THE GENERAL PENROSE INEQUALITY: LESSONS FROM NUMERICAL EVIDENCE

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Formulation of the Penrose inequality becomes ambiguous when the past and future apparent horizons do cross. We test numerically several natural possibilities of stating the inequality in punctured and boosted single- and double-black holes, in a Dain–Friedrich class of initial data and in conformally flat spheroidal data. The Penrose inequality holds true in vacuum configurations for the outermost element amongst the set of disjoint future and past apparent horizons (as expected) and (unexpectedly) for each of the outermost past and future apparent horizons, whenever these two bifurcate from an outermost minimal surface, regardless of whether they intersect or remain disjoint. In systems with matter the conjecture breaks down only if matter does not obey the dominant energy condition.

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1. Introduction

The Penrose–Hawking [1,2] singularity theorems point at the incompleteness of the classical general relativity. The cosmic censorship hypothesis [4] can be regarded as an attempt to contain the damage, by demanding that the **genuine** singularities are hidden within the black holes. While it is still not clear whether the cosmic censorship hypothesis holds true, there is no doubt that it has shaped the research field and led to many important results concerning evolving systems (for a recent review see [3]). Penrose invented in 1973 an inequality that can constitute a necessary condition for the validity of the cosmic censorship [5,6]. The Penrose inequality has been recently proved, following a scenario suggested in 1973 by Geroch [7] in the important Riemannian case [8, 9], that can be roughly described as a momentarily static data set of Einstein evolution equations. It was known for a long time to hold in spherically symmetric systems [10]; a later analysis allowed one to elucidate the problem of the needed energy conditions and show that it is independent of the foliation conditions [11, 12]. There exist analytical scenarios for the proof of the general Penrose inequality [13] and [14] but their validity is not proven, due to the technical complexity.

While numerics cannot *per se* produce a proof, it may disprove the hypothesis or, more likely, be of help in finding its correct formulations. The latter is the main goal of this paper — Sec. 3 presents possible wordings of the Penrose inequality that are checked in later sections in examples representing several classes of initial data. The obtained results can be described as bringing some surprises. In the spherically symmetric spacetimes the future and past apparent horizons are disjoint and it is well known that the Penrose inequality is valid only for the outermost of all apparent horizons (later on abbreviated as AH) [11], assuming an energy condition. In nonspherical spacetimes the two AH's can intersect in most space-time foliations, including the maximal ones. Notable exceptions are the polar gauge foliations in which the two horizons cannot be separated (although they can bifurcate), but their existence status is unclear. The apparent ambiguity would be resolved by accepting the proposal of Horowitz [15] that one should take a surface of a minimal area enclosing all AH's. It is unexpected in this context that whenever the outermost past and future apparent horizons bifurcate (in a sense specified in Sec. 5) from an outermost minimal surface, then the inequality holds true for each of them. This remark applies to either crossing or disjoint AH's, in all nonspherical vacuum configurations tested by us. One can convert this phenomenological observation into a local analytic proof, as sketched in Sec. 5.

The order of remainder of this paper is following. Next section brings the initial constraint equations and a brief description of the conformal method of constructing initial data. Several versions of the Penrose inequality are given in Sec. 3. Sec. 4 briefly describes the relevant numerical methods. Obtained results are reported in Secs 5–7. They support the various versions of the inequality for the vacuum initial data in Secs 5 (punctured Bowen–York data) and 6 (punctured Dain–Friedrich data). Section 7 deals with nonvacuum spheroidal initial data; in this case the Penrose inequality can be broken, if the dominant energy condition is not valid. Last section presents main conclusions.

