\int\limits_{xx=-1} dx g^{ik}\partial_i S\partial_k S\,, \tag{1}$$

where e^2 is the coupling constant, dx is the invariant measure on the de Sitter hyperboloid $xx=-1$ and $g_{ik}, i,k=1,2,3,$ is the internal metric on this hyperboloid. The above action has the following symmetries: the Lorentz symmetry, which, via the first Noether theorem, gives rise to six constants of motion $M_{\mu\nu}=-M_{\nu\mu}$ and the "gauge" symmetry $S(x)\to S(x)+\text{const}$, which, again via the first Noether theorem, gives rise to the additional constant of motion called the total charge,

$$Q=\frac{-1}{4\pi e}\int\limits_{\text{C.S.}} d\Sigma^i\partial_i S\,. \tag{2}$$

Here C.S. means a Cauchy surface in the de Sitter hyperboloid $xx=-1$ and $d\Sigma^i$ is the integration element on this surface.

A quantum field theory is obtained if we assume that

$$[M_{\mu\nu},\,S(x)]=\frac{1}{i}\left(x_\mu\frac{\partial}{\partial x^\nu}-x_\nu\frac{\partial}{\partial x^\mu}\right)S(x) \tag{3}$$

and that there exists a state $|0\rangle$ such that

$$M_{\mu\nu}|0\rangle=0\,,\quad \langle 0|M_{\mu\nu}=0\,,\quad \langle 0|0\rangle=1\,. \tag{4}$$

Eqs (3) and (4) can hold only if

$$[Q,S(x)]=ie \tag{5}$$

and

$$Q|0\rangle=0\,,\quad \langle 0|Q=0\,,\quad \langle 0|0\rangle=1\,. \tag{6}$$

There are many misleading or erroneous statements in the literature on the vacuum state in de Sitter space-time; for this reason the reader is invited to check the consistency of Eqs (3)–(5), and (6) with the help of Ref. [1].

The quantum field $S(x)$ is an operator valued distribution and cannot be a subject of nonlinear operations. It is, however, a very fortunate circumstance that Cauchy surfaces in the de Sitter space-time $xx=-1$ are compact. For this reason averaging over a Cauchy surface has the quality of smearing out with an arbitrarily smooth function of compact support in QED.

Let us choose a fixed unit time-like vector u. The quantum field

$$S(u) = \frac{1}{4\pi} \int\limits_{xx=-1} dx\delta(ux)S(x) \tag{7}$$

is the average of the field $S(x)$ over the compact section of the space-like plane $ux = 0$ and the de Sitter hyperboloid $xx = -1$ and is smooth enough to be exponentiated. $S(u)$ is a quantum field which lives in the Lobachevsky space of four-velocities $uu = +1$. It is easy to see that

$$[M_{\mu\nu},\, S(u)] = \frac{1}{i}\left(u_\mu \frac{\partial}{\partial u^\nu} - u_\nu \frac{\partial}{\partial u^\mu}\right) S(u) \tag{8}$$

and

$$[Q,\, S(u)] = ie\,. \tag{9}$$

Using the smooth quantum field $S(u)$ we can consistently form a charged state

$$|u\rangle = \exp(-iS(u))|0\rangle \tag{10}$$

which is an eigenstate of the total charge Q:

$$Q|u\rangle = e|u\rangle\,, \quad \langle u|u\rangle = 1\,. \tag{11}$$

We shall investigate in this paper some properties of charged states of the form $\exp(-iS(u))|0\rangle$. In particular, we shall give a completely new proof of the theorem that the spectral contents of the state $\exp(-iS(u))|0\rangle$ in the regions $0 < e^2 < \pi$ and $e^2 > \pi$ are different. By spectral content we mean the way in which a given state can be represented as a superposition of eigenstates of the first Casimir operator

$$C_1 = -\tfrac{1}{2} M_{\mu\nu} M^{\mu\nu}\,. \tag{12}$$

2. Calculation of the matrix element $\langle u|\exp(-\sigma C_1)|u\rangle\,,\ \sigma > 0$

To save space we shall write $\nabla_{\mu\nu}(u)$ instead of

$$u_\mu \frac{\partial}{\partial u^\nu} - u_\nu \frac{\partial}{\partial u^\mu}\,.$$

