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ASYMPTOTIC ALGEBRA OF QUANTUM ELECTRODYNAMICS

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Dedicated to Andrzej Staruszkiewicz on the occasion of his 65th birthday

The Staruszkiewicz quantum model of the long-range structure in electrodynamics is reviewed in the form of a Weyl algebra. This is followed by a personal view on the asymptotic structure of quantum electrodynamics.

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1. Introduction

To write on the occasion of the 65^{th} birthday of Professor Andrzej Staruszkiewicz is a great honor. When I think of all the years I have known him I realize that it is at the same time precisely 30 years ago that I attended, as a first year student, his lectures in linear algebra and geometry. To many of us then he was, and remained for all the years of our physics studies, the most impressive and original teacher. One of the pictures many of us cherish in our minds is the scene in which he tries to demonstrate to us that a circle is nothing else than an interval which has been closed up, using for the purpose, not quite successfully one must say, the pointing stick he happened to have in his hand. Anyway, from that moment on I know the difference between the topology of a line and that of a circle.

This difference, as it happens, becomes prominent in Staruszkiewicz's quantum theory of the infrared degrees of freedom of electrodynamics (more on that below). The theory itself is perhaps the most evident testimony to what some of us had the opportunity to discover later on: that Staruszkiewicz's appeal as a teacher reflected the inherent originality of his thinking on physics, and beyond. The author of these words counts among those whose style of physics-making was greatly influenced, albeit sometimes in

polemics, by Professor Staruszkiewicz. More than that, Staruszkiewicz's ideas on the long-range properties of quantum electrodynamics were among those, which aroused my own steady interest in the field. This encourages me to use this opportunity to sketch a pedagogically oriented review of the Staruszkiewicz's model, as I see it, and to follow this by my own view of the asymptotic algebraic structure of quantum electrodynamics. It should be acknowledged, I hope that Professor Staruszkiewicz will agree, that the issues of infrared structure of electrodynamics remain to be controversial. Thus it is only natural that in addition to important common points in our views there will be other on which we differ.

The problem we want to address is the following:

What are the consequences of the long-range character of the electromagnetic interaction for the algebraic structure of the quantum theory of radiation and charged particles? What is the algebraic formulation of Gauss' law, and can it implement the charge quantization?

The approaches to this question summarized here start from concrete structures rather than from general assumptions on the desirable properties of the electromagnetic theory. To place this work in a wider context we start with some general remarks.

Quantum electrodynamics shares many properties (and difficulties) with other quantum field-theoretical models. However, its most interesting ingredients are those, that in our opinion (apparently shared by Staruszkiewicz) are specific to this theory — its long range structure being especially prominent. By this term we mean the group of properties connected with the masslessness of the photon, existence and quantization of electric charge and Gauss' law (see *e.g.* [1,2]). The lack of complete, mathematically sound formulation of quantum electrodynamics is, of course, an obstacle to conclusive understanding of the long range structure. This structure, however, needs only low energies and asymptotic spacetime regions to manifest itself. Therefore, it only very weakly involves the dynamics of the system, which lends some support to the hope that understanding this structure does not presuppose the complete understanding in detail of the dynamics. This belief lies at the base of the investigation of the long range structure from the "axiomatical" point of view. The main result of this study may be briefly formulated as follows: the flux of the electromagnetic field at spacelike infinity is superselected in irreducible representations of local observables [3]. More precisely, if the leading term of the electromagnetic field is well defined and decays as the Coulomb field in spacelike directions, then its distribution in spacelike infinity is fixed in such representations. One can say, therefore, that this long-range electromagnetic field has a classical spectrum, and elements of this spectrum (functions of the angles) label different sectors. Representations of local observables from different sectors are unitarily inequivalent.

In particular, states differing in total charge value are inequivalent in consequence of Gauss' law. There are also two other important consequences of this superselection structure: *(i)* in charged sectors the Lorentz group is spontaneously broken [4,5]; and *(ii)* the contribution of a charged particle to the spectrum of squared four-momentum (the mass spectrum squared) is not point-like. The former means, that although the Lorentz transformations of the observables are defined, they cannot be obtained by the action of a unitary representation of the Lorentz group in the representation space of the charged state. The consequence of the latter is that a charged particle, being accompanied by its electromagnetic field, is an object far more complex then a "bare", neutral particle. This fact is referred to as "the infraparticle problem" [5, 6]. The concept of an elementary particle has to be revised, in consequence, to be applicable to an electron. Several suggestions for such revisions has been formulated, among them the proposal by Buchholz [7] to use weights on the C*-algebra of observables for the generalization of the particle concept seems to be the most far-reaching.

The great value of the axiomatic approach to the quantum field theory problems lies in discovering strict logical connections between the expected fundamental features of the underlying structure on the one hand, and the interpretational (physical) properties of a theory (model) based on it. Among the basic postulates is the locality of observables: that each observable quantity may be measured locally in a compact subset of spacetime, or be derived as a limit of such local quantities (see [8]). However, physics deals with idealizations, and one could ponder whether in the case of electrodynamics, which includes constraints with nonlocal consequences (the Gauss law), we would not learn something by enlarging the scope of the admitted observables by some "variables at infinity".

In the two models summarized here such variables appear in a natural way. Also, both models include one variable of the phase type (circle topology), whose presence leads to the charge quantization. In other respects they differ. Staruszkiewicz considers the spacelike limit of classical electromagnetic fields and quantizes the resulting structure. The model has the advantage of (relative) simplicity, and in fact is probably a minimal field theoretical structure containing the Coulomb field among its variables. This is sufficient for Staruszkiewicz's main objective, which is to look for the justification of the actual value of elementary charge (or rather, one should say, the dimensionless fine structure constant). Formulation of the model is given in [9], additional discussion of the motivation may be found in [10].

My own aim is different, and the intention is to stay closer to the standard analysis. The object sought is the algebra of the asymptotic fields, in the causal, "in" or "out" sense. If we had a complete quantum theory at our disposal, we could try to obtain the algebra in the respective limits. Lacking

this one tries to make a guess based on intuitions formed by simpler quantum models. Perturbational quantum electrodynamics treats the asymptotic fields as uncoupled. This, however, is a wrong idealization, not respecting Gauss' law. Our method is to quantize the causal limits of the classical fields: timelike for matter and lightlike for electromagnetic fields. For separate free fields this reproduces the usual quantization (for the electromagnetic case: as considered at null infinity by Bramson and Ashtekar [11]). For interacting fields, however, a remnant of interaction survives, which correctly incorporates the consequences of Gauss law, and which is truly nonlocal. This may be interpreted as some form of "dressing" of a charged particle, and thus has relations with earlier works by Kulish and Fadeev [12], Fröhlich [13], Zwanziger [14], and others. However, here we are able to obtain a closed algebra which may be expected to have fairly universal features adequately incorporating the long range structure. A formulation and discussion of the model is to be found in [15].

We use physical units in which $\hbar = 1$, $c = 1$.