2. Einstein constraint equations

Let Σ be an asymptotically flat Cauchy hypersurface endowed with an internal metric g_{ij}, the scalar curvature R and the extrinsic curvature K_{ij}. The initial constraint equations read [16]

$$R = 16\pi\rho + K_{ij}K^{ij} - \left(K_i^i\right)^2 \,,$$
$$\nabla_i \left(K_j^i - g_j^i K_l^l\right) = 8\pi j_j \,, \tag{1}$$

where ρ and $\vec{j}$ are the mass density and current density of initial material fields. In the case of maximal slicing condition, $K_i^i = 0$, the initial data can be found by the conformal method [17]. In what follows we analyze conformally flat classes of solutions, corresponding to vacuum and spheroidal systems with matter. The metric reads $g_{ij} = \phi^4 \hat{g}_{ij}$ where $\hat{g}_{ij}$ is the Euclidean metric.

The global energy-momentum can be found from standard formulae

$$E = \frac{-1}{2\pi} \int\limits_{S_\infty} d^2 S^i \nabla_i \phi \,,$$
$$P_j = \frac{1}{8\pi} \int\limits_{S_\infty} d^2 S^i K_{ij} \,. \tag{2}$$

The asymptotic mass is given by $m = \sqrt{E^2 - P_i P^i}$.

An apparent horizon will be understood later on as a two-dimensional surface S lying in Σ with a normal t satisfying one of the two equations

$$\theta \pm \equiv \nabla_i t^i \pm K_{ij} t^i t^j = 0 \,, \tag{3}$$

where the signs + and - correspond to the past and the future apparent horizons, respectively, and θ's are known as optical scalars.

3. Formulation of the Penrose inequality

The Penrose inequality is expected to hold only for the outermost AH. As explained in [11], in the case of spherical symmetry: *Consider the outermost future trapped surface, the (future) apparent horizon, call it S. Let us assume that S is outside the outermost past trapped surface. In other words, we assume $\theta_+(S) = 0$ and that both θ_+ and θ_- are positive outside S.* A simple analytic argument shows the validity of the following inequality

$$m \geq \sqrt{\frac{S_{\mathrm{H}}}{16\pi}} \,, \tag{4}$$

provided that the dominant energy condition is satisfied by matter located outside S. $S_{\rm H}$ in this formula is the area of S. As stressed in the quoted paper, *of course, an identical argument works if the outermost trapped surface is a past apparent horizon.*

In the spherically symmetric geometries optical scalars have the same level sets, since both of them are constant on centered spheres. Thus the spheres surrounding the outermost AH have positive θ_- and θ_+ and *(i)* possess larger area than the AH. Obviously *(ii)* the future and past horizons do not cross. Therefore there is no ambiguity in defining the Penrose inequality and — since the cosmic censorship hypothesis asserts that AH's are enclosed by the event horizon that asymptotically evolves to a Schwarzschild or Reissner–Nordstroem black hole horizon — it can be regarded as the necessary condition for the cosmic censorship [5].

None of the features *(i)* and *(ii)* becomes obvious in the general nonspherical case, even if our liberal definition of AH's is replaced by the more stringent one due to Penrose. Let us recall that in [1] future trapped surface are assumed to have — in our terminology — a positive scalar θ_- and a negative scalar θ_+. That is, each of the two beams of null geodesics emanating orthogonally outward and inward from a trapped surface, is convergent. Consequently, a future apparent horizon (understood as the outermost boundary in the set of all future trapped surfaces) has vanishing θ_+ but non-negative θ_-. The analogous situation (but with optical scalars reversing their roles) takes place for past apparent horizons. Thus the picture of AH's that emerges here resembles that of spherically symmetric geometries. There exist level sets of, say $\theta_+ < 0$, such that $\theta_- > 0$ (and conversely). The two optical scalars do not possess common level sets, but the sign of one of them is controlled on the level set of the other. This can happen only if one matches in a suitable way the choice of both a Cauchy hypersurface and of the two-dimensional foliation within this slice. (This is inherent also to the scheme of the proof of the Penrose inequality that is proposed in [14].) One can find a two-surface S to be such an AH in one particular foliation, but that may not be true in other space-like slices. Even with this stringent definition, the future and past AH's can intersect and there may exist surfaces of a smaller area surrounding them as pointed out by Horowitz [15].

