Vol. 36 (2005) *ACTA PHYSICA POLONICA B* No 1

QUANTUM MECHANICS OF THE ELECTRIC CHARGE IN A CUT FOCK SPACE

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(Received November 10, 2004)

Dedicated to Andrzej Staruszkiewicz on the occasion of his 65th birthday

Assumption that the phase of the Coulomb field is a dynamical degree of freedom, conjugate to the charge operator, leads to the consistent, Lorentz invariant field theory of photons at spatial infinity, which depends parametrically on the value of the fine structure constant. We confirm existence of the normalizable bound state in the spectrum of the first Casimir operator of the Lorentz group. The state exists only for $e^2 < \pi$ indicating that $e^2 = \pi$ is the singular point of the theory. We also show that the theory has an essential singularity at the origin of the complex e^2 plane.

PACS numbers: 11.30.Cp, 03.65.Fd, 03.70.+k

1. Introduction

All works of Andrzej Staruszkiewicz are invariably characterized by deep and unconventional understanding, extreme clarity of arguments and elegant mathematical solutions. Quantum mechanics of the electric charge is no exception. It emerges from quantum theory of electromagnetic field upon additional assumption that the phase of the Coulomb field at spatial infinity is a dynamical degree of freedom, which is canonically conjugated to the charge operator [1, 2]. Since the theory deals with fields at infinity, it applies literally to the very tip of the light cone in the momentum space. Hence, the Hamiltonian of this system vanishes and one may wonder how anything nontrivial can remain in such a situation. But it does (providing an explicit confirmation of the above introductory remarks). What remains is the Lorentz invariance. As in any quantum theory all observables are build from the elementary harmonic oscillators, and this puts the nontrivial constraints on a construction. Namely the commutation rules required by the symmetries must be consistent with those following from the canonical

quantization. In Staruszkiewicz theory they do and moreover, the elementary electron charge e enters deeply into the algebra of Lorentz generators. This offers a tantalizing possibility that the actual value of the fine structure constant may be related to some properties of the Lorentz group.

It is not our goal here to discuss further such fundamental issue. From the purely pragmatic point of view Staruszkiewicz theory provides an explicit example of relatively simple, yet nonlinear three-dimensional field theory. It was not solved until now, however recent developments seem to suggest that the theory may be exactly solvable.

In the next two sections we define the system and present some early results obtained with recently developed method of cut Hilbert space. Sec. 4 contains more complete, high precision studies of the spectrum and of the singularity structure in the complex e^2 plane.

2. The system

The system consists of the infinitely many photons with angular momentum l described by the creation and annihilation operators $a_{l,m},\ \ a^\dagger_{l,m},\ l \geq 1$, and a quantum charge, described by the charge operator Q, with the commutation rules [3][1]

$$[a_{l,m}, a^\dagger_{l',m'}] = \delta_{ll'}\delta_{m,m'}\,, \qquad [Q, a_{l,m}] = [Q, a^\dagger_{l,m}] = 0\,. \tag{1}$$

Above photons result from the quantization of the phase of the Coulomb field at spatial infinity, while the quantum charge is given by the Gauss law calculated again at spatial infinity. We refer to the original works of Staruszkiewicz for the details of this quantization procedure, in particular for the introduction and discussion of the role of the phase of the Coulomb field [4–7]. For our purpose it is sufficient to know that, as the result of the canonical commutation relations between the phase and the charge operator, the charge is quantized in units of the electron charge e

$$Q = eq\,, \qquad q = 0, \pm 1, \pm 2, \dots\ . \tag{2}$$

Because Q commutes with creation and annihilation operators of photons, the whole Hilbert space splits into sectors of a given charge characterized by the integer q.

In contradistinction with the standard quantization of the electromagnetic field, above procedure is manifestly Lorentz invariant. In particular the generators of the Lorentz transformations have the form [3]

[1] Notice that our creation and annihilation operators have different normalization than those of Ref. [3].

$$
\begin{aligned}
M_{12} &= \sum_{lm} m a^\dagger_{l,m} a_{l,m}\,, \\
M_{23}+iM_{31} &= \sum_{lm} \sqrt{(l-m+1)(l+m)}\, a^\dagger_{l,m} a_{l,m-1}\,, \\
M_{03} &= i\sum_{lm} \Big[p_{lm} a^\dagger_{l,m} a_{l+1,m} - r_{lm} a^\dagger_{l,m} a_{l-1,m}\Big] - \sqrt{\frac{2}{3\pi}}\, Q(a^\dagger_{1,0} + a_{1,0})\,, \\
M_{01}+iM_{02} &= i\sum_{lm} \Big[s_{lm}\, a^\dagger_{l,m} a_{l+1,m-1} + t_{lm}\, a^\dagger_{l,m} a_{l-1,m-1}\Big] \\
&\quad + \sqrt{\frac{4}{3\pi}}\, Q(a^\dagger_{1,1} - a_{1,-1})\,, \qquad (3)
\end{aligned}
$$

with

$$
\begin{aligned}
p_{lm} &= \sqrt{\frac{((l+1)^2-m^2)l(l+2)}{(2l+1)(2l+3)}}\,, \\
r_{lm} &= \sqrt{\frac{(l^2-m^2)(l-1)(l+1)}{(2l-1)(2l+1)}}\,, \\
s_{lm} &= \sqrt{\frac{(l-m+1)(l-m+2)l(l+2)}{(2l+1)(2l+3)}}\,, \\
t_{lm} &= \sqrt{\frac{(l+m-1)(l+m)(l-1)(l+1)}{(2l-1)(2l+1)}}\,, \qquad (4)
\end{aligned}
$$

and satisfy the standard commutation rules of the Lorentz algebra

$$
[M_{\mu\nu}, M_{\sigma\rho}] = i(g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho} - g_{\mu\sigma}M_{\nu\rho} - g_{\nu\rho}M_{\mu\sigma})\,, \qquad (5)
$$

or

$$
\begin{aligned}
[L_k, L_l] &= i\varepsilon_{klm}L_m\,, & L_k &= \tfrac{1}{2}\varepsilon_{klm}M_{lm}\,, \\
[M_{0k}, M_{0l}] &= -i\varepsilon_{klm}L_m\,, & [L_k, M_{0l}] &= i\varepsilon_{klm}M_{0m}\,. \qquad (6)
\end{aligned}
$$

Since $Q = qe$, the fine structure constant enters into this algebra in a rather unconventional way. In fact Staruszkiewicz has shown that the spectrum of the first Casimir operator

$$
C_1 = -\frac{1}{2}M_{\mu\nu}M^{\mu\nu}\,, \qquad (7)
$$

in addition to the usual continuum from the main series of the unitary representations of the Lorentz group, has also a single discrete state which belongs to the supplementary series. Its eigenvalue is known analytically [2]

$$c_1 = \frac{e^2}{\pi}\left(2 - \frac{e^2}{\pi}\right). \tag{8}$$

The state is normalizable only for $0 \leq z = \frac{e^2}{\pi} < 1$. Above $e^2 = \pi$ it merges into the continuum.

