Vol. 36 (2005) *ACTA PHYSICA POLONICA B* No 1

ON ANALYTICITY OF STATIC VACUUM METRICS AT NON-DEGENERATE HORIZONS

Piotr T. Chruściel†

Département de Mathématiques, Faculté des Sciences
Parc de Grandmont, F37200 Tours, France
e-mail: piotr@gargan.math.univ-tours.fr
URL: www.phys.univ-tours.fr/~piotr

(Received September 9, 2004)

Cracow's school of theoretical physics, led by Professor Andrzej Staruszkiewicz, has made deep contributions to our understanding of black holes. It is a pleasure to dedicate to Professor Staruszkiewicz this contribution to the subject, on the occasion of his 65th birthday

We show that static metrics solving vacuum Einstein equations (possibly with a cosmological constant) are one-sided analytic at non-degenerate Killing horizons. We further prove analyticity in a two-sided neighborhood of "bifurcate horizons".

PACS numbers: 04.20.Cv

1. Introduction

It is a classical result of Müller zum Hagen [9] that stationary vacuum metrics are analytic, in appropriate charts, in the region where the Killing vector is timelike. However, analyticity does sometimes stop at Killing horizons, as can be seen by the Scott–Szekeres extensions of the Curzon metric [12, 13]; compare [3] for examples with a cosmological constant. The aim of this note is to point out that one-sided analyticity always holds at *non-degenerate* static Killing horizons. We also prove analyticity in a (two-sided) neighborhood of "bifurcate horizons". In addition to their intrinsic interest, our results have applications to the classification of static solutions[1] of the

† Partially supported by the Polish State Committee for Scientific Research grant 2 P03B 073 24.

[1] In his proof of Israel's theorem, Robinson [11] appeals to analyticity up-to-boundary of the metric, which has not been justified until this work. While Robinson's proof has been superseded by more complete results [4,5], it remains the simplest one in the connected non-degenerate case, and it seems of interest to have a complete argument along his lines.

Einstein equations [2, 11], or to discussions of cosmic censorship [8] (compare [7]).

We assume an arbitrary space-time dimension $n+1$, and vacuum Einstein equations, perhaps with a cosmological constant. It should be clear that the argument generalises to certain couplings of matter fields to the geometry via Einstein equations.

We expect the result to remain valid for stationary, not necessarily static, Killing horizons, we plan to return to this question in a near future.

2. The method

The proof of the above turns out to be rather simple, and relies on the "Wick rotation" method. It is well known that a metric g with timelike Killing vector field X may locally be written in the form

$$g = -u^2(dt + \theta_i dx^i)^2 + h_{ij}dx^i dx^j \,, \tag{2.1}$$

where $\theta_i dx^i$ is a connection 1-form on the space of x^i's, u is the length of the Killing field $X = \partial/\partial t$, and h is a Riemannian metric. All the fields above are t-independent. (Since all considerations here are strictly local the range of the function t, and the associated question of completeness of the orbits of X, are completely irrelevant for our purposes.) The simplest case to consider is that of static metrics, where θ can be set to zero, so that (2.1) becomes

$$g = -u^2 dt^2 + h_{ij}dx^i dx^j \,. \tag{2.2}$$

Suppose that g solves the vacuum Einstein equations (possibly with a cosmological constant), it is well known that the Riemannian counterpart of g,

$$u^2 d\tau^2 + h_{ij}dx^i dx^j \,, \tag{2.3}$$

also satisfies those equations. A simple way of seeing that is as follows: for $\alpha \in \mathbb{C}^*$ consider the family of complex valued tensor fields

$$g(\alpha) = -\alpha^2 u^2 dt^2 + h_{ij}dx^i dx^j \,.$$

Let $\mathrm{Ric}(\alpha)$ be the complex valued tensor field obtained by calculating the Ricci tensor of $g(\alpha)$ using the usual formulae. Since the Ricci tensor is a rational function of the $g_{\mu\nu}$'s and their derivatives, all the coordinate components $R(\alpha)_{\mu\nu}$ of $\mathrm{Ric}(\alpha)$ are meromorphic functions of α. For $\alpha \in \mathbb{R}^*$ we have $R(\alpha)_{\mu\nu} = 0$, since for those values the metric $g(\alpha)$ can be obtained by a coordinate transformation $t \to \tau = \alpha t$ from the metric $g = g(1)$. Uniqueness of analytic extensions implies that $R(\alpha)_{\mu\nu} = 0$ for all $\alpha \in \mathbb{C}^*$, setting $\alpha = i$ one obtains the desired result for the Riemannian metric $g(i)$.