2. Asymptotic fields at spacelike infinity

We start with a discussion of the spacelike limit of classical fields. Suppose that $A(x)$ is a classical field satisfying the wave equation. Its Fourier representation is then given by

$$A(x) = \frac{1}{\pi} \int a(k)\delta(k^2)\varepsilon(k^0)e^{-ix\cdot k}\, d^4k\,, \tag{1}$$

where δ is the Dirac delta function and ε is the sign function. If $A_b(x)$ is an electromagnetic vector potential in Lorentz gauge of a free electromagnetic field, then $a_b(k)$ is a vector function satisfying $k \cdot a(k) = 0$ on the light-cone, and the reality of $A_b(x)$ is equivalent to

$$\overline{a_b(k)} = -a_b(-k)\,. \tag{2}$$

If $a_b(k)$ is a smooth function then $A_b(x)$ decreases rapidly in spacelike directions. However, as is well-known, the spacelike decay of the actual radiation fields produced in real processes is determined by the rate of decrease of the Coulomb fields of the sources. Thus one considers a wider class of potentials, those with well-defined spacelike scaling limit:

$$A_b^{\mathrm{as}}(x) := \lim_{\lambda\to\infty} \lambda A_b(\lambda x)\,, \quad x^2 < 0\,, \tag{3}$$

which is expressed in terms of the Fourier transform as the existence of the limit

$$a_b^{\mathrm{as}}(k) = \lim_{\mu\searrow 0} \mu\, a_b(\mu k)\,. \tag{4}$$

Note that both $A^{\rm as}_b(x)$ and $a^{\rm as}_b(k)$ are homogeneous functions of degree -1.

Before proceeding further let us remind the reader that if $f(l)$ is a function of a future-pointing null vector l, homogeneous of degree -2, written as $f(l) = f(l^0, \vec{l})$ in a given Minkowski basis, then the following integral

$$\int f(l)\, d^2 l := \int f(1, \hat{l})\, d\Omega(\hat{l}) \tag{5}$$

is Lorentz invariant, *i.e.* independent of the basis (here $\hat{l}$ is a unit vector in 3-space and $d\Omega(\hat{l})$ is the solid angle measure). We also note for later use that the differentiations of functions on the cone in tangent directions may be conveniently expressed by the application of the operators

$$L_{ab} := l_a \partial_b - l_b \partial_a\,, \qquad \text{where} \quad \partial_a := \frac{\partial}{\partial l^a}\,, \tag{6}$$

and that

$$\int L_{ab} f(l)\, d^2 l = 0\,. \tag{7}$$

Note that also the operator $l \cdot \partial$ is intrinsically defined on the cone, as $l_a\, l\cdot\partial = l^c L_{ac}$. Furthermore, if $h(l)$ is a regular function on the cone (except, possibly, its tip) and for the sake of differentiation one extends it in a regular, but otherwise arbitrary way to a neighborhood of the cone (outside the tip), then one shows that on the cone itself one has

$$[L_{ab} L_c{}^b + L_{ac}]h = [l_a l_c \partial^2 - (l_a \partial_c + l_c \partial_a + g_{ac}) l\cdot\partial]h\,. \tag{8}$$

As the operator on the l.h. side is intrinsically defined on the cone, the same must be true for the r.h. side. In particular, if $h(l)$ is homogeneous of degree 0, then on the cone one has

$$[{}^*L_{cb} {}^*L_a{}^b + L_{ac}]h = [L_{ab} L_c{}^b + L_{ac}]h = l_a l_c \partial^2 h\,, \tag{9}$$

where star denotes the dual of an antisymmetric tensor. This shows that in this case the expression $\partial^2 h(l)$ determines a homogeneous function of degree -2 intrinsically on the cone, which in each Minkowski basis may be represented by

$$\partial^2 h = (l^0)^{-2} {}^*L_{0b} {}^*L_0{}^b h\,. \tag{10}$$

In a similar way one shows that for two functions $h_1(l)$ and $h_2(l)$ homogeneous of degree 0 one has

$${}^*L_{cb} h_1 {}^*L_a{}^b h_2 = L_{ab} h_1 L_c{}^b h_2 = l_a l_c\, \partial h_1 \cdot \partial h_2\,. \tag{11}$$

Thus $\partial h_1 \cdot \partial h_2$ is intrinsically defined on the cone, and in each Minkowski basis there is

$$\partial h_1 \cdot \partial h_2 = (l^0)^{-2} {}^*L_{0b} h_1 {}^*L_0{}^b h_2 \,. \tag{12}$$

Taking into account that ${}^*L_{0b} l^0 = 0$ and integrating by parts with the use of (7) one has now

$$\int \partial^2 h_1 \, h_2 \, d^2 l = -\int \partial h_1 \cdot \partial h_2 \, d^2 l = \int h_1 \partial^2 h_2 \, d^2 l \,. \tag{13}$$

We can now return to the discussion of the asymptotic field. Calculating the asymptotic spacelike limit for the Fourier representation one shows that it becomes in this limit

$$A^{\mathrm{as}}_b(x) = \frac{-i}{2\pi} \int \frac{a^{\mathrm{as}}_b(l)}{x \cdot l - i0} \, d^2 l + \text{compl. conj.} \,, \tag{14}$$

which yields the asymptotic electromagnetic field

$$F^{\mathrm{as}}_{bc}(x) = \frac{i}{2\pi} \int \frac{l_b a^{\mathrm{as}}_c(l) - l_c a^{\mathrm{as}}_b(l)}{(x \cdot l - i0)^2} \, d^2 l + \text{compl. conj.} \tag{15}$$

(both $(x \cdot l - i0)^{-1}$ and $(x \cdot l - i0)^{-2}$ are well-defined homogeneous distributions). We stress that here, and throughout the paper, l always denotes a *future-pointing* null vector. Now, one can show that there exist unique up to additive constants, homogeneous of degree 0 complex functions $a(l)$ and $b(l)$ such that

$$l_b a^{\mathrm{as}}_c(l) - l_c a^{\mathrm{as}}_b(l) = L_{bc} a(l) - {}^*L_{bc} b(l) \tag{16}$$

— this follows from homogeneity of degree -1 of $a^{\mathrm{as}}_b(l)$ and its orthogonality to l^b, and can be shown most easily with the use of spinor formalism. We can thus separate F^{as}_{ab} into two parts:

$$F^{\mathrm{as}}_{ab} \;=\; F^{\mathrm{E}}_{ab} + F^{\mathrm{M}}_{ab} \,, \tag{17}$$

where

$$F^{\mathrm{E}}_{ab}(x) \;=\; \frac{i}{2\pi} \int \frac{L_{ab} a(l)}{(x \cdot l - i0)^2} \, d^2 l + \text{compl. conj.} \,, \tag{18}$$

$${}^*F^{\mathrm{M}}_{ab}(x) \;=\; \frac{i}{2\pi} \int \frac{L_{ab} b(l)}{(x \cdot l - i0)^2} \, d^2 l + \text{compl. conj.} \tag{19}$$