One of the goals of this work is to study other characteristic of this state, and ultimately its wave function which is not known up to date. To this end we shall briefly discuss now the cut Fock space method which turned out to be quite useful in studying quantum systems of various complexity [8].

3. The cut Fock space and early results

Consider any quantum mechanical system with finite number of degrees of freedom. Its basic objects like, position or momentum can be written in terms of the creation and annihilation operators a and $a^\dagger$. Similarly, more complex observables like Hamiltonian, angular momentum *etc.*, can also be expressed by these creation and annihilation operators[2]. It is, therefore, natural (and apparently very useful) to use as a basis of the Hilbert space the eigenbasis of the occupation number operators $N_d = a_d^\dagger a_d$, where the subscript d labels all degrees of freedom present in the system[3]. In principle one needs the infinite number of basis vectors, however it turns out that in many cases, very accurate results, *e.g.* the spectrum of lower and intermediate states, are obtained with finite bases only [10]. Moreover, monitoring dependence of the results on the size of the basis gives us the model independent criterion of the convergence of the whole procedure.

It remains to decide how we do cut the basis. It turns out that the very convenient cutoff consists of limiting the total number of quanta

$$N_Q = \sum_d N_d < N_{\max}, \tag{9}$$

in the system. The reason for this is that the sum in Eq. (9) is usually invariant under the symmetries present in the problem. If so, the cutoff (9) respects these symmetries and one can take full advantage of them in the cut system as well. Moreover, the occupation number basis and the cutoff (9) are particularly suited for the computer applications since they are characterized by the discrete, dimensionless numbers. No dimensional scale appears in this regularization.

[2] The method is most efficient for the polynomial interactions, however it can be extended to arbitrary potentials.

[3] Extension for fermions is simple. In fact the original motivation of this approach came from supersymmetric systems [9].

The present problem is more challenging since the number of degrees of freedom is infinite. We have, therefore, introduced the double cutoff $(N_{\max}, l_{\max})$ which limits the total number of quanta (here photons at spatial infinity) and their angular momentum. Recovering the true "continuum limit" requires then taking both of these parameters to infinity. This may seem like a tall order, however, only practical experience with the particular system can tell whether such a program is feasible.

In practice we use the algebraic program like Mathematica. Any quantum state is written as the superposition of N_s elementary states

$$|\mathrm{st}\rangle = \sum_{I}^{N_s} \alpha_I \left| \vec{n}^{(I)} \right\rangle , \tag{10}$$

where I labels all states in the basis and $\vec{n}^{(I)}$ is the vector of the occupation numbers for the I-th basis state. After fixing $l_{\max}$ the length of $\vec{n}$ is $(l_{\max}+1)^2$. In Mathematica above state is represented by a flexible list

$$|\mathrm{st}\rangle \leftrightarrow \left\{ N_s, \{\alpha_1, ..., \alpha_{N_s}\}, \left\{ \left\{ n_1^1, ..., n_{l_{\max}+1}^1 \right\}, ..., \left\{ n_1^{N_s}, ..., n_{l_{\max}+1}^{N_s} \right\} \right\} \right\} . \tag{11}$$

whose length can vary dynamically during the calculations. The first element of this list gives the number of basis states in the superposition. The second is the list of complex amplitudes with which the elementary (*i.e.* basis) states appear. Finally, the third element is the list of N_s lists specifying occupation numbers of basis states. The simplest elementary state, the empty state is given by

$$|0, 0, ..., 0\rangle \leftrightarrow \{1, \{1\}, \{\{0, ..., 0\}\}\} . \tag{12}$$

One then implements all quantum operations like addition, multiplication by a number, scalar product on these lists. Next, we define list valued operations corresponding to the creation and annihilation operators, and finally implement the action of more complex operators like the Lorentz generators, Eqs. (3), or the Casimir operators. This approach allows to calculate analytically matrix elements of various observables as a functions of e. Final diagonalization is done numerically.

3.1. Identification of the bound state and its eigenvalue

Table I shows the eigenvalues (and their degeneracies) of the two Casimir operators: C_1 given by Eq. (7) and

$$C_2 = -\frac{1}{8}\, \varepsilon^{\mu\nu\sigma\rho} M_{\mu\nu}\, M_{\sigma\rho} , \tag{13}$$

TABLE I

Eigenvalues, and their degeneracies, g, of the two Casimir operators (14), (15) in a cut Hilbert space.

c_1	g	c_2	g
−24.00	13		
19.20	1		
16.47	3		
14.09	5		
13.06	1		
−11.20	11		
9.97	3		
9.60	5		
9.59	1		
8.00	3	−7.06	7
−8.00	9	−5.29	9
7.20	9	−3.47	5
−6.89	7	−3.34	5
6.82	7	−2.63	7
6.41	5	−2.19	3
6.40	3	0.00	93
−6.40	9	2.19	3
5.86	7	2.63	7
4.80	5	3.34	5
2.94	1	3.46	5
−2.82	7	5.29	9
−2.40	5	7.06	7
2.36	3		
1.91	5		
1.83	7		
−1.60	7		
0.80	5		
−0.80	9		
0.00067	5		
0.00037	3		
0.00030	1		

for $N_{\max} = 3$ and $l_{\max} = 2$. Even with such a dramatic cutoffs the basis has 165 vectors, since the cut system already has 9 degrees of freedom. It is interesting, however, that even in such a severely restricted Hilbert space we can recognize signs of the Lorentz symmetry and in particular identify the Staruszkiewicz state from the supplementary series.