An identical argument applies of course to the family of complex tensor fields

$$g(\alpha) = -u^2(\alpha dt + \theta_j dx^j)^2 + h_{jk}dx^j dx^k \,, \tag{2.4}$$

so that if g were an Einstein Lorentzian metric,

$$\mathrm{Ric}(g) = \lambda g \tag{2.5}$$

for some constant λ, then the complex tensor field $g(\alpha)$ again satisfies the set of equations

$$\mathrm{Ric}\,(g(\alpha)) = \lambda g(\alpha) \tag{2.6}$$

for all values of $\alpha \in \mathbb{C}^*$. In particular if $\alpha = i$ we obtain that the complex tensor field

$$g(i) = u^2(d\tau + i\theta_j dx^j)^2 + h_{jk}dx^j dx^k \,, \tag{2.7}$$

solves the set of complex equations (2.5). In this work we will, however, concentrate on the static case, so that this fact is irrelevant for the remainder of this paper.

3. One-sided analyticity near a Killing horizon

From now on we restrict ourselves to the static case, locally $\theta = df$. Recall that a Killing horizon is a null hypersurface $\mathcal{K}$ such that X is tangent to the generators of $\mathcal{K}$. As is well known (see, *e.g.*, [5, Proposition 3.2]), a non-degenerate $\mathcal{K}$ corresponds to a smooth totally geodesic boundary, say $\partial\Sigma$, for the metric h. Further, if $\rho = \rho(p)$ denotes the distance from p to $\partial\Sigma$ in the metric h then, in Gauss coordinates around $\partial\Sigma$, all the functions appearing in the metric are smooth[2] functions of ρ^2 and of the remaining coordinates. Moreover, u vanishes on $\partial\Sigma = \{\rho = 0\}$, with non-zero gradient there. This implies (the well known fact) that the set $\{\rho = 0\}$ for the Riemannian metric (2.3) corresponds to a smooth axis[3] of rotation of a Killing vector Y. Now Y is the obvious counterpart of X under the transition from (2.2) to (2.3), and this transition preserves hypersurface-orthogonality, hence Y satisfies

$$Y^\flat \wedge dY^\flat = 0\,,$$

where $Y^\flat := g(i)(Y,\cdot)$. Since $g(i)$ is a Riemannian Einstein metric, its coordinate components $g(i)_{ij}$, with respect to harmonic coordinates, satisfy an

[2] Throughout we assume smoothness of the manifold and of the metric. However, there exists $k < \infty$ such that if the metric is C^k, then the methods here apply, leading to analyticity. The exact value of k can be found by chasing losses of differentiability that arise in the constructions here, as well as in those of [5].

[3] By this we mean a submanifold of codimension two invariant under the flow of Y, with Y generating rotations in the normal bundle.

elliptic quasilinear system of PDEs and are, therefore, real analytic. Further, the harmonic coordinates are smooth in the original smooth atlas. The geodesic coordinates around the rotation axis $\partial\Sigma$ are also analytic because (1) the axis of rotation is a totally geodesic submanifold (of co-dimension two) in the Riemannian manifold $(M, g(i))$, hence analytic; (2) normal coordinates around an analytic submanifold are analytic in an atlas in which the metric is analytic. (This follows from the analytic impicit function theorem [14].) It should be clear that this provides the desired *one-sided* analytic atlas in the Lorentzian solution near the horizon, by running backwards the calculations of, *e.g.*, [5, Proposition 3.2]. Since there is a major subtlety here, as

one obtains analytic coordinates only in the region $g(X,X) \leq 0$,

we provide the details: Consider a covering of $\{\rho = 0\}$ by domains of definition $\mathcal{O}_i$, $i = 1, \cdots, N$, of analytic coordinate systems x^a, $a = 3, \ldots, n+1$, and for $q \in \mathcal{O}_a$ let x^A, $A = 1, 2$, denote geodesic coordinates on $\exp_q\{(T_q\partial\Sigma)^\perp\}$. Set $(x^\mu) = (x^A, x^a)$. From what has been said it follows that the x^μ–coordinate components of the Riemannian metric tensor $g(i)$ are analytic functions of the x^μ's. We have the following local form of the metric