Using this form one finds that $F^{\mathrm{E}}_{[ab} x_{c]} = {}^*F^{\mathrm{M}}_{[ab} x_{c]} = 0$. This follows from the identity $x_c (x \cdot l - i0)^{-2} = -\partial_c (x \cdot l - i0)^{-1}$ and the following transformations of the integral

$$\int L_{[ab} \, a \, \partial_{c]} \frac{1}{x \cdot l - i0} \, d^2 l = \int \partial_{[b} a \, L_{ac]} \frac{1}{x \cdot l - i0} \, d^2 l = -\int \frac{L_{[ac} \partial_{b]} a}{x \cdot l - i0} \, d^2 l = 0 \,,$$

and similarly for $b(l)$. In consequence

$$F^{\mathrm{E}}_{ab}(x) = x_a K^{\mathrm{E}}_b(x) - x_b K^{\mathrm{E}}_a(x)\,, \quad {}^*F^{\mathrm{M}}_{ab}(x) = x_a K^{\mathrm{M}}_b(x) - x_b K^{\mathrm{M}}_a(x)\,, \quad (20)$$

where

$$K^{\mathrm{E}}_a(x) = \frac{1}{x^2}\, x^c F^{\mathrm{E}}_{ca}\,, \quad K^{\mathrm{M}}_a(x) = \frac{1}{x^2}\, x^{c*} F^{\mathrm{M}}_{ca}\,. \quad (21)$$

This form shows that F^{E}_{ab} and F^{M}_{ab} are fields of electric and magnetic type respectively: one can check directly that the long range tail produced by scattered electric charges is of type F^{E}_{ab}; by duality, F^{M}_{ab} would appear in scattering of magnetic monopoles.

Thus, being interested in the actual electrodynamics, we do not need to include long-range fields of the magnetic type in the theory, and from now on we assume that

$$F^{\mathrm{M}}_{ab} = 0\,, \quad \text{that is} \quad b = 0\,. \quad (22)$$

In that case we have

$$l_b a^{\mathrm{as}}_c(l) - l_c a^{\mathrm{as}}_b(l) = L_{bc} a(l)\,, \quad (23)$$

so

$$a^{\mathrm{as}}_b(l) = \partial_b a(l) + l_b \alpha(l)\,, \quad (24)$$

where $a(l)$ has been extended for the sake of differentiation to a homogeneous function in a neighborhood of the lightcone, and $\alpha(l)$ is a homogeneous function of degree -2. The second term does not contribute to the field F^{as}_{ab}, so it must yield a gauge term in the potential, and indeed:

$$\frac{-i}{2\pi} \int \frac{l_b \alpha(l)}{x \cdot l - i0}\, d^2 l = \nabla_b \frac{-i}{2\pi} \int \alpha(l) \log \Big[\frac{x \cdot l - i0}{t \cdot l}\Big]\, d^2 l\,, \quad (25)$$

where t is any future-pointing unit timelike vector and $\nabla_b := \partial/\partial x^b$. However, we note that the omission of this term does not leave an unambiguously defined gauge invariant expression for the asymptotic potential. Although a and $\partial^2 a$ are intrinsically defined on the cone, the expression $\partial_b a$ is not, and depends on the choice of homogeneous extension of a to the neighborhood of the cone: two different homogeneous extensions yield two $\partial_b a(l)$'s differing by a term of the form $l_b \beta(l)$.[1] This corresponds to a change of gauge in A^{as}_b, therefore not all information on the potential A^{as}_b may be encoded in the light-cone function $a(l)$.

[1] For instance, for the homogeneous function $f(l) = l^2/(t \cdot l)^2$ we have on the cone: $f = 0$, but $\partial_a f(l) = 2 l_a/(t \cdot l)^2$.

The electromagnetic field $F^{\rm as}_{ab}$ is most compactly expressed with the use of Eqs. (20) and (21). We can now identify $F^{\rm as}_{ab} = F^{\rm E}_{ab}$ and write $K_a = K^{\rm E}_a$, so by the homogeneity properties we have

$$x^2 K_a(x) = x^c F^{\rm as}_{ca}(x) = \nabla_a[-x\cdot A^{\rm as}(x)] = \frac{1}{e}\nabla_a S(x)\,, \tag{26}$$

where e is the elementary charge, and following Staruszkiewicz we have denoted

$$S(x) = -e\, x\cdot A^{\rm as}(x)\,. \tag{27}$$

For any future-pointing unit timelike vector t there is

$$\frac{x\cdot a^{\rm as}(l)}{x\cdot l - i0} = \partial a(l)\cdot\partial \log\Big[\frac{x\cdot l - i0}{t\cdot l}\Big] + \frac{t\cdot a^{\rm as}(l)}{t\cdot l}\,, \tag{28}$$

so using (14) one finds that

$$S(x) = \frac{e}{2}\int \partial^2 {\rm Re}\, a(l)\, \varepsilon(x\cdot l)\, d^2 l + \frac{e}{\pi}\int \partial^2 {\rm Im}\, a(l) \log\left[\frac{|x\cdot l|}{t\cdot l}\right]\, d^2 l + S_t\,, \tag{29}$$

$$S_t \equiv -\frac{e}{\pi}\int \frac{t\cdot {\rm Im}\, a^{\rm as}(l)}{t\cdot l}\, d^2 l\,. \tag{30}$$

This scalar function, homogeneous of degree zero, contains the whole information on the field $F^{\rm as}_{ab}(x)$, and in addition has an additive constant S_t not contributing to this field. This constant is both gauge- and t-dependent:

$$\text{if}\quad \tilde a^{\rm as}_b(l) = a^{\rm as}_b(l) + l_b\beta(l)\,,\quad \text{then}\quad \tilde S_t = S_t - \frac{e}{\pi}\int \beta(l)\, d^2 l\,, \tag{31}$$

and if t' is another future-pointing unit timelike vector, then

$$S_{t'} = S_t + \frac{e}{\pi}\int \partial^2 {\rm Im}\, a(l) \log\left[\frac{t'\cdot l}{t\cdot l}\right]\, d^2 l\,. \tag{32}$$

The last transformation property confirms that the t-dependence of the formula (29) is spurious. On the other hand, the whole function $S(x)$ also undergoes the gauge transformation:[2]

$$\tilde S(x) = S(x) - \frac{e}{\pi}\int \beta(l)\, d^2 l\,. \tag{33}$$

[2] We should acknowledge here that Staruszkiewicz regards $S(x)$ as gauge-independent. This is a consequence of his apparent treating $\partial_b a(l)$ as an unambiguously defined quantity. Note also, that in general the contraction of the gauge term (25) with x^b does not vanish.

We stated above that not the whole information on $a_b^{\rm as}(l)$ is contained in $a(l)$. However, as it turns out, the freedom of adding a constant to $a(l)$ may be used to choose this function so as to contain the whole information on $S(x)$. Namely, given $a_b^{\rm as}(l)$, a special solution of Eq. (23) for $a(l)$ may be shown to be

$$a(l) = \frac{1}{4\pi} \int \frac{l \cdot a^{\rm as}(l')}{l \cdot l'} \, d^2 l' \,. \tag{34}$$

This solution has the following remarkable property: for each unit timelike vector t there is

$$\int \frac{a(l)}{(t \cdot l)^2} \, d^2 l = \int \frac{t \cdot a^{\rm as}(l)}{t \cdot l} \, d^2 l \,, \tag{35}$$

so with this choice, which will always be assumed from now on, we have

$$S_t = -\frac{e}{\pi} \int \frac{\operatorname{Im} a(l)}{(t \cdot l)^2} \, d^2 l \,. \tag{36}$$

The function $S(x)$ is now seen to be determined completely and uniquely by $\partial^2 \operatorname{Re} a(l)$ and $\operatorname{Im} a(l)$.