First, the cut system is also rotationally symmetric. The Casimir operators, when written explicitly in terms of the generators[4]

$$C_1 = M_{01}^2 + M_{02}^2 + M_{03}^2 - M_{12}^2 - M_{23}^2 - M_{31}^2, \tag{14}$$

$$C_2 = -M_{01}M_{23} - M_{02}M_{31} - M_{03}M_{12}, \tag{15}$$

are clearly rotationally invariant. Since our cutoffs respect SO(3) rotations, we expect to see rotationally invariant spectrum. And indeed this is the case: all eigenvalues in Table I group into familiar $2l+1$ multiplets. Therefore, the spherically symmetric states, *i.e.* the SO(3) singlets, can be easily identified.

It is known that the spectrum of C_1 has the form

$$c_1 = 1 + \nu^2 - n^2, \tag{16}$$

for any real ν and integer n [11]. It is also known that the spectrum of C_2 is proportional to n. On the other hand, it follows from Eq. (15) that C_2 annihilates SO(3) singlets. Therefore, spherically symmetric states must have $n = 0$ and due to Eq. (16) have positive eigenvalues. This is readily confirmed by Table I.

This is not the end. According to (16) all eigenvalues of the singlet states should be greater than 1. In Table I all *but one* singlet eigenvalues of C_1 are bigger than 1. Is something wrong? No, Eq. (16) applies only to the principal series of the unitary representations of the Lorentz group. For the supplementary series one has [11]

$$c_1 = 1 - \sigma^2\,, \qquad 0 < \sigma < 1\,. \tag{17}$$

Hence the last singlet state in the table can be readily identified as the one from the supplementary series. We conclude that our cutoff is very gentle indeed and allows to see many nontrivial signatures of the Lorentz algebra. In particular we can unambiguously locate the state from the supplementary series which in the infinite cutoff limit tends to the bound state found by Staruszkiewicz.

The spectrum in Table I was obtained for very small $z = 0.00015$. For other values of $0 < z$ the pattern is the same: all spherically symmetric states have positive eigenvalue of C_1 and for $0 < z < 0.8$ only one of them

[4] We use $\varepsilon^{0123} = 1$.

has $c_1 < 1$. The last range of z depends on how accurately can we predict the actual value of the eigenvalue with presently available cutoffs.

Fig. 1 compares the z dependence of the above eigenvalue obtained for the two lowest cutoffs with the exact result, Eq. (8). Clearly there is a room to improve — this is not a surprise. On the other hand, even for $z \sim 0.4$ this, very rough, approximation gives already reasonable results. Second, the proper trend with increasing $l_{\max}$ is observed — things seem to converge monotonously from the very beginning. Finally, because of the poor approximation for larger z, obtained eigenvalue is not smaller than 1 which seems to hamper identification with the state from the supplementary series. However, continuity in z implies that this is indeed the "supplementary" state which is just badly approximated with present cutoffs.

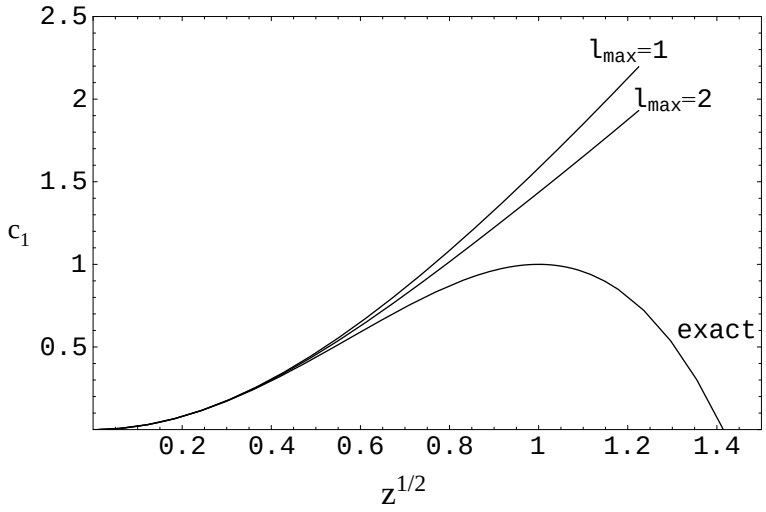

Fig. 1. Dependence of the lowest singlet eigenvalue of C_1 on z for two cutoffs $l_{\max}$. Exact result, Eq. (8) is also shown.

3.2. Other observables

Obviously the cut Fock space method is not restricted solely to the eigenvalues. One easily obtains eigenvectors — a task which cannot be realized within the original analytic approach. In fact any other observable, or quantum average, can be readily calculated at finite cutoffs. As one example we quote the multiplicity distribution of photons in an eigenstate of C_1. Given the previous eigenstate $|\psi\rangle$ in terms of our Fock basis (10), we have the complete information of its wave function. In particular, one can easily obtain the multiplicity distribution of photons dressing the bare Coulomb field

$$P_k(e^2) = \sum_I |\,\langle\{n_{lm}\} = k|\psi\rangle\,|^2, \tag{18}$$

and study its dependence on e^2. Similarly, one can explore the average values of the "magnetic field" which couples to the charge operator in Eqs. (3), *etc.*

3.3. First elements of the Jacobi matrix of C_1

Interestingly, there exists a class of averages which can be calculated exactly even with present, relatively small, cutoffs. These are the lower matrix elements of the first Casimir operator in the Jacobi basis — a basis generated by iterative action of the C_1 itself onto the pure Coulomb state $|0\rangle$. This basis will be discussed in details shortly in connection with more advanced methods. Here we only point out that the Hilbert space cut to $N_{\max} = 6$ and $l_{\max} = 3$ is sufficient to calculate first nine matrix elements of C_1 exactly. We obtain

$$C = \begin{pmatrix} 2z & \sqrt{\frac{8}{3}}z & 0 \\ \sqrt{\frac{8}{3}}z & \frac{14}{3}z+4 & \frac{4}{3\sqrt{5}}\sqrt{(5z+3)(5z+9)} \\ 0 & \frac{4}{3\sqrt{5}}\sqrt{(5z+3)(5z+9)} & \frac{2}{3}\frac{275z^3+1200z^2+1485z+486}{(5z+3)(5z+9)} \end{pmatrix}. \tag{19}$$

This result confirms analytic calculations of leading in z terms performed some time ago with entirely different method [12], and serves as the benchmark for the high order calculations described in the next section.