In expressions like $\nabla_{\mu\nu}(u)S(u)$ one can even drop the first argument u because the argument u of $S(u)$ indicates the variable with respect to which the differentiation is carried out. Let us note first that

$$[S(u)\,,\ S(v)] = 0 \tag{13}$$

for each pair of points u, v in the Lobachevsky space $uu = +1$. This is so because the definition (7) of $S(u)$ picks up only the even part of the field $S(x)$ *i.e.* the part $(1/2)[S(x)+S(-x)]$ and even parts do commute with each other on the strength of canonical commutation relations (3). As an obvious consequence we have that

$$[S(u)\,,\ \nabla_{\mu\nu}S(u)] = 0\,. \tag{14}$$

Consider now $[S(u)\,,\,C_1]$. We have

$$\begin{aligned}[S(u)\,,\,C_1] &= \frac{1}{2}[M_{\mu\nu}M^{\mu\nu}\,,\,S(u)]\\ &= \frac{1}{2}\left\{\frac{1}{i}\nabla_{\mu\nu}S(u)\cdot M^{\mu\nu} + M^{\mu\nu}\cdot\frac{1}{i}\nabla_{\mu\nu}S(u)\right\}.\end{aligned} \tag{15}$$

Therefore

$$[S(u)\,,\ [S(u)\,,\,C_1]] = \nabla_{\mu\nu}S(u)\nabla^{\mu\nu}S(u) \tag{16}$$

and

$$[S(u)\,,\,[S(u)\,,\,[S(u)\,,\,C_1]]] = 0\,. \tag{17}$$

Consider now the matrix element $\langle v|\exp(-\sigma C_1)|u\rangle$, where v is a fixed four-velocity different from u and $\sigma > 0$. Using Eqs (14)–(16) and (17) in an obvious way we have

$$-\frac{\partial}{\partial\sigma}\langle v|\,e^{-\sigma C_1}|u\rangle = \langle v|\,e^{-\sigma C_1}\,e^{-iS(u)}\frac{-1}{2}\nabla_{\mu\nu}S(u)\nabla^{\mu\nu}S(u)|0\rangle\,. \tag{18}$$

On the other hand, let us apply the Laplace–Beltrami operator $\varDelta(u) = -(1/2)\nabla^{\mu\nu}(u)\nabla_{\mu\nu}(u)$ to the matrix element $\langle v|\exp(-\sigma C_1)|u\rangle$. Taking into account that $\varDelta(u)S(u) = 0$ as a consequence of the equation of motion $\varDelta(x)S(x) = 0$ we have

$$\varDelta(u)\langle v|\,e^{-\sigma C_1}|u\rangle = \langle v|\,e^{-\sigma C_1}\,e^{-iS(u)}\frac{-1}{2}\nabla_{\mu\nu}S(u)\nabla^{\mu\nu}S(u)|0\rangle\,. \tag{19}$$

Comparing (18) and (19) we see that

$$\left\{\frac{\partial}{\partial\sigma} - \varDelta(u)\right\}\langle v|\,e^{-\sigma C_1}|u\rangle = 0\,. \tag{20}$$

This means that the matrix element $\langle v|\exp(-\sigma C_1)|u\rangle$ is a solution of the heat transport equation in the Lobachevsky space of four-velocities $uu = +1$. The initial value for this solution is obviously the matrix element $\langle v|u\rangle$ which was calculated in Ref. [1] as $\exp(-(e^2/\pi)(\lambda\coth\lambda - 1))$, where λ is the hyperbolic angle between u and v:

$$g_{\mu\nu}u^{\mu}v^{\nu} = \cosh\lambda\,. \tag{21}$$

To solve the Cauchy problem for the heat transport equation (20) we apply the standard procedure: we represent the initial value as a superposition of plane waves, solve the heat transport equation for each plane wave, and represent the final value as a superposition of time evolved plane waves. However, since we are in the Lobachevsky space of four-velocities $uu = +1$, we have to apply plane waves in Lobachevsky space which Gelfand, Graev, and Vilenkin described in their great book [2]. This means that we have to apply Eqs (20) and (21) which Gelfand, Graev, and Vilenkin give on page 477 of their book. These equations give the Fourier transform and its inverse in Lobachevsky space. The result of this obvious procedure is summarized in the following lemma: suppose that $f(u;0)$ is the initial value for the function $f(u;\sigma)$ which solves the heat transport equation in the Lobachevsky space $uu = +1$,

$$\left\{\frac{\partial}{\partial\sigma} - \Delta(u)\right\} f(u;\sigma) = 0\,.$$