$$g(i) = \sum_{i=1}^{2}(dx^i)^2 + h + \sum_{A,a} O(\rho)dx^A dx^a + \sum_{A,B} O(\rho^2)dx^A dx^B + \sum_{a,b} O(\rho^2)dx^a dx^b\,, \tag{3.1}$$

with h — the metric induced by g on $\partial\Sigma$. The $O(\rho^2)$ character of the $dx^A dx^B$ error terms is standard; the $O(\rho^2)$ character of the $dx^a dx^b$ error terms follows from the totally geodesic character of $\partial\Sigma$. The Killing vector field Y takes the form $Y = x^1\partial_2 - x^2\partial_1 = \partial_\varphi$, where

$$(x^1, x^2) = (\rho\cos\varphi, \rho\sin\varphi)\,. \tag{3.2}$$

When expressed in terms of ρ and φ, the functions $g(i)_{\mu\nu} := g(i)(\partial_{x^\mu}, \partial_{x^\nu})$ are analytic functions of the x^μ's, hence (by composition) of ρ and of φ. Let R_π denote a rotation by π in the (x^A)-planes, R_π is obtained by flowing along Y a parameter-time π and is therefore an isometry, leading to

$$g(i)_{ab}(-x^1, -x^2, x^a) \;=\; g(i)_{ab}(x^1, x^2, x^a)\,, \tag{3.3a}$$
$$g(i)_{AB}(-x^1, -x^2, x^a) \;=\; g(i)_{AB}(x^1, x^2, x^a)\,, \tag{3.3b}$$
$$g(i)_{Aa}(-x^1, -x^2, x^a) \;=\; -g(i)_{Aa}(x^1, x^2, x^a)\,. \tag{3.3c}$$

In particular all odd-order derivatives of g_{ab} with respect to the x^B's vanish at $\{x^A = 0\}$, *etc.* Those symmetry properties together with analyticity imply (using, *e.g.*, Osgood's lemma) that there exist analytic $b_{ab}(s, x^a)$,

$\gamma_a(s,x^a)$, $\psi(s,x^a)$, with $\psi(0,x^a)=1$, such that

$$g(i)_{ab}(x^1,x^2,x^a) = b_{ab}(\rho^2,x^a)\,,$$
$$\left(g(i)_{Ab}Y^A\right)(x^1,x^2,x^a) = \rho^2\gamma_b(\rho^2,x^a)\,.$$
$$u(x^1,x^2,x^a) := \sqrt{\left(g(i)(Y,Y)\right)(x^1,x^2,x^a)} = \rho\left(1+\rho^2\psi(\rho^2,x^a)\right)\,.$$

Similarly, let $n = x^A\partial_A$, then $g(i)_{AB}Y^An^B$ and $g(i)_{AB}n^An^B$ are analytic functions invariant under the flow of Y, with $g(i)_{AB}Y^An^B = (g(i)_{AB} - \delta_{AB})Y^An^B = O(\rho^4)$, $g(i)_{AB}n^An^B = \rho^2 + O(\rho^4)$, hence there exist analytic functions $\alpha(s,x^a)$ and $\beta(s,x^a)$ such that

$$\begin{aligned}\left(g(i)_{AB}Y^An^B\right)(x^1,x^2,x^a) &= \rho^4\alpha(\rho^2,x^a)\,,\\ \left(g(i)_{AB}n^An^B\right)(x^1,x^2,x^a) &= \rho^2 + \rho^4\beta(\rho^2,x^a)\,.\end{aligned}$$

One similarly finds existence of an analytic one-form $\lambda_a(s,x^b)dx^a$ such that

$$\left(g(i)_{Aa}n^A\right)(x^1,x^2,x^b) = \rho^2\lambda_a(\rho^2,x^b)\,.$$

In polar coordinates (3.2) one therefore obtains

$$Y^\flat := g(i)(Y,\cdot) = \rho^2\left((1+\rho^2\psi)^2d\varphi + \alpha\rho d\rho + \gamma_a dx^a\right)\,.$$

Writing $g(i)$ in the form[4]

$$g(i) = u^2(d\varphi + \theta_j dy^j)^2 + h_{jk}dy^jdy^k\,, \tag{3.4}$$

with $y^j = (\rho^2,x^a)$, one has $Y^\flat = u^2(d\varphi+\theta_jdy^j)$ leading to

$$\theta := \theta_jdy^j = \frac{\alpha}{2(1+\rho^2\psi)^2}d(\rho^2) + (1+\rho^2\psi)^{-2}\gamma_adx^a\,,$$

$$h_{jk}dy^jdy^k = (1+\rho^2\beta)\left(\frac{d(\rho^2)}{2\rho}\right)^2 + b_{ab}dx^adx^b + \lambda_ad(\rho^2)dx^a - u^2\theta_i\theta_jdy^idy^j\,, \tag{3.5}$$

in particular all the functions h_{jk} are analytic functions of ρ^2 and x^a, except for the singular term $(2\rho)^{-2}\left(d(\rho^2)\right)^2$.