3. Staruszkiewicz's model

At this point one observes that $S(x)$ satisfies the wave equation

$$\Box S(x) = 0 \,, \tag{37}$$

and that Eq. (29) almost gives the most general function homogeneous of degree zero satisfying this equation.[3] The reservation "almost" is due to the fact that in place of $\partial^2 \operatorname{Re} a(l)$ one can have an arbitrary function $c(l)$ homogeneous of degree -2. This makes a difference of only one degree of freedom. Namely, if t is any timelike, unit, future-pointing vector, and one denotes

$$c_t(l) = c(l) - \frac{\int c(l') \, d^2 l'}{4\pi (t \cdot l)^2} \,, \quad \text{then} \quad \int c_t(l) \, d^2 l = 0 \,. \tag{38}$$

But each function satisfying the last equation may be represented as a result of applying ∂^2 to a homogeneous function of degree 0, so the only quantity lacking from (29) is $\int c(l) d^2 l$. Following Staruszkiewicz we now add this degree of freedom. Thus we:

$$\text{replace} \quad \partial^2 \operatorname{Re} a(l) \to -\frac{1}{2\pi} c(l) \,, \quad \text{and denote} \quad \operatorname{Im} a(l) \equiv -\frac{1}{4} D(l) \,, \tag{39}$$

[3] Eq. (37) together with the homogeneity are equivalent to the wave equation on the hyperboloid $x^2 = -1$.

where the choice of constants is a mere convention. Our function $S(x)$ becomes now

$$S(x) = -\frac{e}{4\pi}\int c(l)\,\varepsilon(x\cdot l)\,d^2l - \frac{e}{4\pi}\int \partial^2 D(l)\log\left[\frac{|x\cdot l|}{t\cdot l}\right]\,d^2l + S_t\,, \quad (40)$$

$$S_t = \frac{e}{4\pi}\int \frac{D(l)}{(t\cdot l)^2}\,d^2l\,. \quad (41)$$

If thus extended function $S(x)$ is now used in (26) to determine the asymptotic field $F^{\text{as}}_{ab}(x)$, then the new degree of freedom added to $S(x)$ produces a charged field, with charge given by

$$Q = \frac{1}{4\pi}\int c(l)\,d^2l \quad (42)$$

— this is shown by integrating the flux of electric field over a sphere.

Staruszkiewicz's model now rests upon two main suppositions: that one can base a model of the long-range structure on the field $S(x)$ alone, and that S_t should be interpreted as a phase variable. For the motivation we refer the reader to the original papers by Staruszkiewicz. Consider the first supposition. One looks for a quantization condition for $\hat{S}(x)$ of the form $[\hat{S}(x),\hat{S}(y)]\propto \mathrm{id}$, where "hats" indicate the quantum versions of these variables. This should be expressible as $[\hat{c}(l),\hat{D}(l')]\propto \mathrm{id}$. Let $D(l)$ and $c(l)$ be now classical test functions, homogeneous of degree 0 and -2, respectively, and denote

$$\hat{c}(D) = \frac{1}{4\pi}\int \hat{c}(l)D(l)\,d^2l\,, \quad \hat{D}(c) = \frac{1}{4\pi}\int \hat{D}(l)c(l)\,d^2l\,. \quad (43)$$

Then the only Lorentz-covariant quantization condition, up to a multiplicative constant on the r.h. side, is

$$[\hat{c}(D),\hat{D}(c)] = \frac{i}{4\pi}\int D(l)c(l)\,d^2l\ \mathrm{id} \quad (44)$$

— the choice of the particular constant will be justified in a moment. A straightforward calculation with the use of (40) yields now

$$\begin{aligned}[\hat{S}(x),\hat{S}(y)] \;=\; & i2e^2\varepsilon\left(\frac{x^0}{\sqrt{-x^2}}-\frac{y^0}{\sqrt{-y^2}}\right)\\ & \times\theta\left(\Big[\frac{x}{\sqrt{-x^2}}-\frac{y}{\sqrt{-y^2}}\Big]^2\right)\frac{x\cdot y}{\sqrt{(x\cdot y)^2-x^2y^2}}\ \mathrm{id}\,, \quad (45)\end{aligned}$$

which is the relation obtained by another method by Staruszkiewicz.[4] This commutation relation guarantees causality when restricted to the unit hyperboloid $x^2 = -1$.

Consider now the second supposition, that $\hat{S}_t$ is a phase variable. Using (41), (42) and (43) one finds that

$$\hat{S}_t = \hat{D}\Big(\frac{e}{(t\cdot l)^2}\Big)\,, \quad \hat{Q} = \hat{c}(1)\,, \tag{46}$$

so by (44) one has

$$[\hat{Q}, \hat{S}_t] = ie \text{ id}\ . \tag{47}$$

The supposition means that in this relation $\hat{S}_t$ should be used, in fact, in the form $\exp[-i\hat{S}_t]$, and the above commutation relation should be understood as

$$\hat{Q}e^{-i\hat{S}_t} = e^{-i\hat{S}_t}(\hat{Q} + e\,\text{id})\,. \tag{48}$$

The precise formulation of the commutation relations thus obtained has the following Weyl form derived by the heuristic substitution

$$W(D) = \exp\big[i\hat{c}(D)\big]\,, \quad R(c) = \exp\big[-i\hat{D}(c)\big]\,, \tag{49}$$

and by admitting in $R(c)$ only those test functions c for which there is

$$n_c := \frac{1}{4\pi e}\int c(l)\,d^2l \in \mathbb{Z}\,. \tag{50}$$

The algebraic relations are

$$\begin{aligned} W(D)W(D') &= W(D+D')\,, \quad R(c)R(c') = R(c+c')\,,\\ W(D)R(c) &= \exp\Big\{\frac{i}{4\pi}\int D(l)c(l)\,d^2l\Big\}R(c)W(D)\,,\\ W^*(D) = W(-D)\,, &\quad R^*(c) = R(-c)\,, \quad W(0) = R(0) = \text{id}\,, \end{aligned} \tag{51}$$

which defines an abstract Weyl algebra. To consider a physical realization of the system one needs a *-representation of this algebra by operators in a Hilbert space.[5] Before choosing a particular representation we make some comments on the structure of the algebra.

First of all, one should observe that the algebra could be formulated in terms more directly connected with the spacetime relations. Namely, for any two homogeneous solutions of the wave equation (37) the formula

$$\{S_1, S_2\} := \sqrt{(x^0)^2+1}\int \big[S_1\nabla_0 S_2 - S_2\nabla_0 S_1\big](x^0, \sqrt{(x^0)^2+1}\,\hat{x})d\Omega(\hat{x})\,, \tag{52}$$

[4] There is a misprint of a sign on the r.h. side of this relation in [9].

[5] Not to burden notation we shall keep the same symbol for the operator, as for the abstract element itself.

where $\hat{x}$ is a vector on a unit sphere in 3-space and $d\Omega(\hat{x})$ is the solid angle measure, defines a symplectic form conserved under the evolution and independent of the reference system.[6] On the other hand one can show that

$$\{S_1, S_2\} = e^2 \int \big[c_1 D_2 - c_2 D_1\big](l)\, d^2 l\,, \tag{53}$$

if $S_i(x)$ are represented as in (40). Thus the initial values $S(0,\hat{x})$, $\nabla_0 S(0,\hat{x})$ could be used instead of $c(l)$, $D(l)$ as test fields of the algebra elements. This leads to relativistic locality of the commutation relations on the hyperboloid $x^2 = -1$, but we do not go in any further details.