Then

$$f(u;\sigma) = \frac{1}{2\pi^2}\int du' f(u';0)\frac{1}{\sinh\lambda}\int_0^{\infty} d\nu\nu\, e^{-\sigma(1+\nu^2)}\sin(\nu\lambda)\,, \tag{22}$$

where du is the invariant measure in the Lobachevsky space $uu = +1$ and λ is the hyperbolic angle between the observation point u and the integration point u'. The second integral in (22) can be calculated. In this way we obtain

$$f(u;\sigma) = \frac{e^{-\sigma}}{(4\pi\sigma)^{3/2}}\int du' f(u';0)\frac{\lambda}{\sinh\lambda}\, e^{-\frac{\lambda^2}{4\sigma}}\,, \tag{23}$$

where λ is the hyperbolic angle between the observation point u and the integration point u'.

Now, we took the matrix element $\langle v|\exp(-\sigma C_1)|u\rangle$ in order to be able to differentiate with respect to u while leaving v untouched. In fact, however, we are interested in the matrix element $\langle u|\exp(-\sigma C_1)|u\rangle$. Geometrically this means that we have to take in Eq. (23) the observation point u at the origin of the spherically symmetric distribution $f(u';0)$. Introducing the spherical coordinates

$$\begin{aligned} u'^{\,0} &= \cosh\psi\,,\\ u'^{\,1} &= \sinh\psi\sin\vartheta\cos\varphi\,,\\ u'^{\,2} &= \sinh\psi\sin\vartheta\sin\varphi\,,\\ u'^{\,3} &= \sinh\psi\cos\vartheta\,, \end{aligned} \tag{24}$$

and taking the solution at the origin of spherical symmetry we have finally ($z = e^2/\pi$)

$$\langle u|\, e^{-\sigma C_1}|u\rangle = \frac{1}{2\sqrt{\pi}}\frac{e^{-\sigma}}{\sigma^{3/2}}\int\limits_0^\infty d\psi \sinh\psi\, e^{-z(\psi\coth\psi-1)}\cdot\psi\, e^{-\frac{\psi^2}{4\sigma}}\,. \tag{25}$$

3. The differential equation satisfied by the matrix element $\langle \boldsymbol{u}|\exp(-\boldsymbol{\sigma} \boldsymbol{C}_1)|\boldsymbol{u}\rangle$

It is remarkable that the matrix element $\langle u|\exp(-\sigma C_1)|u\rangle$ satisfies a certain partial differential equation. This equation, as well as the equation arrived at later on, satisfied by the resolvent $\langle u|(C_1-\lambda)^{-1}|u\rangle$, is obviously a trace of some deeper structure which, for the time being, we fail to understand.

In fact let us put

$$f(\nu,z) = \int\limits_0^\infty d\psi \sinh\psi\, e^{-z\psi\coth\psi}\cdot\psi\, e^{-\nu\psi^2}\,, \quad \nu>0\,. \tag{26}$$

Then

$$\begin{aligned}\frac{\partial f}{\partial z} &= -\int\limits_0^\infty d\psi\, e^{-z\psi\coth\psi}\cosh\psi\cdot\psi^2\, e^{-\nu\psi^2}\\ &= \int\limits_0^\infty d\psi \sinh\psi \frac{d}{d\psi}\left\{e^{-z\psi\coth\psi}\cdot\psi^2\, e^{-\nu\psi^2}\right\}\\ &= z\frac{\partial f}{\partial z} + z\frac{\partial^2 f}{\partial z^2} + z\frac{\partial f}{\partial \nu} + 2f + 2\nu\frac{\partial f}{\partial \nu}\end{aligned} \tag{27}$$

which means that the function $f(\nu,z)$ is a solution of the partial differential equation

$$(z-1)\frac{\partial f}{\partial z} + z\frac{\partial^2 f}{\partial z^2} + z\frac{\partial f}{\partial \nu} + 2\nu\frac{\partial f}{\partial \nu} + 2f = 0\,. \tag{28}$$