---

[4] Note that θ_j here is real, arising from the potential failure of hypersurface orthogonality of the polar coordinates associated to the harmonic ones, and *not* from the introduction of a complex constant in the metric as in Section 2. The introduction of the complex constant i there was done only to justify that the Riemannian metric denoted by $g(i)$ is Einstein; this last fact can be checked by direct calculations in any case.

Note that hypersurface-orthogonality has not been used anywhere in the calculation above (except for the initial justification of analyticity)[5] so that quite generally we have proved:

Proposition 3.1 *θ extends smoothly to the rotation axis $\{Y = 0\}$, analytically when the metric is analytic.*

Let us return to the static case, the hypersurface-orthogonality condition $Y^\flat \wedge dY^\flat = 0$ is equivalent to $d\theta = 0$, hence there exists, locally, a function τ such that

$$d\tau = d\varphi + \theta \,.$$

(The function τ is clearly analytic in the y^i's, but this is irrelevant for our purposes, since all the functions in (3.4) are φ-independent.) Writing

$$h_{jk}dy^j dy^k = \left(\frac{d(\rho^2)}{2\rho}\right)^2 + \hat{h}_{jk}dy^j dy^k \,, \tag{3.6}$$

where the $\hat{h}_{ij}$'s are defined by subtracting the first term at the right-hand-side of (3.6) from (3.5), the Lorentzian equivalent of the metric (3.4) reads now

$$g = g(1) = -y^1(1 + y^1\psi)^2 dt^2 + \frac{(dy^1)^2}{4y^1} + \hat{h}_{jk}dy^j dy^k \,. \tag{3.7}$$

Introducing a new coordinate u replacing t,

$$u = t + \tfrac{1}{2}\ln(y^1) \,,$$

the undesirable singular term in (3.7) cancels out. This provides the required analytic atlas in a one-sided neighborhood of the Killing horizon, covering the $\{g(X,X) \le 0\}$ region, compatible with the initial smooth structure, in which the metric functions are analytic up-to-boundary on the set where X is timelike or null.

4. Static initial data: global analyticity

In the previous section the starting point of our considerations was a static space-time. However, one can start with static initial data and ask about regularity of those. More precisely, consider a triple (M, γ, ϕ), where

[5] It should be emphasised that, from a space–time point of view, the hypothesis of non-degeneracy of the horizon has been made. We are not aware of any results about the behavior of θ near degenerate horizons.

(M, γ) is a smooth[2] n-dimensional Riemannian manifold and ϕ is a smooth function on M, satisfying the following set of equations

$$\Delta_\gamma \phi = -\lambda\phi \,, \tag{4.1a}$$

$$\phi(R(\gamma)_{ij} - \lambda\gamma_{ij}) = D_i D_j \phi \,. \tag{4.1b}$$

Here D is the Levi–Civita connection of γ, $\Delta_\gamma := D_k D^k$ its Laplace–Beltrami operator, $R(\gamma)_{ij}$ the Ricci tensor of γ, $R(\gamma)$ the scalar curvature of γ, while $\lambda \in \mathbb{R}$ is a constant related to the cosmological constant Λ. The function ϕ is allowed to change sign. It is well known that the set of zeros of a non-trivial ϕ, solution of (4.1), forms a smooth, embedded, totally geodesic submanifold of M, if not empty. Again, it is well known [9] that $M \setminus \{\phi = 0\}$ can be endowed with an analytic atlas, with respect to which γ and ϕ are analytic — this is a relatively straightforward consequence of the underlying elliptic features of the system of equations (4.1) for γ and ϕ, in regions where ϕ does not contain zeros.

