COTTON BLEND GRAVITY pp WAVES

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Dedicated to Andrzej Staruszkiewicz, a pioneer in d = 3 gravity [Acta Phys. Pol. XXIV, 735 (1963)], on his 65th birthday

We study conformal gravity in $d = 2 + 1$, where the Cotton tensor is equated to a — necessarily traceless — matter stress tensor, for us that of the improved scalar field. We first solve this system exactly in the pp wave regime, then show it to be equivalent to topologically massive gravity.

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The gravitational properties of $d = 2 + 1$ worlds have been studied intensively, both in normal Einstein theory [1] and in its topologically massive extension [2]. For the latter, the Einstein tensor $G_{\mu\nu}$ is supplemented by the Cotton conformal curvature tensor $C_{\mu\nu}$ [3]; the usual Weyl tensor vanishes identically here. Like the latter, $C_{\mu\nu}$ is symmetric, traceless and vanishes if and only if space is conformally flat. It is also identically conserved.

Our purpose here is to examine the pure Cotton model, in which gravity is entirely governed by $C_{\mu\nu}$, with field equations

$$C_{\mu\nu} = \kappa\, T_{\mu\nu}\,, \tag{1}$$

where the matter stress tensor $T_{\mu\nu}$ must be traceless and

$$\begin{aligned}\sqrt{g}\, C^{\mu\nu} &\equiv \tfrac{1}{2}\,\varepsilon^{\mu\alpha\beta}\, D_\alpha\, R^\nu_\beta + \tfrac{1}{2}\,\varepsilon^{\nu\alpha\beta}\, D_\alpha\, R^\mu_\beta \\ &= \varepsilon^{\mu\alpha\beta}\, D_\alpha\, (R^\nu_\beta - \tfrac{1}{4}\,\delta^\nu_\beta\, R)\,. \end{aligned} \tag{2}$$

That the two expressions for $C^{\mu\nu}$ are equal follows from the Bianchi identity. [Our conventions are $R_{\mu\nu} = +\partial_\alpha\, \Gamma^\alpha_{\mu\nu} + \ldots$, and signature $(1,\ -1,\ -1)$.]

We choose the simplest continuous source[1]: a conformally coupled scalar field ψ whose action is

$$I = \tfrac{1}{2}\int d^3x\, \sqrt{g}\, (g^{\mu\nu}\, \partial_\mu\psi\partial_\nu\psi + \tfrac{1}{8}\, R\, \psi^2) \tag{3}$$

with (improved) energy-momentum tensor [4]

$$T_{\mu\nu} = \partial_\mu\psi\partial_\nu\psi - \tfrac{1}{2}\, g_{\mu\nu} g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi + \tfrac{1}{8}\,\psi^2 G_{\mu\nu} + \tfrac{1}{8}(g_{\mu\nu}D^2 - D_\mu D_\nu)\psi^2\,, \tag{4}$$

$T_{\mu\nu}$ is covariantly conserved and traceless on the matter shell,

$$\left(D^2 - \tfrac{1}{8}\, R\right)\psi = 0\,. \tag{5}$$

We shall solve this system in the plane-fronted parallel ray (pp) Ansatz for the geometry: with $u \equiv \frac{1}{\sqrt{2}}\,(t+x),\ v = \frac{1}{\sqrt{2}}\,(t-x)$,

$$ds^2 = F(u,y)du^2 + 2dudv - dy^2 \tag{6}$$

so that

$$g_{\mu\nu} = \begin{array}{c|ccc} & u & v & y \\ \hline u & F & 1 & 0 \\ v & 1 & 0 & 0 \\ y & 0 & 0 & -1 \end{array} \qquad g^{\mu\nu} = \begin{array}{c|ccc} & u & v & y \\ \hline u & 0 & 1 & 0 \\ v & 1 & -F & 0 \\ y & 0 & 0 & -1 \end{array} \tag{7}$$

Note that vanishing $g_{uy},\ g_{vy}$ and g_{vv} can be achieved by a coordinate choice, while g_{uv} can be set to unity by a conformal transformation. Thus our Ansatz consists in the requirement that g_{uu} be v-independent and that g_{yy} be unity. The Ricci and Cotton tensors each possess only one non-vanishing component

$$R_{uu} = \tfrac{1}{2}\, F''\,, C_{uu} = \tfrac{1}{2}\, F''' \tag{8}$$

and R vanishes, so that $G_{\mu\nu}$ coincides with $R_{\mu\nu}$. (We denote derivation with respect to y by a dash; with respect to u, by an over-dot.)