In the rest of this paper by AH's are understood two-surfaces satisfying one of the conditions of (3), which might be weaker than the notion employed in the singularity theorems (but see a discussion following the point *(ii)* below). That means that the failure of a particular version (or all of them) formulated below of the Penrose inequality does not necessarily negate the cosmic censorship hypothesis (CCH). And conversely, their validity lends even more credence in CCH. It is not without significance that such AH's are easier to find numerically than the standard objects defined by Penrose.

The three versions of the Penrose inequality read as follows, (assuming the dominant energy condition [2] for nonvacuum initial data):

(i) The minimalistic one (PIM henceforth); it was borrowed from a proposition first put forward by Horowitz [15]. *The surface A_M of the smallest area S_M surrounding regions with horizons* satisfies the inequality

$$m \geq \sqrt{\frac{S_{\mathrm{M}}}{16\pi}}\,. \tag{5}$$

It appears in many of the numerical cases reported later that A_{M} coincided with that constructed from apparent horizons (see *(ii)*), but in a number of configurations it consisted also of segments of minimal surfaces. The existence of configurations having portions of minimal surfaces extending outside AH's, means that it is not excluded that the actual area of an event horizon — if there is one — is smaller than that of the AH. In such a case PIM constitutes the necessary condition for the validity of CCH.

(ii) The standard one (PIS later on). *The closed 2-surface A_H is either the outermost apparent horizon (if AH's do not intersect) or a union of segments of outermost future and/or past apparent horizons.* Then its area S_{H} satisfies the inequality

$$m \geq \sqrt{\frac{S_{\mathrm{H}}}{16\pi}}\,. \tag{6}$$

A_{H} does not manifestly satisfy the assumptions of the singularity theorems, but its importance lies in the fact that it may do so in another foliation (say, the polar gauge one). The heuristic argument is as follows. The product of two optical scalars is *(i)* invariant and *(ii)* vanishes on A_{H}. If there exists a local boost to a polar gauge foliation (that is, a foliation with $\theta_- = \theta_+$; apparent horizons correspond here to minimal surfaces) of the space–time, then on a polar gauge slice the two-surface A_{H} would become an apparent horizon (that is, the outermost minimal surface) in the sense of Penrose, and then the CCH demands the existence of an event horizon. The area of the intersection of the event horizon with the actual polar gauge slice would have to be bigger than of A_{H}. Accepting that, PIS seems to be just right one, as a necessary condition, from the point of view of CCH. Unfortunately, there is a gap in the argument. Namely, there is no possibility to rule out the existence of minimal surfaces that extend outward of the outermost apparent horizon. There are reasons to expect (basing on the analogy to spherical symmetry) that the polar gauge slice does not penetrate regions with minimal surfaces and the surface A_{H} would not be seen on the slice.

(iii) In the cases with intersecting apparent horizons $A_{\rm A}$'s we compared also their area related quantities $\sqrt{S_{\rm A}/16\pi}$ with the asymptotic mass. Invariably it was found that

$$m > \sqrt{\frac{S_{\rm A}}{16\pi}} \tag{7}$$

for each of the horizons, and with a significant safety margin. There is no obvious reason why the area $S_{\rm A}$ should be bigger than $S_{\rm H}$, but this is what was found to be true in all analyzed examples. That suggests that *(iii)* is stronger than PIS and it comes as a surprise that numerics supports the inequality (7).

Data concerning the two stronger *(ii)* and *(iii)*) of the above conjectures are given in Sections 5 and 6; their validity implies PIM. Only in the first part of Sec. 5 we report data concerning the version PIM, to show that the results are close for all three statements. The horizons do not cross in the case of spheroidal initial data and there is no minimal surface outside the outermost AH; therefore conjectures PIS and PIM do coincide in the examples considered in Sec. 7.