4. High precision study

In spite of its obvious limitations the previous approach, has provided some results which were not available with the direct analytic methods. Most important, however, is the new formulation of the problem, namely as the "infinite volume" limit of the simple quantum mechanics in the finite, cut Hilbert space. To move further one has to attack the question of the rapidly growing number of degrees of freedom.

Fortunately there exists another cutoff leading to much more quantitative results. Consider acting with the Casimir operator C_1 on the bare state of the Coulomb field $|q, 0\rangle$ in any sector of the normalized charge q. For simplicity we will work with $q = 1$, omit the q eigenvalue and rename $|q, 0\rangle \to |0\rangle$. The state $C_1|0\rangle$ contains finite number of photons with their angular momenta limited as follows from Eqs. (3). The next state, $(C_1)^2|0\rangle$, will contain

correspondingly more quanta within larger range of l's and so on. Let us then define the cut Hilbert space as the one spanned by N states

$$(C_1)^n|0\rangle \;\text{ with }\; n=0,\dots N-1\,. \tag{20}$$

The cutoff N has three advantages: (a) it is given in terms of a single parameter, (b) it is Lorentz invariant, and (c) it can be pushed quite high (into the range of $N\sim 100$) due to the recent analytical developments. All results described below are obtained with this method referred to as the Jacobi matrix approach. It splits into two steps: *(i)* constructing an orthonormal basis (Jacobi basis) from states (20) together with the matrix representation C_{mn} of C_1 in this basis, and *(ii)* calculating the vacuum averages of powers of C_1

$$\langle 0|(C_1)^n|0\rangle\,, \tag{21}$$

referred to as the *moments* of C_1. The two steps are independent and will be discussed separately.

4.1. Recursive scheme for the Jacobi matrix of any observable

Except for the eigenbasis, the matrix representation of any observable H is in general not sparse. However, there exists a basis in which H is tridiagonal. To construct such a basis take any state $|0\rangle$, act on it with H and then orhonormalize resulting set of vectors. This can be cast into the following recursive procedure

$$\begin{aligned} &|a_0\rangle = |0\rangle, \qquad |b_0\rangle = |a_0\rangle, \qquad |e_0\rangle = |b_0\rangle, \\ &|a_n\rangle = H|a_{n-1}\rangle\,, \quad |b_n\rangle = \prod_{k=0}^{n-1} P_k|a_n\rangle, \quad |e_n\rangle = \frac{1}{\sqrt{Z_n}}|b_n\rangle, \quad n>0, \end{aligned} \tag{22}$$

where

$$P_k = 1-|e_k\rangle\langle e_k|, \qquad \text{and} \qquad Z_k=\langle b_k|b_k\rangle\,,\;\; k=0,1,\dots,n\,. \tag{23}$$

It is easy to see that the matrix representation $\langle e_m|H|e_n\rangle \equiv H_{mn}$ is tridiagonal (hence the term: Jacobi matrix of H). Indeed, it follows from the construction that $|e_m\rangle$ has non-zero components only in the space V_m spanned by the first m powers of H: $\{|0\rangle, H|0\rangle,\dots,H^m|0\rangle\}$. In fact $|e_m\rangle$ is the orthogonal complement of V_{m-1} to V_m. Consequently $H|e_m\rangle$ is entirely contained in V_{m+1} and, therefore, it is orthogonal to $|e_n\rangle$ for $m+1<n$. It follows from the hermiticity that H_{mn} is tridiagonal, QED.

Since the starting point of the recursion (22) consists of a pair $(H,|0\rangle)$, all numbers which result from this procedure, *e.g.* the matrix elements of H, its

eigenvalues or eigenvectors (in $|e_n\rangle$ representation), are universal functions of *moments* of H defined as

$$\langle 0|H^n|0\rangle \equiv H_n\,. \tag{24}$$

Universal, meaning that they are the same functions of H_n independently of the nature of H and $|0\rangle$.

To derive these relations define the sequence of A_n operators such that

$$|b_n\rangle = A_n|0\rangle\,. \tag{25}$$

Since $|e_n\rangle$'s are orthonormal, the product of projectors in Eq. (22) can be rewritten as the sum, and consequently A_n satisfy

$$A_n = H^n - \sum_{k=0}^{n-1} \frac{A_k}{Z_k}\,\langle 0|A_k H^n|0\rangle\,. \tag{26}$$

Using (22), (23), (26) one obtains the following expression for the normalization constants

$$Z_n \;=\; H_{2n} - \sum_{k=0}^{n-1} \frac{\langle 0|A_k H^n|0\rangle\;\langle 0|A_k H^n|0\rangle}{Z_k}, \quad n>0\,, \tag{27}$$

in terms of the "vacuum" averages $\langle 0|A_k H^n|0\rangle$. In fact they turn out to be a part of a more general structure. To see this, define the off-diagonal Z factors

$$Z_{mn} \equiv \langle 0|A_m H^n|0\rangle\,. \tag{28}$$

It follows from (26) that they satisfy recursive relations in the first index

$$\begin{aligned} Z_{mn} \;&=\; H_{m+n} - \sum_{k=0}^{m-1} \frac{Z_{km}Z_{kn}}{Z_{kk}}\,, \quad m>0, \\ Z_{0n} \;&=\; H_n\,. \end{aligned} \tag{29}$$

For $m=n$ it reduces to (27), hence the diagonal elements

$$Z_{nn} = Z_n, \tag{30}$$

coincide with the normalization factors defined in Eq. (22).