[1] In previous studies of (1), point particle sources were considered [5]; the sourceless equation, but with a dimensional Kaluza–Klein reduction, was also solved [6].

As is shown in Appendix A, the field equations require ψ to depend only on u. We simplify our procedure by using this fact, which implies that all energy-momentum tensor components but T_{uu} vanish. Furthermore (5) is identically satisfied. Thus there is only one equation to solve: $C_{uu} = \kappa T_{uu}$ or

$$\frac{1}{2}\, F''' = \frac{\kappa\psi^2}{16}\, \left(F'' + 2\, \frac{\ddot{\sigma}}{\sigma} \right) , \qquad \sigma = \frac{1}{\psi^2}\, . \tag{9}$$

The solution is immediate,

$$F(u,y) = f\, \exp\left[\frac{\kappa\psi^2 y}{8} \right] - \frac{\ddot{\sigma}}{\sigma}\, y^2 + \alpha y + \beta\, , \tag{10}$$

where f, α and β are three integration constants — actually functions of u — arising from solving the third-order equation (9). Evidently the Ricci (equivalent, in $d = 3$, to the full) curvature

$$R_{uu} = \frac{1}{2}\, F'' = \frac{\kappa^2\psi 4}{128}\, f\, \exp\left[\frac{\kappa\psi^2 y}{8} \right] - \frac{\ddot{\sigma}}{\sigma} \tag{11}$$

does not depend on (α, β) and in Appendix B we show that a coordinate transformation removes them.

Thus we have established that a pp wave geometry is supported by the Cotton tensor with a conformally coupled scalar field source, which also propagates as a wave:

$$\begin{aligned} g_{uu} \;&=\; f(u) \exp\left[\frac{\kappa\psi^2 y}{8} \right] - \frac{\ddot{\sigma}}{\sigma}\, y^2\, , \qquad g_{uv} = 1, \qquad g_{yy} = -1\, , \\ \psi \;&=\; \psi(u)\, . \end{aligned} \tag{12}$$

Note that the scalar field is not further specified beyond depending on retarded time, as is appropriate for a free field; $f(u)$ is arbitrary, but its vanishing would imply that of $C_{\mu\nu}$. Also note that the exponent is proportional to ψ^2, so the curvature always blows up exponentially as $\kappa y \to \infty$.

The equations obeyed by a Killing vector X_μ in our original (u, v, y) coordinates, where there is no v-dependence, require that $X_v = \text{const}$, $X_y = X_y(u)$ while X_u obeys

$$\dot{X}_u - \tfrac{1}{2}\, F' X_y - \tfrac{1}{2} \dot{F} X_v \;=\; 0\, , \tag{13}$$

$$X'_u + \dot{X}_y - F' X_v \;=\; 0\, . \tag{14}$$

For generic f and ψ, the geometry supports only one Killing vector: $X_1^\alpha = (0, 1, 0)$, corresponding to a constant shift of v, which clearly is an isometry

of the v-independent metric (12). However, with constant ψ (vanishing $\ddot{\sigma}/\sigma$) and special form for f, $f = (A + Bu)^n e^{mu}$, so that

$$g_{uu} = (A + Bu)^n \, \exp\left[mu + \frac{\kappa\psi^2 y}{8}\right] \tag{15}$$