Next, we note two important symmetries of the algebra. For $\lambda \in \mathbb{R}$ we have a group of automorphisms of the algebra defined by

$$\gamma_\lambda(A) = W(\lambda) A W(-\lambda)\,, \quad \gamma_\lambda \gamma_{\lambda'} = \gamma_{\lambda+\lambda'}\,. \tag{54}$$

By the basic commutation relations we have

$$\gamma_\lambda\big(W(D)\big) = W(D)\,, \quad \gamma_\lambda\big(R(c)\big) = e^{i\lambda n_c e} R(c)\,. \tag{55}$$

In representations in which $W(\lambda)$ is regular we have $W(\lambda) = \exp[i\lambda\hat{Q}]$, where $\hat{Q}$ has the interpretation of the charge operator. Therefore the automorphism γ_λ should be regarded as a (global) gauge transformation. Accordingly, the algebra of observables is the subalgebra of (51) consisting of elements invariant under γ_λ, which is generated by the elements of the form $W(D)R(\partial^2 F)$ with $F(l)$ homogeneous of degree 0 (recall that if $n_c = 0$ then there exists such F that $c = \partial^2 F$). Elements $R(c)$ with $n_c \neq 0$ are field variables interpolating between superselection sectors and creating the charge $n_c e$. This confirms our earlier statement that S_t is a gauge dependent quantity, which should not be regarded as an observable. Note, however, that $R(c)^* R(c') = R(c' - c)$ *is* an observable if $n_c = n_{c'}$, so sectors are labelled only by charge value. Note, moreover, that $\gamma_{(2\pi/e)} = \mathrm{id}$. Thus if the representation of (51) is irreducible then $\exp[i2\pi\hat{Q}/e] \propto \mathrm{id}$. If in addition 0 is in the spectrum of $\hat{Q}$, then the spectrum is equal to $e\mathbb{Z}$. This leads to the quantization of charge and justifies the choice of the multiplicative constant in the quantization condition (44).

Another symmetry group of the algebra (51) is the Lorentz group, which acts on the algebra by the automorphisms (Λ is a Lorentz transformation):

$$\alpha_\Lambda\big[W(D)\big] \;=\; W(T_\Lambda D)\,, \quad \alpha_\Lambda\big[R(c)\big] = R(T_\Lambda c)\,,$$

where

$$[T_\Lambda D](l) \;=\; D(\Lambda^{-1} l)\,, \quad [T_\Lambda c](l) = c(\Lambda^{-1} l)\,. \tag{56}$$

[6] This is the symplectic form for solutions of the wave equation on the hyperboloid $x^2 = -1$, *cf.* footnote 3.

There is no nontrivial translation symmetry in the algebra.

One looks for representations which have a cyclic vector Ω (that is the closure of the linear span of all vectors $W(D)R(c)\Omega$ is the whole representation space), in which the Lorentz symmetry is implementable, *i.e.* there exists a unitary representation of the Lorentz group $U(\Lambda)$ such that for each operator A in the representation of the algebra there is

$$\alpha_\Lambda(A) = U(\Lambda)AU^*(\Lambda)\,, \tag{57}$$

and in which Ω is Lorentz-invariant:

$$U(\Lambda)\Omega = \Omega\,. \tag{58}$$

A class of such representations may be obtained by the Fock method (we are not aware of a proof that this exhausts the set of covariant representations). Assume that the operators of the observable elements $W(D)$ and $R(\partial^2 F)$ are regular, that is there exist selfadjoint $\hat{c}(D)$ and $\hat{D}(\partial^2 F)$ such that for $\lambda \in \mathbb{R}$ there is $W(\lambda D) = \exp[i\lambda\hat{c}(D)]$ and $R(\lambda\partial^2 F) = \exp[-i\lambda\hat{D}(\partial^2 F)]$. Let κ be any real positive number. Suppose that in the representation space there exists a vector Ω_κ which is cyclic and for each $F(l)$ homogeneous of degree 0 satisfies

$$\Big[\sqrt{\kappa}\,\hat{c}(F) + \frac{i}{\sqrt{\kappa}}\hat{D}(\partial^2 F)\Big]\,\Omega_\kappa = 0\,. \tag{59}$$

One shows that these conditions determine a unique (up to a unitary equivalence) representation. We sketch the proof. Suppose first that such Ω_κ exists. From the condition (59) for $F = 1$ we have in particular $\hat{Q}\,\Omega_\kappa = 0$. Moreover, from the commutation relations we get

$$\hat{Q}\,W(D)R(c)\Omega_\kappa = n_c e\,W(D)R(c)\Omega_\kappa\,. \tag{60}$$

Therefore the representation space is

$$\mathcal{H} = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n\,, \quad \text{where} \quad \hat{Q}\mathcal{H}_n = ne\mathcal{H}_n\,, \tag{61}$$

and $\mathcal{H}_n$ is the closure of the linear span of vectors $W(D)R(c)\Omega_\kappa$ with $n_c = n$. It is now easy to see that all matrix elements of operators $W(D)R(c)$ between arbitrary vectors from the set $W(D')R(c')\Omega_\kappa$ are reduced with the use of commutation relations either to zero, or to a matrix element in the space $\mathcal{H}_0$. It is thus sufficient to show the existence and uniqueness of representation of observable elements $W(D)R(\partial^2 F)$ in $\mathcal{H}_0$. For that purpose for each complex function $F(l)$ homogeneous of degree 0 let us denote by $[F]$ its equivalence

class with respect to the addition of a constant. Let $\mathcal{K}$ be the Hilbert space of such classes with the scalar product

$$([F],[G])_{\mathcal{K}} = \frac{1}{4\pi}\int(-\partial F\cdot\partial G)\,d^2l\,, \tag{62}$$

and let $\mathcal{H}_0$ be the Fock space based on the "one-excitation" space $\mathcal{K}$. Denote by Ω the "Fock vacuum" vector and by $d([F])$ the annihilation operator in that Fock space:

$$d([F])\Omega = 0\,,\qquad \big[d([F]),d^*([G])\big] = ([F],[G])_{\mathcal{K}}\,\mathrm{id}\ . \tag{63}$$

We set $\Omega_\kappa = \Omega$ and for real F

$$\begin{aligned}\hat{c}(F)|_{\mathcal{H}_0} &= \frac{1}{\sqrt{2\kappa}}\Big\{d([F])+d^*([F])\Big\}\,,\\ \hat{D}(\partial^2F)|_{\mathcal{H}_0} &= -i\sqrt{\frac{\kappa}{2}}\Big\{d([F])-d^*([F])\Big\}\,.\end{aligned} \tag{64}$$

It is easy to show that this ensures the correct commutation relations and that Eq. (59) is now satisfied, so the existence of the representation is proved. Furthermore, it follows from (59) alone that

$$(\Omega_\kappa, W(D)R(\partial^2F)\Omega_\kappa) = \exp\frac{1}{4}\Big[-\kappa^{-1}\|[D]\|^2_{\mathcal{K}} - \kappa\|[F]\|^2_{\mathcal{K}} + i2([D],[F])_{\mathcal{K}}\Big]\,. \tag{65}$$