We wish to show analyticity up-to and across the set of zeros of ϕ: Consider, thus, a one-sided local neighborhood of $\{\phi = 0\}$, replacing ϕ by $-\phi$ if necessary it suffices to consider the case $\phi \geq 0$. An appropriate periodic identification of an angular variable ϕ shows that the Riemannian metric, which we shall call $g(i)$,

$$g(i) = \phi^2 d\varphi^2 + \gamma$$

has a smooth[6] axis of rotation at $\{\varphi = 0\}$ for a Killing vector field $Y = \partial_\varphi$, and is Einstein. Then the argument leading from (3.1) to (3.6) applies, and is actually somewhat simpler because in the Riemannian case there is no need to introduce a new coordinate $y^1 = \rho^2$, the coordinate system (ρ, x^a) being the one in which the metric is analytic. Eq. (3.4) shows that h is the metric on the space of orbits of the Killing vector Y, so is γ, hence h is isometric to γ. This proves one-sided analyticity of γ in an appropriate atlas. Similarly considering the region $\{\phi \leq 0\}$, one obtains an analytic atlas on $\{\phi \leq 0\}$ with respect to which $-\phi$ and γ are analytic. Thus, ϕ and γ are analytic up-to-boundary both on $\{\phi \geq 0\}$ and on $\{\phi \leq 0\}$. Smoothness implies that the power series on both sides of $\{\phi = 0\}$ coincide, establishing analyticity near $\{\phi = 0\}$, and hence throughout M.

---

[6] This is established by first introducing normal coordinates (x, v^a) near $\{\phi = 0\}$, and using the fact that $u = \kappa x + O(x^2)$, for some non-zero constant κ. This provides continuity of the metric. To obtain smoothness one can prove directly, using (4.1), the parity properties of $g(i)$ as in (3.3a) with $x^2 = 0$, with a similar equation for ϕ/x. Alternatively, it follows from (4.1) that $R(\gamma) = (n-1)\lambda$, therefore the set $(M, \gamma, K := 0)$ is a vacuum initial data set (with cosmological constant) for the vacuum Einstein equations. Letting $(\mathcal{M}, g)$ be the maximal globally hyperbolic development of the data, if $\{\phi = 0\}$ is not empty then on $(\mathcal{M}, g)$ there exists a static hypersurface-orthogonal Killing vector X with a non-degenerate Killing horizon, and (3.3) follows from the analysis in [5].

5. Analyticity in a neighborhood of a "bifurcate Killing horizon"

The one-sided analyticity of the metric up-to-and-including the event horizon suffices for several purposes; that is, *e.g.*, the case for most issues concerning the properties of static domains of outer communications, which under the usual conditions coincide with the set $g(X,X)<0$. It is, nevertheless, interesting to enquire about extendibility of analyticity across the horizon. In the current context the following examples should be borne in mind:

1. Consider any smooth vacuum space-time $(\mathcal{M},g)$ with a non-degenerate Killing horizon $\mathcal{N}_r^+$ associated to a Killing vector field X, and suppose that $\mathcal{M}$ contains the "bifurcation surface"
$$\mathcal{S} := \{X=0\}\cap\overline{\mathcal{N}_r^+}\neq\emptyset\,. \tag{5.1}$$
Let $\mathcal{N}_l^+$ a second Killing horizon associated with $\mathcal{S}$, so that $\dot{J}^+(\mathcal{S}) = \mathcal{N}_l^+\cup\mathcal{N}_r^+$, see figure 1. (In case of unusual global causality properties of $(\mathcal{M},g)$, the notions of future and past here should be understood locally near $\mathcal{S}$.) Smoothly perturbing the characteristic initial data

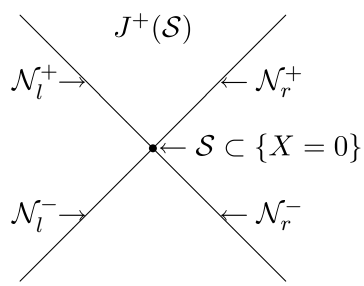

Fig. 1. Four Killing horizons $\mathcal{N}_r^\pm$ and $\mathcal{N}_l^\pm$ meeting at a bifurcation surface $\mathcal{S}$. We have $J^+(\mathcal{S}) = \mathcal{D}^+(\mathcal{N}_l^+\cup\mathcal{N}_r^+)$, at least locally.

on $\mathcal{N}_l^+$, without modifying those on $\mathcal{N}_r^+$, by evolution one will obtain a space-time $(\mathcal{M}',g')$ such that 1) g' smoothly extends to the previous metric g across $\mathcal{N}_r^+$; 2) for generic perturbations there will be no Killing vectors on $J^+(\mathcal{N}_l^+\cup\mathcal{N}_r^+)$. In the new space-time there will still be a locally defined Killing vector field X in a one-sided (past) neighborhood of $\mathcal{N}_r^+$, but X will not extend anymore to a Killing vector field defined on $\mathcal{M}'$. Thus, even the extendibility of a Killing vector field across a one-sided Killing horizon might fail in general (compare, however, [7]). An example of such behavior, with a metric which is explicit except for one function, in the category of C^{557} (but not smooth) metrics, is provided by the family of Robinson–Trautman extensions of the Schwarzschild metric of [6, Corollary 3.1].