there is the additional Killing vector

$$X_2^\alpha = \left(-\frac{\kappa\psi^2}{8}\,(A + Bu)\,, \quad \left(\frac{\kappa\psi_v^2}{8} + my\right) B\,, \quad m(A + Bu) + (n + 2)B\right) \tag{16}$$

whose Lie bracket with X_1^α closes on X_1^α. Thus, since X_1^α is a translation, X_2^α is a dilation. This is seen explicitly when the following coordinate transformation is performed (with $B \neq 0$)

$$\begin{aligned} U &= A + Bu\,, \\ V &= \frac{1}{B}\left(v + \frac{m^2}{2a^2}\,u + \frac{m}{a}\,y - \frac{m^2 A}{2a^2 B} - \frac{2m}{a}\,\ln B\right), \\ Y &= y + \frac{m}{a}\,u - \frac{2}{a}\,\ln B\,. \end{aligned} \tag{17}$$

Here $a \equiv \kappa\psi^2/8$. The line element becomes

$$ds^2 = U^n\, e^{aY} dU^2 + 2dU dV - dY^2$$

and the dilation Killing vector reads

$$X^\alpha = (-aU,\ aV,\ n + 2)\,. \tag{18}$$

Finally, we show that an amusing property of our CS + scalar system is its formal equivalence to the CS + Einstein (=TMG) model of [2]. As is easily checked, the improved scalar's action can be represented as the Einstein action of the rescaled metric $g'_{mn} = \psi^4 g_{mn}$,

$$I[\psi; g] = \int d^3x \sqrt{g'} R(g')\,. \tag{19}$$

This rescaling is purely formal: ψ remains the matter field variable. Consider now Cotton gravity; since the Cotton tensor is conformally invariant, the gravitational field equations are simply those of TMG,

$$C_{\mu\nu}(g') = \kappa G_{\mu\nu}(g')\,. \tag{20}$$

Furthermore, the scalar's equation is already included as the trace of (20), so variation of ψ is unnecessary. Note too that TMG, like the scalar field, has one degree of freedom, while Cotton gravity has none. [This amusing correspondence between the two models is no longer valid in presence of generic matter since traceful $T_{\mu\nu}$ are forbidden here, but permitted in TMG.] The TMG form also explains why the scalar field was rather secondary in our pp example: the only part of its stress tensor that contributes is the improvement term $\sim R_{\mu\nu}$, hence the homogeneous nature, $F''' \sim F''$, of the field equations. Since improved scalar actions rescale to Einstein's in any D, the properties we have just noted also carry over when they are coupled to the corresponding conformal gravity models, which of course also require traceless sources. In $D = 4$, for example, we find that adding Weyl gravity ($\int d^4x\sqrt{-g}\, C^2$) recovers Weyl plus Einstein gravity [7].

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Appendix A

No v, y dependence of ψ

We record the vanishing components of $T_{\mu\nu}$ in (4). As in (9), it is convenient to work in terms of $\sigma = \psi^{-2}$.

$$T_{vv} \;=\; \frac{1}{8\sigma^2}\,\partial_v^2\,\sigma = 0\,, \tag{A.1}$$

$$T_{vy} \;=\; \frac{1}{8\sigma^2}\,\partial_v\,\sigma' = 0\,. \tag{A.2}$$

These have the consequence that

$$\sigma = g(u,y) + h(u)v\,. \tag{A.3}$$

Next

$$T_{yy} = \frac{1}{8\sigma^3}\left(\sigma\sigma'' + \dot\sigma\partial_v\sigma - \frac{1}{2}\,F\partial_v\sigma\partial_v\sigma - \frac{1}{2}\,\sigma'\sigma'\right). \tag{A.4}$$

Inserting (A.3) and separating terms linear in v and v-independent leaves

$$h(g'' + \dot h) \;=\; 0\,, \tag{A.5}$$

$$gg'' + \dot g h - \tfrac{1}{2}\,Fh^2 - \tfrac{1}{2}\,g'g' \;=\; 0\,. \tag{A.6}$$

Continuing, we examine the uv component.