Having this equation we can multiply the function $f(\nu,z) = f(1/4\sigma,z)$ by trivial factors $\sigma^{-3/2}\exp(z-\sigma)$ and obtain the following lemma: let $\langle u|\exp(-\sigma C_1)|u\rangle = c(\sigma,z)$; then the function $c(\sigma,z)$ satisfies the partial differential equation

$$z\frac{\partial^2 c}{\partial z^2} - (z+1)\frac{\partial c}{\partial z} - 2\sigma(1+2\sigma z)\frac{\partial c}{\partial \sigma} - 2\sigma(1+3z+2\sigma z)c = 0\,. \tag{29}$$

4. The eigenvalue and the probability of occurrence of the bound state of the Casimir operator C_1 in the state $|u\rangle = \exp(-iS(u))|0\rangle$

The Casimir operator C_1 is known to have no lower bound; its eigenvalues in the so called main series of irreducible unitary representations of the proper orthochronous Lorentz group are [2] $1 + \nu^2 - n^2$, where ν is a real number and n is an integer. However, n is proportional to the eigenvalue of the second Casimir operator

$$C_2 = M_{01}M_{23} + M_{02}M_{31} + M_{03}M_{12} \tag{30}$$

which obviously annihilates all spherically symmetric states. $|u\rangle = \exp(-iS(u))|0\rangle$ is spherically symmetric in the rest frame of u. Therefore in the subspace of spherically symmetric states the Casimir operator C_1 does have a lower bound. This means that the asymptotic behaviour of solutions of Eq. (29) for $\sigma \to \infty$ is determined by the state with the lowest eigenvalue.

Assume that the spectral decomposition of C_1 does contain a bound state with the lowest eigenvalue. Then Eq. (29) has to have for $\sigma \to \infty$ an asymptotic solution of the form

$$c_0(\sigma, z) = A(z)\, e^{-\sigma B(z)}\,. \tag{31}$$

Putting this into Eq. (29) we obtain on the left hand side a polynomial of second degree in σ with coefficients depending on z. Thus for $\sigma \to \infty$ all three coefficients have to vanish. This gives us three ordinary differential equations for two functions $A(z)$ and $B(z)$. Remarkably enough, all three equations can be simultaneously solved with the result

$$A(z) = (1 - z)\, e^{z}\,, \quad B(z) = z(2 - z)\,. \tag{32}$$

In this way we have the following lemma: the function $c_0(\sigma, z) = (1 - z)\, e^{z} \cdot \exp[-\sigma z(2 - z)]$ is an exact solution of the partial differential equation (29). This obviously suggests that in the state $|u\rangle$ there is a bound state of the Casimir operator C_1 with the eigenvalue $z(2 - z)$ and the probability of occurrence $(1 - z)\, e^{z}$. This probability cannot be negative which means that the state can exist only for $0 < z < 1$. It is thus seen that the coupling constant $z = e^2/\pi = 1$ is critical and separates two kinematically different regimes of the theory. For $0 < z < 1\,,\ 0 < z(2 - z) < 1$ which means that the bound state belongs to the so called supplementary series of irreducible unitary representations of the proper orthochronous Lorentz group [2], since for the main series $1 < C_1 < \infty$ [2].

5. Calculation of the resolvent $\langle u|(C_1 - \lambda)^{-1}|u\rangle$

Let us multiply Eq. (25) by $\exp(\lambda\sigma)$, assume that λ is smaller than the smallest eigenvalue of the Casimir operator C_1 present in the spectral decomposition of the matrix element $\langle u|\exp(-\sigma C_1)|u\rangle$ and integrate both sides over σ, $0 < \sigma < \infty$. All integrals on the right hand side are absolutely convergent and their order can be interchanged. In this way we obtain

$$\begin{aligned}
&\left\langle u\left|\frac{1}{C_1-\lambda}\right|u\right\rangle \\
&= \frac{1}{2\sqrt{\pi}}\int\limits_0^\infty d\psi \sinh\psi\, e^{-z(\psi\coth\psi-1)}\cdot\psi\int\limits_0^\infty d\sigma\sigma^{-3/2}\, e^{-\sigma(1-\lambda)-\frac{\psi^2}{4\sigma}} \\
&= \int\limits_0^\infty d\psi \sinh\psi\, e^{-z(\psi\coth\psi-1)-\psi\sqrt{1-\lambda}}\,. \qquad (33)
\end{aligned}$$

The last integral exists for $1 - z - \sqrt{1-\lambda} < 0$ *i.e.* for $\lambda < z(2-z)$ which was assumed from the very beginning.