Using (26) and (29) one derives recursive relations for the matrix elements of H in the Jacobi basis[5]

$$\begin{aligned} \langle e_{n-1}|H|e_n\rangle \;&=\; \langle e_n|H|e_{n-1}\rangle = \frac{1}{\sqrt{Z_{n-1}Z_n}}(Z_{nn} - H_{n-1}Z_{n1}), \\ \langle e_n|H|e_n\rangle \;&=\; \frac{1}{Z_n}\left(Z_{n\,n+1} - H_n Z_{n1} - \frac{Z_{n-1\,n}}{Z_{n-1}}(Z_n - H_{n-1}Z_{n1})\right). \end{aligned} \tag{31}$$

[5] We need to consider only symmetric matrices here. Generalization to hermitian observables is evident.

Together with the initial conditions

$$\langle e_0|H|e_0\rangle = H_1\,, \quad \text{and} \quad \langle e_0|H|e_1\rangle = \langle e_1|H|e_0\rangle = \sqrt{Z_1}\,, \tag{32}$$

these relations determine all elements of the Jacobi matrix of H in terms of the moments H_n.

4.2. Moments of the first Casimir operator

In order to calculate Jacobi matrix of size N one needs all moments of H up to $2N+1$ order. In the case of C_1 this is now possible for very high N, due to the recent result of Staruszkiewicz who has derived a series of interesting representations for various Green functions of his theory [2,12]. One of them reads

$$\langle 0|e^{i\tau C_1}|0\rangle = \frac{1}{\pi}e^{i\tau}\int\limits_0^\infty d\nu\nu e^{i\tau\nu^2}\int\limits_{-\infty}^{\infty} d\lambda \sinh(\lambda)\sin(\nu\lambda)e^{-z(\lambda\coth(\lambda)-1)}, \tag{33}$$

and can be turned into an explicit expression for the generating function of moments

$$\begin{aligned}\langle 0|e^{tC_1}|0\rangle &= \left(\frac{d}{d\lambda}\exp\left[t(1-\frac{d^2}{d\lambda^2})\right]F(\lambda,z)\right)_{\lambda=0}\,,\\ F(\lambda,z) &= \sinh(\lambda)\exp\left[-z(\lambda\coth(\lambda)-1)\right].\end{aligned} \tag{34}$$

With this formula one can calculate the moments and consequently the Jacobi matrix and its spectrum up to $N\sim 100$. Even more effective algorithms became recently available [13].

4.3. The eigenvalue of the bound state

Above recursive scheme for the Jacobi matrix, $C_{mn}=\langle e_m|C_1|e_n\rangle$, of C_1, together with the result (34), lead to calculations of the whole spectrum of C_1 with much higher precision. To begin with, we have constructed C_{mn} up to the size $N=18$. This is roughly equivalent to the same value of the invariant cutoff (20) and requires 37 moments (21).

Fig. 2 shows the lowest eigenvalue of the cut Jacobi matrix C_{mn}, as the function of $z=e^2/\pi$ for all $1<N<19$, together with the exact result (8). Since the basis created with the procedure (22) spans only the singlet sector, the eigenvalues of C_{mn} are necessarily those of the spherically symmetric states. Therefore, the lowest one should correspond to the bound state discussed in Sec. 2. Indeed we observe nice convergence to the exact solution. Compared to Fig. 1 the evidence is now much more compelling.

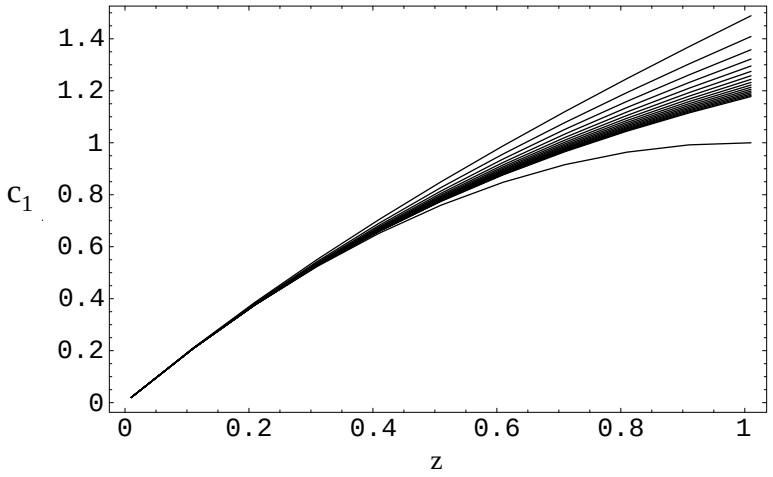

Fig. 2. The lowest eigenvalue of C_1, for first eighteen values of the cutoff, compared with the exact result, Eq. (8) (the lowest curve).

However, at first glance one might worry that there is no convergence at all for higher z. We have checked this more carefully by fitting the N dependence of the lowest eigenvalue for couple of z's. Assuming that $c(N, z)$ is regular at $N = \infty$, one can expand it in the power series in $1/N$

$$c(N, z) = \sum_{k=0}^{K} \frac{\gamma_k(z)}{N^k} \,. \tag{35}$$

Indeed the fits nicely confirm this ansatz. They are stable with increasing the number of terms in the asymptotic expansion. The asymptotic value of $c(\infty, z) = \gamma_0(z)$ is also stable, and converges to the exact solution with increasing K. However, this convergence is fast at small z and is progressively slower when we approach $z = 1$. For example, at $z = 0.41$ we needed only two terms in Eq. (35) to get γ_0 within 1% of the exact result, while at $z = 0.91$ more than 12 terms are required, *cf.* Table II.

TABLE II

Asymptotic value of the lowest eigenvalue of the first Casimir operator as obtained from fits with increasing number of the $1/N^k$ terms. Exact result at this $z(z = 0.91)$ is 0.9919.

K	1	4	8	12
	1.119	1.052	1.035	1.027

Equivalently, the fits show clear increase of the higher coefficients $\gamma_k(z)$ with z which just tells that the convergence with N becomes *slower* with z increasing towards $z = 1$. This is perfectly consistent with $z = 1$ being the critical point of the theory.

To conclude this section, we observed convergence of the discrete eigenvalue with the cutoff in the whole range $0 < z < 1$. The asymptotic form of $c(N, z)$ seems to be regular at $N = \infty$ showing the critical slowing down in the neighborhood of $z = 1$.