As by the GNS construction these expectation values determine the representation up to a unitary equivalence (see *e.g.* [16]), the uniqueness follows. The unitary representation of the Lorentz group with the desired properties (57) and (58) is now obtained by

$$U(\Lambda)W(D)R(c)\Omega_\kappa = W(T_\Lambda D)R(T_\Lambda c)\Omega_\kappa\,. \tag{66}$$

One can easily show that the generators M_{ab} of these transformations, defined by $U(\delta^a{}_b+\omega^a{}_b)\approx\exp[-\frac{i}{2}\omega^{ab}M_{ab}]$ for small antisymmetric ω_{ab}, may be expressed as

$$M_{ab} = -\frac{1}{4\pi}\int :\hat{c}(l)L_{ab}\hat{D}(l): d^2l\,, \tag{67}$$

where normal ordering is determined by point splitting as

$$:\hat{c}(l)L_{ab}\hat{D}(l) := \lim_{l'\to l}\Big\{\hat{c}(l')L_{ab}\hat{D}(l) - (\Omega_\kappa,\hat{c}(l')L_{ab}\hat{D}(l)\Omega_\kappa)\,\mathrm{id}\Big\}\,, \tag{68}$$

and the limit goes over l' linearly independent from l.

The above construction leaves us with the freedom of one real parameter κ in the choice of representation. In the usual situation for quantum fields the selection criterion which often leaves only one representation is the demand that the vacuum state be translation invariant and the total energy be a positive operator. We do not have this criterion for our disposal in the case of present model. However, Staruszkiewicz thinks that the asymptotic field (18) should "remember" that its first term (the one explicitly written in (18)) is obtained from the positive frequency field, which in usual electrodynamics annihilates the vacuum. Thus he demands that the quantum version of the first term in (18) annihilates Ω_κ. Looking at (18), (39) and (59) it is easy to convince oneself that this condition is satisfied if, and only if,

$$\kappa = \frac{2}{\pi}\,. \tag{69}$$

In this way one arrives at an interesting and elegant model, which explicitly depends on the value of elementary charge e and has a charged field among its variables. Staruszkiewicz believes, and in fact this is his main motivation, that some mathematical and physical consistency restrictions will squeeze out of this model an information on the size of the fine structure constant $e^2/\hbar c$. That this hope may, in fact, be justified, is suggested by the structure of the Lorentz group representation $U(\Lambda)$. As it turns out, the breakup of this representation into irreducibles must depend nontrivially on the value of $e^2/\hbar c$ [17].

We hope that the formulation of the Staruszkiewicz model we have discussed here helps to clarify its structure at least for some readers. But it should also help to simplify calculations. We give as an example the calculation of the scalar product of states $R(e/(v\cdot l)^2)\Omega_\kappa$ (in Staruszkiewicz's notation $e^{-iS_0}|0\rangle$ with S_0 the spherically symmetric part of $S(x)$ in the reference system with time axis v). Denote $F_{v,u}(l) = e\log[v\cdot l/u\cdot l]$. Then by (51) and (65) we have

$$\begin{aligned}&(R(e(v\cdot l)^{-2})\Omega_\kappa, R(e(u\cdot l)^{-2})\Omega_\kappa) = (\Omega_\kappa, R(\partial^2 F_{v,u})\Omega_\kappa)\\ &= \exp\big[-(\kappa/4)\|[F_{v,u}]\|^2_{\mathcal{K}}\big] = \exp\big[-(e^2\kappa/2)\big(\chi_{v,u}\coth\chi_{v,u}-1\big)\big]\,,\end{aligned} \tag{70}$$

where $v\cdot u = \cosh\chi_{v,u}$. For $\kappa = 2/\pi$ this reproduces the result obtained in a much more involved way in [9]. We have used in the calculation:

$$\begin{aligned}\|[F_{v,u}]\|^2_{\mathcal{K}} &= -\frac{1}{4\pi}\int[\partial F_{v,u}(l)]^2\,d^2l\\ &= \frac{e^2}{4\pi}\int\left[\frac{2\,v\cdot u}{(v\cdot l)(u\cdot l)} - \frac{1}{(v\cdot l)^2} - \frac{1}{(u\cdot l)^2}\right]d^2l\\ &= 2e^2\left\{\frac{v\cdot u}{\sqrt{(v\cdot u)^2-1}}\log\Big[v\cdot u + \sqrt{(v\cdot u)^2-1}\Big] - 1\right\}.\end{aligned} \tag{71}$$

4. Asymptotic causal algebra

Let us now return again to the discussion of the asymptotic fields considered in Section 2. Recall that the assumption of their behavior as defined in (3) was dictated by the fall-off of Coulomb fields of charges. However, it later turned out that one half of the resulting asymptotic fields, these of magnetic type (19), did not actually appear in real processes, so they could be omitted. This left us with the long-range characteristics of the electric type only. But now we can ask further: do *all* of these characteristics have a role to play in real processes? Our answer is: no, and as we shall see, this is precisely what allows us to construct an algebra which unites both the usual local and the long-range degrees of freedom.

The selection criterion for free electromagnetic fields we want to use is this: we admit only those fields which may be produced as radiation fields in processes involving scattering charged particles or fields, asymptotically moving freely for early and late times. Recall that radiation field is the difference between the retarded and advanced field produced by the current. Take the simplest instant of such field, the radiation field produced by a charge scattered instantaneously at $x = 0$. In this case the radiation potential in spacelike directions is the difference of two Coulomb fields

$$A_b^{\mathrm{rad}}(x) = Q\left(\frac{v_b}{\sqrt{(v\cdot x)^2 - x^2}} - \frac{u_b}{\sqrt{(u\cdot x)^2 - x^2}}\right), \quad x^2 < 0\,, \tag{72}$$

where Q is the charge of the particle, and v and u its initial and final velocity respectively. Note that this potential is homogeneous of degree -1, so its spacelike asymptotic limit (3) is given by the same function. More generally, if the motion of the particle is modified but v and u remain its asymptotic velocities, then the above formula still gives the spacelike asymptotic $A^{\mathrm{as}}(x)$ of the potential. A striking feature of this potential is its evenness:

$$A^{\mathrm{as}}(-x) = A^{\mathrm{as}}(x)\,, \qquad x^2 < 0\,. \tag{73}$$

Now, this property is conserved under the superposition principle, so it remains true for a general field produced by particles. One can show that the same property holds for electromagnetic potential radiated by scattered charged fields. Thus we take (73) as our selection criterion. Compare this with the general asymptotic potential (14). Our condition is then equivalently expressed as

$$\mathrm{Im}\, a_b^{\mathrm{as}}(k) = 0\,. \tag{74}$$

We want to view our selection criterion from yet another viewpoint. For a general Lorentz gauge potential of the form (1) let us denote for a future-

pointing null vector l and $s \in \mathbb{R}$:

$$\dot{V}_b(s,l) = -\int \omega a_b(\omega l) e^{-i\omega s}\, d\omega\,, \tag{75}$$

where dot denotes differentiation with respect to s. It is easy to see that $\dot{V}_b(s,l)$ is a real function, orthogonal to l^b and homogeneous of degree -2 in all its variables:

$$l \cdot \dot{V}(s,l) = 0\,, \qquad \dot{V}_a(\mu s, \mu l) = \mu^{-2}\dot{V}_a(s,l)\,, \quad \mu > 0\,, \tag{76}$$