4. Description of numerical methods

In the conformal method and for vacuum conformally flat initial data, one first solves analytically the equation $\hat{\nabla}_i \hat{K}^i_j = 0$ (the covariant derivatives are in the Euclidean metric) and then the Lichnerowicz–York equation

$$\Delta\phi = -\frac{\hat{K}_{ij}\hat{K}^{ij}}{8}\phi^{-7}, \tag{8}$$

with the flat Laplacian $\mathit{\Delta}$. This is an example of a weakly nonlinear elliptic equation; its leading derivatives are linear (hence the equation is quasilinear) and the nonlinearity is rather weak (*cf.* negative powers of the conformal factor ϕ). Due to the cylindrical symmetry, we search for ϕ as a function of the angle θ and the coordinate radius r. It is necessary to map the problem into one with finite domain; thus the radius r is replaced by another independent variable $v = r/(1+r)$. Adopting $x = \cos\theta$, one has to solve (8) in the rectangular $-1 \le x \le 1, 0 \le v \le 1$. It is solved iteratively by the standard Newton method on the lattice up to 200×5000 points. Due to the weak nonlinearity of the Lichnerowicz–York equation, it was enough to apply at most 4–5 iterations. We used four different solvers, in particular the MUMPS [18] and HYPRE [19] ones.

The apparent horizon equation (3) becomes in our context a nonlinear ordinary equation, for the function $r(\theta)$. This is a classical two-point problem (see a discussion in [20] in a similar context) with $dr/d\theta|_{\theta=0} =$

$dr/d\theta|_{\theta=\pi} = 0$. It is solved by the standard shooting method. We resorted to two numerical packages, ODEPACK [21] and SUBPLEX [22]. The needed extrapolation of the formerly found solution ϕ has been done with the help of the bilinear interpolation [23]. The bilinear method appeared entirely satisfactory, due to the high density of our numerical lattice.

5. Boosted punctured data

Assume $\hat{P}$ to be a constant vector and $\hat{n}$ — a unit normal to a metric sphere in the Euclidean geometry. Assume standard spherical coordinates r, θ and ϕ. One can easily check that the extrinsic curvature

$$K_{ij} = \frac{3}{2r^2\phi^2}\left(\hat{P}_i\hat{n}_j + \hat{P}_j\hat{n}_i - \left(\hat{g}_{ij} - \hat{n}_i\hat{n}_j\right)\hat{P}_l\hat{n}^l\right) \tag{9}$$

satisfies the momentum part of Eq. (1) with the vanishing current (that is the boosting part of the Bowen–York initial extrinsic curvature [24]). The Hamiltonian constraint (the first equation in (1)) reads now, assuming vacuum case ($\rho = 0$) and aligning the z-axis along $\hat{P}$,

$$\Delta\phi = -\frac{9(\hat{P})^2}{16r^4}\left(1 + 2\cos^2\theta\right)\phi^{-7}\,, \tag{10}$$

where Δ is the flat Laplacian. For these boosted data one obtains the global momentum $P_i = \hat{P}_i$, the global energy E is given by (2) and the asymptotic mass reads $\sqrt{E^2 - P^2}$.

There are two established ways of solving the resulting (Lichnerowicz–York) equation.

(i) In the first approach, that takes care about the global topology of the manifold (the so-called conformal imaging method [24]; that actually requires the use of a larger set of extrinsic curvature data), Bowen and York solve Eq. (10) outside $r \geq a$, assuming that the sphere $r = a$ is a minimal surface and that at infinity the conformal factor ϕ goes to 1.

(ii) In the second approach, the puncture method, one splits ϕ into two parts, $\phi = 1 + m_1/(2r) + \phi_1$ and finds a solution ϕ_1 in the whole Euclidean space, demanding that at infinity $\phi_1 \approx d_1/(2r)$ [25]; here m_1 and d_1 are some constants, that are related to the global energy of the manifold.