2. Analyticity alone does not guarantee uniqueness of extensions.

In any case, so far we have only shown one-sided analyticity up-to-boundary, on a set where X is timelike or null, and it is not completely clear that this will guarantee analyticity beyond the Killing horizon in general: in each coordinate chart on which the set $g(X,X) < 0$ is given by $\{x^1 > 0\}$ there exists an analytic extension of the metric to an open subset of the set $\{x^1 < 0\}$, but this extension could fail to coincide[7] with the original metric there.

Let us show that there exists a setting where analyticity necessarily extends beyond the event horizon: suppose, for instance, that $\mathcal{M}$ contains a bifurcation surface $\mathcal{S}$ as in (5.1) (compare figure 1) contained within a spacelike achronal hypersurface Σ. Assuming staticity, we can deform Σ in space-time so that Σ is orthogonal to X. The results in Section 4 show that the initial data induced on Σ are analytic with respect to an appropriate atlas. By [1] the metric g is analytic in wave coordinates, compatible with the analytic atlas on Σ, on the domain of dependence $\mathcal{D}(\Sigma)$. This last set contains a neighborhood of $\mathcal{S}$. Let, now, $\mathcal{N}_r^+$ and $\mathcal{N}_l^+$ be as in figure 1. Section 3 provides analytic characteristic initial data (see, *e.g.*, [10]) there, and we have already established analyticity in a whole space-time neighborhood of $\mathcal{S}$. But analytic characteristic initial data on $\mathcal{N}_r^+$ and $\mathcal{N}_l^+$, compatible at $\mathcal{S}$, lead[8] to an analytic solution in $\mathcal{D}^+(\mathcal{N}_r^+ \cup \mathcal{N}_l^+)$, providing the desired result.

REFERENCES

[1] S. Alinhac, G. Métivier, *Invent. Math.* **75**, 189 (1984).

[2] L. Bessières, J. Lafontaine, L. Rozoy, (2004), in preparation.

[3] J. Bičák, J. Podolský, *Phys. Rev.* **D55**, 1985 (1996).

[4] G. Bunting, A.K.M. Masood-ul-Alam, *Gen. Rel. Grav.* **19**, 147 (1987).

[5] P.T. Chruściel, *Class. Quantum Grav.* **16**, 661 (1999).

[6] P.T. Chruściel, D. Singleton, *Commun. Math. Phys.* **147**, 137 (1992).

[7] Recall the example where the analytic up-to-boundary function $f(x) = 0$ defined for $x \geq 0$ is smoothly extended by $\exp(x^{-1})$ for $x < 0$.

[8] A simple proof is obtained using Garabedian's proof [15, Volume III] of the Cauchy–Kovalevski theorem.

[7] H. Friedrich, I. Rácz, R.M. Wald, *Commun. Math. Phys.* **204**, 691 (1999).

[8] V. Moncrief, J. Isenberg, *Commun. Math. Phys.* **89**, 387 (1983).

[9] H. Müller zum Hagen, *Proc. Cambridge Philos. Soc.* **68**, 199 (1970).

[10] A.D. Rendall, *Proc. Roy. Soc. London A* **427**, 221 (1990).

[11] D.C. Robinson, *Gen. Rel. Grav.* **8**, 695 (1977).

[12] S.M. Scott, P. Szekeres, *Gen. Rel. Grav.* **18**, 557 (1986).

[13] S.M. Scott, P. Szekeres, *Gen. Rel. Grav.* **18**, 571 (1986).

[14] B.V. Shabat, *Introduction to Complex Analysis. Part II, Translations of Mathematical Monographs*, vol. 110, American Mathematical Society, Providence, RI, 1992.

[15] M.E. Taylor, *Partial Differential Equations*, Springer, New York, Berlin, Heidelberg 1996.