It is again remarkable that the last integral which is not given in the Ryzhik and Gradshteyn Tables (I checked it in the VI$^{\text{th}}$ American Edition) can be calculated exactly with the help of the partial differential equation which this integral is a solution of.

In fact, consider the integral

$$F(x,y) = \int\limits_0^\infty d\zeta e^{-x\zeta-y\zeta\coth\zeta} \qquad (34)$$

which exists for $x + y > 0$. Differentiating and integrating by parts, as in Section 3, one can show that this integral fulfills the partial differential equation

$$F + x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + y\left(\frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial x^2}\right) = 0\,. \qquad (35)$$

This is a hyperbolic equation for which the straight line $x + y = 0$ is the boundary of the domain of influence of the positive x axis $y = 0$, $x > 0$. Hence we can try to solve the Cauchy problem with the initial data on the positive x axis $y = 0$, $x > 0$. We have that

$$F(x,0) = \int\limits_0^\infty d\zeta\, e^{-x\zeta} = \frac{1}{x}\,. \qquad (36)$$

This singularity must propagate to the left since $F(x,y)$ is regular for $x+y>0$. The function $1/(x+y)$ is an exact solution of Eq. (35). Therefore $F(x,y)=1/(x+y)$ plus a solution of Eq. (35) which vanishes for $y=0$. We have

$$\left.\frac{\partial F}{\partial y}\right|_{y=0} = \int_0^\infty d\zeta\, e^{-x\zeta}(-)\zeta \coth\zeta\,. \tag{37}$$

Subtracting from this function the function

$$\left.\frac{\partial}{\partial y}\frac{1}{(x+y)}\right|_{y=0} = -\frac{1}{x^2} \tag{38}$$

I can say that $F(x,y)=1/(x+y)$ plus a solution of Eq. (35) which vanishes at $y=0$ and whose y derivative at $y=0$ is equal to

$$\frac{1}{x^2} - \int_0^\infty d\zeta\zeta\, e^{-x\zeta}\coth\zeta = -2\sum_{n=0}^\infty \frac{1}{(x+2n+2)^2}\,. \tag{39}$$

From the superposition principle it is seen that the problem is thus reduced to the following one: find a solution of Eq. (35) which vanishes for $y=0$ and whose y derivative at $y=0$ is equal to $-2/(x+2n+2)^2$. One can check that this solution is equal to

$$-2y\frac{(x+2n+2-y)^n}{(x+2n+2+y)^{n+2}}\,. \tag{40}$$

Therefore for $x+y>0$

$$\int_0^\infty d\zeta\, e^{-x\zeta-y\zeta\coth\zeta} = \frac{1}{x+y} - 2y\sum_{n=0}^\infty \frac{(x+2n+2-y)^n}{(x+2n+2+y)^{n+2}} \tag{41}$$

which is a nice result not to be found in the Ryzhik and Gradshteyn Tables.

The result (41) allows us to calculate the resolvent (33) since $\sinh\psi = (1/2)(\exp\psi - \exp(-\psi))$ and the integral (33) is seen to be of the form (34). Making the obvious substitutions we obtain for $0<z<1$:

$$\begin{aligned}\left\langle u\left|\frac{1}{C_1-\lambda}\right|u\right\rangle &= \frac{(1-z)e^z}{z(2-z)-\lambda}\\ &\quad +2z^2e^z\sum_{n=0}^\infty \frac{(\sqrt{1-\lambda}+2n+1-z)^{n-1}}{(\sqrt{1-\lambda}+2n+1+z)^{n+2}}\,.\end{aligned} \tag{42}$$

This formula shows at once the eigenvalue of the bound state, the probability of its occurrence and the cut $1 \leq \lambda < \infty$ which reflects the contribution from the main series of irreducible unitary representations of the proper orthochronous Lorentz group.