$$T_{uv} = \frac{1}{8\sigma^3}\left(\sigma\partial_v\dot{\sigma} - \dot{\sigma}\partial_v\sigma + \frac{1}{2}\,F\partial_v\sigma\partial_v\sigma + \frac{1}{2}\,\sigma'\sigma'\right) = 0\,. \tag{A.7}$$

With (A.3) and (A.5) this requires

$$g(g'' + \dot{h}) = 0\,. \tag{A.8}$$

Finally, we consider the uu component equation,

$$\begin{aligned} T_{uu} \;=\; & \frac{1}{8\sigma^3}\left(\sigma\ddot{\sigma} - F\dot{\sigma}\partial_v\sigma + \frac{1}{2}\,F^2\partial_v\sigma\partial_v\sigma + \frac{1}{2}\,F\sigma'\sigma' - \frac{1}{2}\dot{F}\sigma\partial_v\sigma\right. \\ & \left. - \frac{1}{2}\,F'\sigma\sigma' + \frac{1}{2}\,F''\sigma^2\right) = \frac{1}{2\kappa}\,F'''\,. \end{aligned} \tag{A.9}$$

Upon multiplication by σ^3 and decomposition according to powers of v, the v^3 term requires $h^3F''' = 0$ or $h = 0$, since we assume that the Cotton tensor is non-vanishing. It then follows from (A.6) and (A.8) that $g' = 0$, so σ and therefore ψ depend only on u. Other components of $T_{\mu\nu}$, as well as the matter field equation (5), do not provide independent restrictions.

Appendix B

Removing integration "constants"

In the line element associated with (10)

$$ds^2 = \left(f\exp\left[\frac{\kappa\psi^2 y}{8}\right] - \frac{\ddot{\sigma}}{\sigma}\,y^2 + \alpha y + \beta\right)du^2 + 2dudv - dy^2\,, \tag{B.1}$$

we pass to the new coordinates

$$u = U\,, \qquad v = V + A(U)Y + B(U)\,, \qquad y = Y + C(U)\,, \tag{B.2}$$

so that (B.1) becomes

$$\begin{aligned} ds^2 \;=\; & \left[f\exp\left[\frac{\kappa\psi^2 C}{8}\right]\,\exp\left[\frac{\kappa\psi^2 Y}{8}\right] - \frac{\ddot{\sigma}}{\sigma}\,Y^2 + \left(\alpha - 2\frac{\ddot{\sigma}}{\sigma}\,C + 2\dot{A}\right)Y\right. \\ & \left. + \beta + \alpha C - \frac{\ddot{\sigma}}{\sigma}\,C^2 - \dot{C}^2 + 2\dot{B}\right]dU^2 + 2dUdV \\ & +2\,(A - \dot{C})\,dYdU - dY^2\,. \end{aligned} \tag{B.3}$$

The procedure then is: with given $\ddot{\sigma}/\sigma$ and α, solve the equation

$$\ddot{C} - \frac{\ddot{\sigma}}{\sigma}\, C = -\frac{\alpha}{2}\,, \tag{B.4}$$

set $A = \dot{C}$ and determine B by quadrature:

$$\begin{aligned} B \;&=\; \frac{1}{2}\int^{U} du' \left(\frac{\ddot{\sigma}}{\sigma}\, C^2 + \dot{C}^2 - \alpha C - \beta\right) \\ &=\; \frac{1}{2}\, C\dot{C} - \frac{1}{2}\int^{U} du' \left(\frac{\alpha}{2}\, C + \beta\right)\,. \end{aligned} \tag{B.5}$$

Upon redefining the arbitrary f to absorb $\exp[\kappa\psi^2 C/8]$, the line element becomes

$$ds^2 = \left(f \exp\left[\frac{\kappa\psi^2 Y}{8}\right] - \frac{\ddot{\sigma}}{\sigma}\, Y^2\right) dU^2 + 2dUdV - dY^2 \tag{B.6}$$

in agreement with (12).

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