A straightforward calculation then shows that the Fourier representation (1) may be written as

$$A_b(x) = -\frac{1}{2\pi}\int \dot{V}_b(x\cdot l, l)\, d^2 l\,. \tag{77}$$

If $a_b(k)$ has a scaling limit (4) then taking into account the reality condition (2) one finds that $\omega \mathrm{Re}\, a_b(\omega l)$ is continuous in $\omega = 0$, while $\omega \mathrm{Im}\, a_b(\omega l)$ has a jump of magnitude $2\mathrm{Im}\, a_b^{\mathrm{as}}(l)$. This leads to the estimate

$$\dot{V}_b(s,l) = -\frac{2}{s}\mathrm{Im}\, a_b^{\mathrm{as}}(l) + O(|s|^{-1-\epsilon}) \quad \text{for} \quad |s| \to \infty \tag{78}$$

for some $\epsilon > 0$. Now, consider the null asymptotics of the potential, more precisely, take an arbitrary point in spacetime x and consider the asymptotics of $A(x + Rl)$ for $R \to \infty$. One shows that if the leading term in (78) does not vanish, then the dominating term of this asymptotics is $2\mathrm{Im}\, a_b^{\mathrm{as}}(l) \log R/R$. As it turns out, in that case the leading term for the angular momentum density at $x + Rl$ is of order $\log R/R^2$. This means that even the *differential* flux of angular momentum radiated into infinity cannot be defined, which is our second reason to reject those fields.

We want now to consider an interacting theory, and we take for definiteness the classical theory of the electron–positron Dirac field coupled by local gauge principle to the electromagnetic field, with the intention of later "quantization". In perturbative calculations one uses an approximation in which the fields are free at very early and very late times, (matter is completely decoupled from radiation). This procedure is assisted by some preliminary regularization, such as restricting the interaction to some subset of spacetime, which may be an effective tool to do practical calculations, but is unable to satisfactorily clarify the infrared structure. We want to improve on that approximation so as to take into account the infrared degrees of freedom and the Gauss law.

The selection criterion for the electromagnetic fields may still be taken over to the interacting case in the following sense. If A_b is the Lorentz

potential of the total field, then one defines in standard way the incoming and outgoing free fields by $A_b = A_b^{\mathrm{ret}} + A_b^{\mathrm{in}} = A_b^{\mathrm{adv}} + A_b^{\mathrm{out}}$, where A_b^{ret} and A_b^{adv} are the retarded and the advanced potential of the sources, respectively. Then it may be consistently assumed that both A_b^{in} and A_b^{out} satisfy the selection criterion.

Our aim is to consider fields at causally remote regions, "in" or "out", and we restrict attention to the "out" case. This is usually taken to mean: on a spacelike hyperplane, which is taken to the limit of time tending to $+\infty$. However, due to the different propagation speeds of matter and radiation one can exchange this for: matter field far away in the future timelike directions, and electromagnetic field far away in the future null directions. Consider the electromagnetic field first. With our assumptions one shows that there is a function $V_b(s,l)$ homogeneous of degree -1 such that

$$\lim_{R\to\infty} RA_b(x+Rl) = V_b(x\cdot l, l)\,. \tag{79}$$

This function is homogeneous of degree -1, satisfies

$$l\cdot V(s,l) = Q\,, \tag{80}$$

where Q is the charge of the field, and is bounded by

$$|\dot{V}_b(s,l)| \leq \frac{\mathrm{const}}{(t\cdot l)^2}\left(1+\frac{|s|}{t\cdot l}\right)^{-1-\epsilon} \tag{81}$$

(only the constant depends on t). The "out" field may be recovered from this asymptotics by (77), and its null asymptotics is given by (79) with $V_b(s,l)$ replaced by $V_b^{\mathrm{out}}(s,l) = V_b(s,l) - V_b(+\infty,l)$. The limit value $V_b(+\infty,l)$ is completely determined by the outgoing currents, and determines according to (79) the null asymptotics of the advanced potential. The spacelike asymptotics of the "out" field is governed by

$$a_b^{\mathrm{as}}(l) = -\frac{1}{2\pi}\int \dot{V}_b(s,l)\,ds = \frac{1}{2\pi}V_b^{\mathrm{out}}(-\infty,l)\,, \tag{82}$$

but the spacelike asymptotics of the *total* field is determined by $V_b(-\infty,l)$, and for any point x and spacelike vector y one has

$$\lim_{R\to\infty} R^2F_{ab}(x+Ry) = \frac{1}{2\pi}\int \big(l_aV_b(-\infty,l) - l_bV_a(-\infty,l)\big)\delta'(y\cdot l)\,d^2l\,. \tag{83}$$

Note also, that the second and the third terms in the function $S(x)$ as given by (29) now vanish, so here one could not construct an analogy of the Staruszkiewicz model — function $D(l)$ in (39) is identically zero. There is

no need nor space for the extension given by the first replacement in (39) either. On the other hand, the constant in $\mathrm{Re}\, a(l)$ *will* appear in our model, and will be related to a phase variable. We denote

$$\Phi(l) = \frac{1}{4\pi} \int \frac{l \cdot V^{\mathrm{out}}(-\infty, l')}{l \cdot l'} \, d^2 l \,. \tag{84}$$

Consider now the timelike asymptotics of the Dirac field $\psi(x)$. One shows that with an appropriate choice of a local gauge (locally related to the Lorentz gauge) one has for $v^2 = 1$, v future-pointing:

$$\psi(\lambda v) \sim -i\lambda^{-3/2} e^{-i(m\lambda + \pi/4)\gamma \cdot v} f(v) \quad \text{for} \quad \lambda \to \infty \,, \tag{85}$$

where γ^a are the Dirac matrices. Define, provisionally, the free outgoing Dirac field by

$$\psi_f^{\mathrm{out}}(x) = \Big(\frac{m}{2\pi}\Big)^{3/2} \int e^{-imx \cdot v \gamma \cdot v} \gamma \cdot v \, f(v) d\mu(v) \,, \tag{86}$$

where $d\mu(v)$ is the invariant measure d^3v/v^0 on the hyperboloid $v^2 = 1$, $v^0 > 0$, and the formula is a concise form of the Fourier representation of $\psi_f^{\mathrm{out}}(x)$, reproducing in the free field case the original field $\psi(x)$. The outgoing current of the Dirac field is determined by $f(v)$, and one shows that the lacking component $V_b(+\infty, l)$ of the total electromagnetic potential is given by

$$V_a(+\infty, l) = \int n(v) V_a^e(v, l) \, d\mu(v) \,, \tag{87}$$

where $n(v) = \overline{f(v)} \gamma \cdot v f(v)$ is the asymptotic density of particles moving with velocity v and

$$V_a^e(v, l) = \frac{e v_a}{v \cdot l} \tag{88}$$

is the null asymptotics (79) of the Lorentz potential of the Coulomb field surrounding a particle with charge e moving with constant velocity v. Therefore, the above relation is the implementation of the Gauss constraint on the space of classical asymptotic variables.