4.4. Singularity structure in a complex e^2 plane

The eigenvalue (8) is regular at $z = 0+$. This, however, does not preclude the nontrivial singularity structure of the whole theory at small z. We have looked at this question in some details by examining singularities generated by the recursion (29). It is easy to see that the normalization constants Z_n are rational functions of z

$$Z_n = z^2 \frac{W_n(z)}{W_{n-1}(z)} \,. \tag{36}$$

With the z^2 factored out, polynomials $W_n(z)$ do not vanish at $z = 0$. Singularity structure of the theory is encoded in these polynomials. For example, singularities of the Jacobi matrix (31) coincide with those of the normalization constants.

We have, therefore, examined the zeroes of the polynomials $W_n(z)$ in the complex z plane looking for some structures around the origin. Figs. 3–5 show maps of these zeroes for $n = 6, 8, 10$, respectively. $W_n(z)$ is of the order of $n(n-1)$ hence there are enough zeros to see some regularities already at these cutoffs. Indeed we see the clustering of zeroes around the negative real axis, with the tip of the cluster extending, with increasing n, towards the origin. Since the full theory is well defined on the positive real axis, this result may only mean two things: (a) either the tip reaches the origin asymptotically at infinite cutoff, or (b) it converges to some small negative value z^*.

To decide which case it is, we have done dedicated study of the first zero of $W_n(z)$ for n's as high as 90. In doing so the full power of Mathematica turned out to be very useful. We had to deal with rapidly oscillating polynomials of the order up to 8000 which were varying over hundreds orders of magnitude. Cancellations were tremendous, nevertheless with the Mathematica ability to do calculations in arbitrary precision, we were able to extract stable, hence meaningful, results.

Fig. 6 shows the distance of the first zero from the origin for $n \leq 91$. Nothing indicates any flattening, *i.e.* the cluster of zeros seen in Figs. 3–5

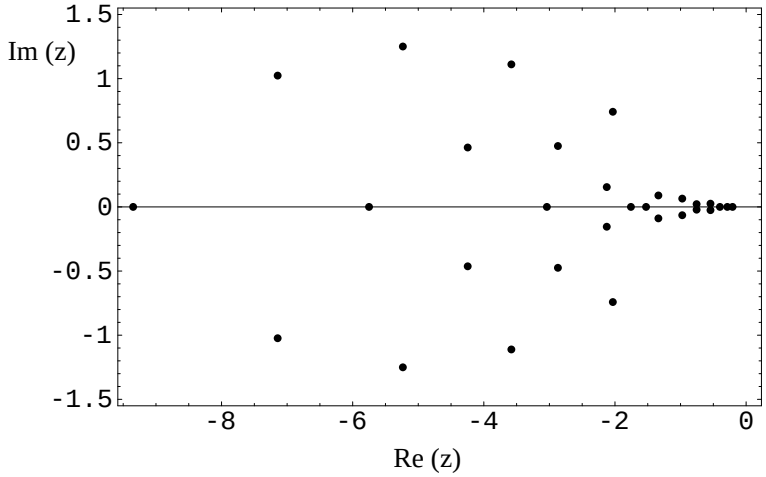

Fig. 3. A map of the zeroes of $W_6(z)$ on the complex z plane.

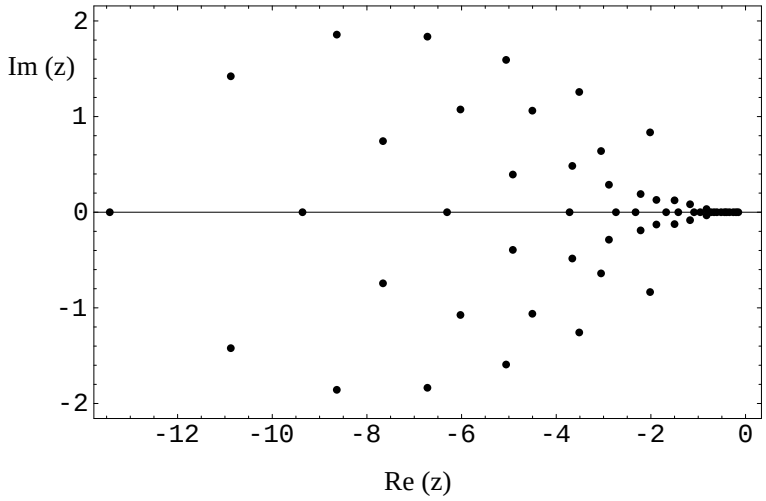

Fig. 4. As Fig. 3 but for $W_8(z)$.

seems to extend all the way to the origin at infinite $n_{\max}$. This means that the theory has the essential singularity at $e^2 = 0$. To check that more quantitatively we have extrapolated the data shown in Fig. 6 to infinity by fitting them to the asymptotic expansion in powers of $1/n$.

$$-z_0(n) = \zeta_0 + \frac{\zeta_1}{n} + \frac{\zeta_2}{n^2} + \frac{\zeta_3}{n^3} + \frac{\zeta_4}{n^4}. \tag{37}$$

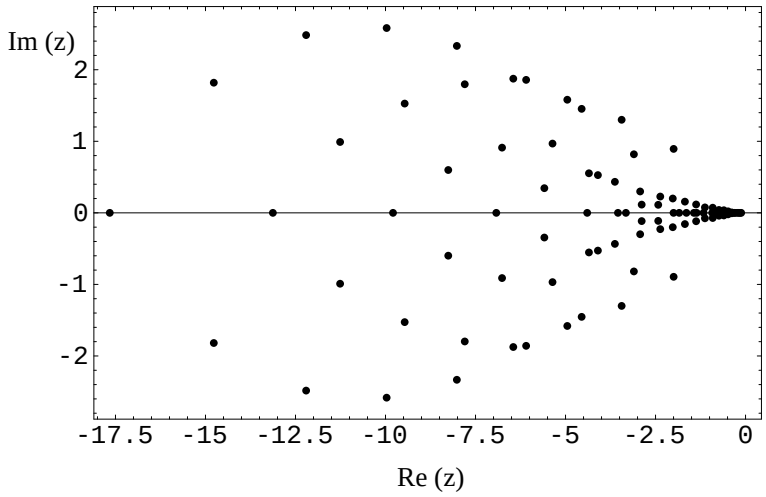

Fig. 5. Zeroes of $W_{10}(z)$.