In this paper we use the second approach in order to treat vacuum configurations with one or two black holes. The data for three exemplary configurations (chosen from a much bigger sample) with single black holes are presented in the first table. The first column is the parameter m_1 appearing in the preceding splitting, the second column is the linear momentum $\hat{P}$, and the third, fourth and fifth columns, respectively are the global mass m and the "horizon" masses $m_{\mathrm{M}} = \sqrt{S_{\mathrm{M}}/16\pi}$, $m_{\mathrm{H}} = \sqrt{S_{\mathrm{H}}/16\pi}$ and $m_{\mathrm{A}} = \sqrt{S_{\mathrm{A}}/16\pi}$. In the case of single boosted black holes the asymmetry causes the horizons to intersect the minimal surface, and this is why we consider the case PIM with A_{M}. The surfaces in the fourth and fifth columns do not coincide with apparent horizons, but consist of two (A_{H}) parts (of an apparent horizon to the future and to the past) or three segments (A_{M}) (of an apparent horizon to the future, a minimal surface and an apparent horizon to the past) — as explained in Sect. 3. It happens that the areas S_{A} of the apparent horizons to the past and to the future are equal; the sixth column brings corresponding data which (unexpectedly) obey the Penrose inequality. Each row describes a different configuration.

m_1	$\hat{P}$	m	m_{M}	m_{H}	m_{A}
8	2	8.061855	8.059402	8.059426	8.05948097
4	2	4.122407	4.110666	4.110751	4.11093771
4	5	4.707092	4.499335	4.500002	4.50143757

In this case areas of the past and future AH's are equal. It is clearly seen that all versions, the weaker (PIM) and the stronger (PIS) as well the last one *(iii)* of the Penrose inequality are satisfied. It is noticeable that the areas of A_{M}, A_{H} and A_{A} are very close. The fact of interest is the numerical evidence for the existence of parts of minimal surfaces that extend outward of outmost apparent horizons.

The corresponding results for two black hole configurations are comprised in the next table. Now the puncture method requires that $\phi = 1 + \sum_{i=1}^{2} m_i/(2|\vec{r} - \vec{r}_i|) + \phi_1$ (the two black holes are located at $\boldsymbol{r_i}, i = 1, 2$). The extrinsic curvature reads

$$K_{ij} = \frac{3}{2r^2\phi^2} \sum_{s=1,2} \left(\hat{P}_i^{(s)} \hat{n}_j^{(s)} + \hat{P}_j^{(s)} \hat{n}_i^{(s)} - \left(\hat{g}_{ij} - \hat{n}_i^{(s)} \hat{n}_j^{(s)} \right) \hat{P}_l^{(s)} \hat{n}^{(s)l} \right) , \quad (11)$$

where $\hat{P}^{(s)}$ aligned along the z-axis can be interpreted as the linear momentum of the $s - th$ black hole and $\vec{n}^{(s)} = (\vec{r} - \vec{R}_s)/|\vec{r} - \vec{R}_s|$. As before, one finds a solution ϕ_1 of the Lichnerowicz–York equation in the whole Euclidean space, demanding that at infinity $\phi_1 \approx d_1/(2r)$ [25]; as before m_1 and d_1 are some constants, that are related to the global energy of the manifold. The first and second columns describe parameters ("mass" m_1 and "momentum" $\hat{P}^{(1)}$) of the first black hole, the third and fourth columns give the same information about the second black hole. The fifth and sixth columns, respectively, are the global mass and the areal mass $m_{\mathrm{H}} = \sqrt{S_{\mathrm{H}}/16\pi}$. S_{H}

is the area of A_{H}, the 2-surface constructed according to the recipe *(ii)* of Sec. 2. The 2-surface A_{H} surrounds both black holes (located at $r = 1$ and $\theta = 0$ or $\theta = \pi$). In two cases (fifth and ninth) the past and future AH's do cross; the seventh column presents relevant values of $m_{\mathrm{A}} = \sqrt{S_{\mathrm{A}}/16\pi}$, where S_{A} is the larger of the two areas in question. They satisfy all formulations of the inequality.