The question now arises: do the fields A^{out} and ψ_f^{out} separate completely in the "out" region? We interpret this question as: can the total energy momentum and angular momentum of the system be separated into contributions from A^{out} and ψ_f^{out}? The answer is 'yes' in the case of energy momentum, but 'no' in the case of angular momentum — in this case there is a term which couples the infrared degrees of freedom $V_b^{\mathrm{out}}(-\infty, l)$ with

$f(v)$. However, as it turns out, the full separation may be achieved if one introduces a new variable $g(v)$ by

$$g(v) = \exp\Big(\frac{ie}{4\pi}\int \frac{\Phi(l)}{(v\cdot l)^2}\, d^2l\Big)\, f(v)\,, \tag{89}$$

and defines the "dressed" free Dirac field by

$$\psi^{\rm out}(x) = \Big(\frac{m}{2\pi}\Big)^{3/2}\int e^{-imx\cdot v\gamma\cdot v}\gamma\cdot v\, g(v) d\mu(v)\,. \tag{90}$$

We draw attention of the reader to the following circumstances. First, the transformation (89) is a very nonlocal one. The asymptotics of the local Dirac field in the timelike direction of v is multiplied by a factor containing information on the spacelike asymptotics of the outgoing electromagnetic field $A_b^{\rm out}$. Next, as the conserved quantities have been completely separated, the field $\psi^{\rm out}$ should be regarded as describing the charged particles *together* with their Coulomb fields. Finally, as announced earlier, the constant in $\Phi(l)$ does appear in the model. However, this constant appears only in the exponentiated form given by (89). Thus we put forward the interpretation

$$\frac{e}{4\pi}\int \frac{\Phi(l)}{(v\cdot l)^2}\, d^2l = \text{phase variable}\,. \tag{91}$$

Note that this definition involves only the free electromagnetic characteristics, and is independent of particular matter field.

This classical asymptotic model has a natural "quantization" based on the heuristic demand that the total conserved quantities generate Poincaré transformations. The model is formulated in terms of the quantities which have direct physical meaning in the asymptotic region, that is the asymptotics of the total field $\hat{V}_b(s,l)$, and the asymptotics of the Dirac field with the accompanying Coulomb fields of the particles $\hat{g}(v)$ ("hats" indicate the quantum versions). We introduce the following structures on the space of asymptotic variables: the symplectic form

$$\{V_1, V_2\} = \frac{1}{4\pi}\int \big(\dot{V}_1\cdot V_2 - \dot{V}_2\cdot V_1\big)(s,l)\, ds\, d^2l \tag{92}$$

and the scalar product

$$(g_1, g_2) = \int \overline{g_1(v)}\gamma\cdot v g_2(v) d\mu(v)\,. \tag{93}$$

Let $g(v)$ and $V_b(s,l)$ be classical test fields describing asymptotics of *free* fields, thus, in particular, $V_b(+\infty, l) = 0$. The basic elements of the quantum

model are functionals of those test fields: $W(V)$ and $B(g)$. Loosely, one can think of them as

$$W(V) = e^{-i\{V,\hat{V}\}}\,, \quad B(g) = (g,\hat{g})\,. \tag{94}$$

Elements $W(V)$ and $W(V')$ are identified if the test potentials $V_b(s,l)$ and $V'_b(s,l)$ give the same electromagnetic test field asymptotics and the same phase variable (91), that is

$$l_{[a}V'_{b]}(s,l) = l_{[a}V_{b]}(s,l)\,, \quad \Phi'(l) = \Phi(l) + n\frac{2\pi}{e}\,, \tag{95}$$

where $\Phi(l)$ is related to $V_b(s,l)$ by (84). The algebra is then defined by

$$\begin{gathered} W(V_1)W(V_2) = e^{-\frac{i}{2}\{V_1,V_2\}}W(V_1+V_2)\,, \\ W(V)^* = W(-V)\,, \qquad W(0) = \mathrm{id}\,, \\ [B(g_1),B(g_2)]_+ = 0\,, \quad [B(g_1),B(g_2)^*]_+ = (g_1,g_2)\,\mathrm{id}\,, \\ W(V)B(g) = B(S_\Phi g)W(V)\,, \end{gathered} \tag{96}$$

where

$$\big(S_\Phi g\big)(v) = \exp\left(i\frac{e}{4\pi}\int \frac{\Phi(l)}{(v\cdot l)^2}\,d^2l\right) g(v)\,. \tag{97}$$

With a proper technical formulation of conditions on the scope of test functions the above relations generate a C^*-algebra, which I interpret as the algebra of asymptotic fields in quantum electrodynamics.

The only relation in which the above algebra diverges from the usual tensor product of independent algebras of the two fields separately is the last relation in (96), but this is the key to the physics of the model. We note that for the Coulomb field asymptotics (88) one has

$$\{V^e(v,.),V\} = \frac{e}{4\pi}\int \frac{\Phi(l)}{(v\cdot l)^2}\,d^2l\,. \tag{98}$$

The commutation relation between the fermionic operator $B(g)$ and the electromagnetic operator $W(V)$ may be therefore written in loose terms as

$$e^{-i\{V,\hat{V}\}}\hat{g}(v) = \hat{g}(v)e^{-i\{V,\hat{V}-V^e(v,.)\}}\,. \tag{99}$$

This means that the operator $\hat{g}(v)$, beside its fermionic role which is to annihilate a particle with charge e or create one with the opposite charge, also annihilates or creates the particle's Coulomb field, respectively.

Within the model formulated here the following results are obtained.

(i) The spectrum of the charge operator is quantized in units of elementary charge. This is the consequence of the appearance of the quantum phase. As this phase variable is tied to the free electromagnetic potential, this quantization law is universal.

(ii) In representations of the asymptotic algebra satisfying Borchers' criterion (spacetime translations implementable by unitary operators with the energy-momentum spectrum in the future lightcone) the analogue of the functional form of Gauss' constraint (87) is satisfied.

(iii) The importance of the regularity of representations with respect to all Weyl operators is stressed. The vacuum representation is shown to be non-regular with respect to Coulomb field operators ($W(V)$ with infrared singular test functions V), which leads to the loss of the Coulomb field and to a nonphysical superselection structure. A class of "infravacuum" representations is constructed, which are "close to the vacuum" but regular at the same time. Each irreducible representation of the field algebra in this class leads to the superselection structure of observables characterized by the electric charge. There is neither a zero-energy vector state nor mass-shell charged vector states in these representations.

Finally, to make some contact with the Staruszkiewicz model again, one can consider a kind of adiabatic limit (slowly varying fields) of a Weyl model based on the symplectic form (92) alone (with no fermionic fields, but with charged test fields $V_b(s,l)$ admitted instead). That was done in [18]. The mathematics of the resulting model is identical with that of Staruszkiewicz's model, and in fact our formulation of the latter as a kind of Weyl algebra given in Section 3 was based on that paper. However, the interpretation of variables is different in the two cases. In particular, the quantity (91) survives the adiabatic limit as a phase variable, which is different from Staruszkiewicz's phase.

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