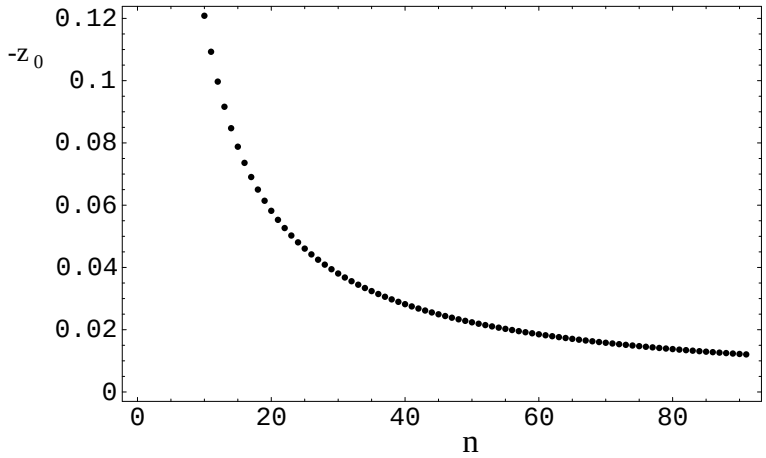

Fig. 6. The distance of the first zero of $W_n(z)$ from the origin.

Results are shown in Table III. Fits are stable with respect to removing first 20–40 data points. This supports the choice of the asymptotic form (37). In addition, when we have fitted the expansion with integer and half-integer powers, coefficients of half-integer powers were in general smaller confirming again an ansatz (37). The extrapolated value at infinity, ζ_0, is consistent with zero. Mathematica routine for nonlinear fits was used to estimate errors, $\Delta(\zeta_k)$, of fitted parameters.

TABLE III

Fits of the n dependence of the distance of the first zero from the origin of the complex z plane. The last row gives a fit to the faked data with a small term ($= -0.0023$) added to z_0.

$n_{\min}$	ζ_0	$\Delta(\zeta_0)$	ζ_1	$\Delta(\zeta_1)$	ζ_2	$\Delta(\zeta_2)$
20	−0.000016	0.000017	1.067	0.004	3.38	0.26
30	−0.000024	0.000053	1.069	0.013	3.31	1.10
40	−0.000087	0.000056	1.085	0.015	1.85	1.47
20	0.00228	0.000018	1.068	0.004	3.38	0.26

The range of the fitted data spans one order of magnitude, that is the last data point reaches the value 0.012. In the absence of any scale in the problem it is hard to decide whether this is far or close to zero, consequently we do not know wether attempted extrapolation "has a short or a long way to go". Therefore, we have also fitted the faked data, with the ten times smaller number 0.0023 added to the original results, in order to see if the fit is able to resolve such a small term. The last row of Table III shows that indeed such a contributions can be identified. Apparently amount of information which is contained in 91 data points is sufficient to resolve such a small constant if it were there. This confirms our conclusion that the zeros of W_n polynomials extend all the way to the origin of the complex z plane.

4.5. The spectrum of the first Casimir operator

As mentioned in Sec. 3 the cut Fock space approach can provide not only the eigenvalues of Casimir operators, but virtually any information about the quantum system. However, it is hard to construct a basis for larger cutoffs. The Jacobi matrix method allows to reach high cutoffs, but amount of information is limited only to the observable used to create the basis. To make contact with the former approach, one would have to construct the unitary transformation which relates the two bases.

On the other hand the Jacobi approach opens another interesting possibility. Namely, given an exact eigenvalue, it allows to construct the exact expansion of the eigenstate in terms of the Jacobi basis[6]. This follows simply from the tri-diagonal nature of C_1 in the Jacobi representation (31). If $\langle e_m|C_1|e_n\rangle \equiv C_{mn}$ is tri-diagonal, the eigenequation

$$C_{mn}\psi_n = c\psi_n \,, \tag{38}$$

[6] I thank A. Staruszkiewicz for bringing this property to my attention.

turns over into the recursion relation for ψ_n providing we know an eigenvalue c. This is precisely the case of the first Casimir operator C_1. We know exactly its all eigenvalues and, therefore, can construct exactly all eigenstates in the Jacobi basis. It is perhaps worth to emphasize the difference with the cutoff method used in Sec. 4.3. Diagonalizing the *cut* Jacobi matrix, as done previously, results in the *approximate* eigenvalues and eigenstates which gradually converge to the exact results. Here, given an exact eigenvalue we can generate the exact eigenvector. In practice if we know only the first N^2 elements of C_{mn} we are able to construct the first N components of the exact eigenvector.

We have turned the problem around and used Eq. (38) to examine the spectrum of the C_1 for a sample of z values. The idea is basically that of the shooting method well known from the elementary quantum mechanics. Literally equation (38) is satisfied for *any* constant c. What determines the actual spectrum is the normalizability condition of the eigenstate. We have, therefore, scanned the whole range od c values, for each c we have constructed recursively corresponding solution ψ_n and checked its dependence on n.