We see from the formula (42) that the bound state cannot exist for $z > 1$ since the probability of occurrence cannot be negative. This can also be seen from the calculation leading to the formula (42). For $x + y > 1$ we have instead of (41)

$$\begin{aligned} &\int_0^\infty d\zeta \sinh\zeta\, e^{-x\zeta - y\zeta\coth\zeta} \\ &= \frac{1/2}{x+y-1} - \frac{1/2}{x-y+1} + 2y^2 \sum_{n=0}^{\infty} \frac{(x+2n+1-y)^{n-1}}{(x+2n+1+y)^{n+2}} \,. \end{aligned} \tag{43}$$

This formula can be derived in the same way as the previous one, given in Eq. (41). Making the obvious substitutions we obtain for $z > 1$:

$$\begin{aligned} \left\langle u \left| \frac{1}{C_1 - \lambda} \right| u \right\rangle \;=\; e^z \Bigg\{ & \frac{1}{2(\sqrt{1-\lambda}+z-1)} - \frac{\sqrt{1-\lambda}+3z+1}{2(\sqrt{1-\lambda}+z+1)^2} \\ & + 2z^2 \sum_{n=1}^{\infty} \frac{(\sqrt{1-\lambda}+2n+1-z)^{n-1}}{(\sqrt{1-\lambda}+2n+1+z)^{n+2}} \Bigg\} . \end{aligned} \tag{44}$$

This resolvent has only the cut $1 \leq \lambda < \infty$ which reflects contribution from the main series.

6. A method to calculate the averages $\langle u|(C_1)^n|u\rangle$ for integer n

Differentiating both sides of Eq. (42) with respect to λ and putting $\lambda = 0$ we can calculate all the averages of the form $\langle u|(C_1)^{-n}|u\rangle$, $n = 1, 2, 3, \ldots$. On the other hand there is no simple way to calculate the averages $\langle u|(C_1)^n|u\rangle$, $n = 1, 2, 3, \ldots$. Professor Wosiek and dr Rostworowski calculated from first principles the following averages ($z = e^2/\pi$):

$$\begin{aligned} \langle u|C_1|u\rangle &= 2z \\ \langle u|(C_1)^2|u\rangle &= \frac{20}{3} z^2 \\ \langle u|(C_1)^3|u\rangle &= \frac{8}{9} z^2 (12 + 35z) \\ \langle u|(C_1)^4|u\rangle &= \frac{16}{45} z^2 (192 + 560z + 525z^2) \end{aligned}$$

$$\begin{aligned}
\langle u|(C_1)^5|u\rangle &= \frac{32}{9}z^2(192 + 704z + 840z^2 + 385z^3) \\
\langle u|(C_1)^6|u\rangle &= \frac{64}{945}z^2(147456 + 647808z + 977760z^2 + 646800z^3 \\
&\quad +175175z^4) \\
\ldots\ldots\ldots\ldots\ldots & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (45)
\end{aligned}$$

One can see from these expressions that these averages increase so quickly that the autocorrelation function $\langle u|\exp(i\sigma C_1)|u\rangle$ cannot have a convergent Taylor series in σ. This is not a problem, of course. Autocorrelation functions do not have to be analytic at the origin. Nevertheless, we have observed the following "experimental" fact: the averages (45) can be recovered from the differential equation (29) in the following way.

We write formally

$$\langle u|e^{-\sigma C_1}|u\rangle = c(\sigma, z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\sigma^n c_n(z)\,; \tag{46}$$

we put this into Eq. (29) and obtain the recurrence relation

$$zc_n'' - (z+1)c_n' - 2nc_n = 4n(n-1)zc_{n-2} - n[4z(n-1) + 2(1+3z)]c_{n-1}\,. \tag{47}$$

Knowing that $c_0(z) = 1$, $c_1(z) = 2z$ and assuming that $c_n(z)$ is a polynomial of degree n one recovers the polynomials (45), which have been correctly calculated from first principles. We fail to see the mathematical justification of this procedure and therefore state it simply as an "experimental" fact which does allow us to calculate the averages $\langle u|(C_1)^n|u\rangle$, $n = 1, 2, 3, \ldots$. This calculation is much easier than the calculation which starts from first principles.

I am greatly indebted to Professor Pawel O. Mazur from the Department of Physics and Astronomy, University of South Carolina, for many useful discussions and for having created for me excellent working conditions at Columbia, SC, where the most important parts of this paper were written. I am also indebted to Professor Jacek Wosiek and dr Andrzej Rostworowski from the Department of Physics, Jagellonian University, for many useful discussions.

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