m_1	$\hat{P}^{(1)}$	m_2	$\hat{P}^{(2)}$	m	m_{H}	m_{A}
5	–5	5	5	10.040901	10.033511	
5	–10	5	10	10.159675	10.132316	
5	–10	5	8	10.176694	10.153778	
5	–10	5	5	10.379353	10.347249	
5	5	5	5	11.146633	10.916395	10.916399
5	–0.25	5	0.25	10.000103	9.999632	
5	–2.5	5	2.5	10.010290	10.008087	
4	–1	5	1.5	9.006743	9.005594	
5	–2.5	5	0.5	10.052687	10.051593	10.051593

In the remaining seven cases the past and future horizons do not cross and there is a minimal surface in between them. Surprisingly — and in a sharp contrast with the corresponding case in spherically symmetric configurations — the Penrose inequality is valid simultaneously for past and future AH's. Another interesting observation is that, in all cases, when the parameter $\hat{P}$ tends to zero then AH's tend to the minimal surface. In this sense, the AH's bifurcate from the minimal surface. This is true, as matter of fact, in all numerical examples studied in Secs. 5 and 6. It happens that there always exists at least one minimal surface; those AH's horizons that bifurcate from the outermost one do satisfy the inequality. On the other hand, AH's that branch from an innermost minimal surfaces (there are several such cases in our sample of data) do break all aforementioned versions.

We show below an analytic argument that these observations remain true (with some reservations) for initial data with AH's that arise from small perturbations of data with minimal surfaces.

Theorem. Let l be a real parameter, $l\hat{K}_{ij}$ be the Bowen–York or Dain–Friedrich extrinsic curvature (multi-puncturized) and ϕ_l be a solution of the Lichnerowicz–York equation on R^3

$$\Delta\phi_l = -\frac{l^2\hat{K}_{ij}\hat{K}^{ij}}{8}\phi_l^{-7}\,. \tag{12}$$

Then ϕ_l, $K_{ij} = \hat{K}_{ij}/\phi_l^2$ constitute initial data of the Einstein equations; assume that for each l there exist apparent horizons that in the limit $l \to 0$ coincide with a nonspherical outermost minimal surface S_0. Then there exists l_0 such that for $|l| < l_0$ the Penrose inequality is satisfied.

Sketch of the proof. In the case of a nonspherical surface S_0 given by $r = r_0(\theta)$, one has a strict inequality, $\epsilon \equiv m_0 - \sqrt{A_0/(16\pi)} > 0$. One can easily show that: *(i)* (using arguments of [17]) $\phi_l \geq \phi_0$ and $m_l = m_0 + c_1 l^2$; *(ii)* (using the Green function of the flat Laplacian) $\phi_l < \phi_0 + c_2 l^2$. Here and below c_i $(i = 1, 2, \ldots)$ are some positive constants. The apparent horizon equations $\theta_+ = 0$ or $\theta_- = 0$ depend on l through the extrinsic curvature terms and the conformal factor ϕ. Since by assumption horizons bifurcate from S_0, they must be located within annulus $(r_0(\theta,\phi) - c_3 l, r_0(\theta,\phi) + c_4 l)$. (This is due to the implicit function theorem.) At an apparent horizon one should compare $m_l = m_0 + c_1 l^2$ with the area of AH's, which is bounded from above by $A_0 + c_5 l$. It is clear that by choosing l_0 small enough one can ensure that $m_l \geq \sqrt{A_{\mathrm{H}}/16\pi}$.

This proof is insensitive on the sign of l and therefore it is valid for both past and future apparent horizons. Any attempt to convert this local result into global one would have to be preceded by a careful estimate of the dependence of the location $r(\theta)$ of an AH on the bifurcation parameter l. Notice that this theorem does not apply to single-puncture initial data, since in this case the geometry corresponding to $l = 0$ is spherically symmetric and $\epsilon = 0$. On the other hand, this result should hold for the two-puncture solutions. There is also a possibility of generalizing the above onto case with initial data given in the exterior of a two-surface S_1, instead of R^3.