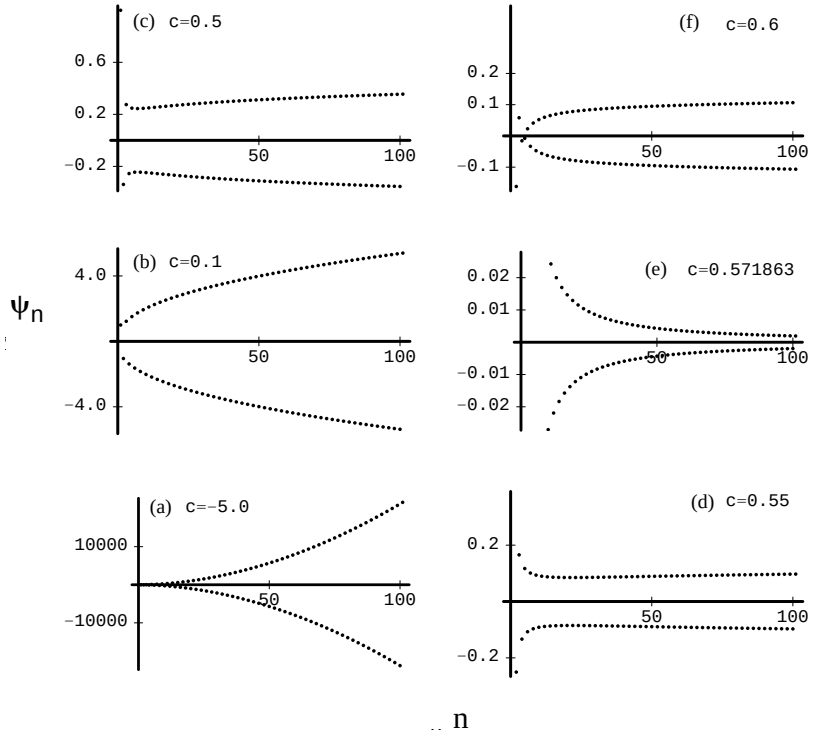

Fig. 7. Solution ψ_n of Eq. (38), as a function of n, for various values of the parameter c and $z = 0.34567$.

Results are shown in Figs. 7–8 (bottom to top, left to right). Beginning with large negative c (Fig. 7(a)) we see that coefficients grow rapidly with n precluding any normalizability. For larger c (Fig. 7(b)) the growth with n becomes weaker (note change of the scale on the vertical axes), however, the states still are not normalizable. There are no negative eigenvalues of C_1 in the scalar sector. In fact this situation holds up to $c \sim 0.5$. Around this value the n dependence begins to change (Fig. 7(c)–7(d)) leading to the normalized state at $c = 0.572\ldots$ (Fig. 7(e)) — as predicted by Eq. (8). Moving on towards higher c we again see non-normalizable solutions (Fig. 7(f))–8(b)) until we reach the threshold of the continuum spectrum at $c = 1$ (Fig. 8(c)). Beyond that point the oscillating behavior sets in. This is typical to the non-localized states. The frequency of the oscillations is not constant in agreement with the power behavior characteristic for the representations of the Lorentz group. Similarly to the usual scattering states the frequency of oscillations increases with the "energy" c. The behavior at the threshold (Fig. 8(c)) is also very similar to that of the zero momentum plane wave expanded in the harmonic oscillator basis.

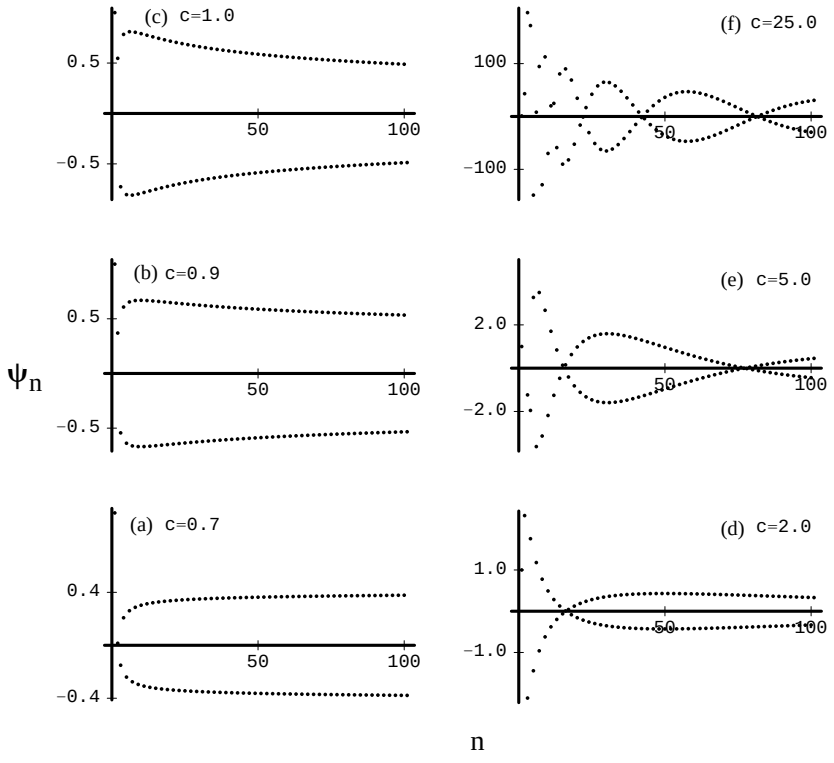

Fig. 8. Same as Fig. 7 but for yet higher c's.

This analysis was done for $z = 0.34567$. For $z > 1$, only continuous spectrum with $c > 1$ exists, as expected.

To conclude, the Jacobi approach allows to construct exactly all eigenstates of C_1 in the Jacobi basis. The spectrum found in this way agrees with the one obtained by Staruszkiewicz. In particular, for $0 < z < 1$, we have confirmed the existence of a single normalized state with the eigenvalue $c = z(2-z)$.

5. Summary

Quantum mechanics of the electric charge, proposed by Staruszkiewicz some time ago, is in fact a very interesting and nontrivial *field theory* of the quantized Coulomb field and photons at spatial infinity. The number of degrees of freedom is infinite, but this infinity is countable. In this respect the theory is simpler than the others, nevertheless it is far from being trivial. A number of analytic results have accumulated over last years, indicating that it may be exactly solvable. However, no complete solution exists up to date. The most intriguing result is the existence of a single bound state — the state of the Coulomb field dressed with photons at infinity. This state belongs to the supplementary series of Lorentz group and exists only for $e^2 < \pi$. In fact the whole theory reveals a fascinating connection between the value of the elementary charge and representations of the Lorentz group.

We have studied some old and some new questions with the new technique of the cut Hilbert space. We have confirmed Staruszkiewicz results on the spectrum and extended his study of Jacobi matrix of the first Casimir operator. First 90 components of the exact bound state were constructed in the Jacobi basis.

The singularity structure around the origin of the complex e^2 plane was studied in details. It was found that the origin is the accumulation point of poles clustering on the negative real axis. This demonstrates that the theory has an essential singularity at $e^2 = 0$ in agreement with the Dyson argument for quantum electrodynamics and other field theories.

I would like to thank A. Staruszkiewicz for numerous, enlightening discussions. This work is supported by the Polish State Committee for Scientific Research under grant no. 1 P03B 024 27 (2004–2007).

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