6. Dain–Friedrich conformally flat initial data

The main feature of these initial data is that the spatial part of the metric is conformally flat (as before), the momentum flow density vanishes (again, as before) and the extrinsic curvature is given, in spherical coordinates, by (the forthcoming formulae are translated from the language of the Newman–Penrose formalism, originally used in [26]).

$$
\begin{aligned}
K_i^3 &= 0\,, \quad \text{for } i = 1, 2 \\
K_1^1 &= \frac{1}{r^3\phi^6}\partial_x^2 W\,, \\
K_1^2 &= \frac{1}{\sin\theta r^3\phi^6}\partial_r\partial_x W\,, \\
K_2^2 &= \frac{1}{r^2\sin^2\theta\phi^6}\left[\partial_r\left(r\partial_r W\right) + \frac{1}{r}\left(x\partial_x W - W\right)\right].
\end{aligned}
\qquad (13)
$$

Here $x = \cos\theta$ and W is an arbitrary function of r and θ. The K_3^3 component can be found from the maximal slicing condition $K_i^i = 0$. Let us point out that this solution generalizes the Bowen–York solution of momentum constraint; the latter corresponds to a particular choice of W [27]. The data of (9), for instance, correspond to $W = r\hat{P}\left(x^3 - 3x\right)/2$. Extrinsic curvature (13) constitutes a partial case of the general solution found by Dain and Friedrich in 2001 [26].

The conformal factor ϕ satisfies the Lichnerowicz–York equation $\Delta\phi = -\frac{1}{8}K_{ij}K^{ij}\phi^5$. We seek a solution ϕ, using the puncture method, of the form $\phi = 1 + m_1/(2r) + \phi_1$.

The numerical calculations have been performed in the following cases:

(i) $\partial_x W = Pr(-x + x^3)$. It is noticeable that here the global momentum is nonzero. Numerical results are given in the forthcoming table.

m_1	P	m	m_{H}	m_{A}
4	5	5.622456	4.296457	4.296680
4	1	4.078569	4.015156	4.015260

(ii) $\partial_x W = \frac{Pr}{2}(1 - x^2)(3x^2 - 1)$. In this case the global momentum vanishes. The table presents the obtained results.

m_1	P	m	m_{H}	m_{A}
4	5	6.073919	4.366950	4.372253
4	1	4.109128	4.021109	4.021207

In both cases *(i)* and *(ii)* the two stronger versions, PIS and *(iii)*, of the Penrose inequality holds true. The 2-surface A_{H}, whose areal mass is depictured in the last but one column, is built from many sections of the intersecting past and future horizons; the number of the intersections seems to depend (for a given nonzero P) on the shape of W as a function of θ. There are more crossings in the case *(i)* (three) than in the case *(ii)* (only two).

7. Spheroidal systems with matter

Assume a foliation of the Euclidean space by oblate spheroids,

$$\frac{x^2 + y^2}{a^2\left(1 + \sigma^2\right)} + \frac{z^2}{a^2\sigma^2} = 1\,. \tag{14}$$

The variable σ changes from 0 to ∞, and angle variables are τ (changing from -1 to 1) and ϕ (varies as usual from 0 to 2π). Assume that there exists a normal flow of matter with the only nonzero component

$$j_\sigma = \frac{1}{8\pi}\phi^{-6}\frac{\sigma\left(\tau^2 - 1\right)}{\left(\sigma^2 + 1\right)\left(\sigma^2 + \tau^2\right)^{5/2}}\,. \tag{15}$$

Let $\hat{n}_i$ denote the unit normal (in the Euclidean metric) to a spheroid. The related traceless extrinsic curvature reads

$$K_i^j = -\phi^{-6}\left(\hat{n}_i\hat{n}^j - \frac{1}{3}g_i^j\right)\frac{1}{\left(\sigma^2+\tau^2\right)^{3/2}}\,; \tag{16}$$

this pair, K_{ij} and j_i, solves the momentum constraint part of Eq. (1). Let us remark, that one could always add to the extrinsic curvature the diagonal components $K_r^r = C/(\phi^6 r^3), K_\theta^\theta = K_\phi^\phi = -K_r^r/2$ (where C is a constant and $r = \sqrt{x^2+y^2+z^2}$) without changing the momentum flow. We do not do this, because our primary intention is to study the influence of the energy conditions onto the validity of the Penrose inequality, and the aforementioned part of the extrinsic curvature is irrelevant from this point of view.

The energy density ρ can be chosen in an arbitrary way, but the simplest possibility — that eases the analysis of the energy conditions — is to assume

$$\rho = C\times\phi^{-8}\times\frac{1}{8\pi}\frac{\sigma\left(1-\tau^2\right)}{\left(\sigma^2+1\right)\left(\sigma^2+\tau^2\right)^{5/2}}\,. \tag{17}$$

Later we shall put either $C = 1$ — which ensures the dominant energy condition — or $C = 0$, which breaks the energy conditions. Notice that

$$K_{ij}K^{ij} = \frac{2}{3}\times\phi^{-12}\times\frac{1}{\left(\sigma^2+\tau^2\right)^3}\,. \tag{18}$$

The Lichnerowicz–York equation takes now the form

$$\Delta\phi = -\frac{1}{12}\frac{1}{\left(\sigma^2+\tau^2\right)^3}\phi^{-7} - \frac{C}{4}\frac{\sigma\left(1-\tau^2\right)}{\left(\sigma^2+1\right)\left(\sigma^2+\tau^2\right)^{5/2}}\phi^{-3}. \tag{19}$$

This equation has been solved with ϕ tending to 1 at infinity and bearing a constant value at the inner boundary, that is assumed to be a unit sphere $r = 1$ in the background Euclidean geometry. In the examples shown below, horizons do not intersect and in all cases the outermost apparent horizon (future or past) has been located outside the minimal surface. (There appear also initial data with intersecting horizons, but they are not of particular interest.) Its area S_{H} enters the forthcoming data through the formula $m_{\mathrm{H}} = \sqrt{S_{\mathrm{H}}/16\pi}$.

We shall present data corresponding to:
(i) $C = 1$,

$\phi(r=1)$	m	m_{H}
2.4	2.801054	2.800734
2.5	3.0009593	3.0006660
3.5	5.0004402	5,0003797834

It is clear that $m_{\mathrm{H}} < m$ and that the version PIS holds true.

(ii) $C = 0$. The energy density vanishes and therefore the dominant energy condition is broken. One expects that the Penrose inequality (PIS) may be broken now, and in fact this is what happens, albeit only in the first two examples.

$\phi(r=1)$	m	m_{H}
2.4	2.80004295	2.80011252
2.5	3.00003632	3.000604868
3.5	5.000009651	5.000005477

8. Concluding remarks

The weakest form of the Penrose inequality due to Horowitz is that $m \geq \sqrt{S_{\mathrm{M}}/16\pi}$, where S_{M} is the smallest area of a two-surface encompassing a region with apparent horizons satisfying the assumptions of the singularity theorems. This paper deals with three other formulations, of which even the weakest (PIM) is stronger than the Horowitz's one, because our notion of the outermost apparent horizon is weaker than that required by the singularity theorems. Despite this fact, all investigated statements of the Penrose inequality are confirmed by our numerical analysis for vacuum initial data and for those systems with matter that satisfy an energy condition. The only negative examples correspond to data with matter that does not satisfy an energy condition.

There exist minimal surfaces in all investigated examples with vacuum; these are mostly singlets but in a number of cases also doublets. It is observed that the two (past and future) AH's, that bifurcate (with the momentum being the bifurcation parameter — see the end of Sec. 5) from the outermost minimal surface, do satisfy — regardless of whether they cross or do not cross — all versions of the Penrose inequality. An analytic proof that this is true (with some reservations), at least for initial data with small extrinsic curvature, is sketched at the end of Sec. 5.

In summary, results of this work show that the Penrose inequality does not hold only when expected not to hold; that points strongly in favour of the validity of the Penrose's conjecture in physically interesting cases.

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