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ON STABILITY OF RENORMALISED CLASSICAL ELECTRODYNAMICS

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Dedicated to Professor Andrzej Staruszkiewicz in honour of his 65-th birthday

It is shown that the total energy of the static "field + particle" system, defined in the framework of classical, renormalised electrodynamics of particles and fields, depends in an unstable way upon the field boundary data. It is argued that this phenomenon may be also an origin of the unstable dynamical behaviour of the system (*i.e.* existence of "runaway solutions"). It is proved that a suitable polarisation mechanism of the particle restores the stability, at least on the level of statics. Whether or not it restores also the full, dynamical stability of the theory is still an open question.

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1. Introduction

Classical electrodynamics in its present form is unable to describe interaction between charged particles intermediated by electromagnetic field. Indeed, typical well posed problems of the theory are of the contradictory nature: either we solve partial differential equations for the unknown field, knowing (*a priori* !) trajectories of charged particles (and, therefore, knowing the field sources), or we solve ordinary differential equations for the trajectories of test particles, knowing (*a priori* !) the field (and, therefore, knowing the forces acting on particles). Combining these two procedures into a single theory leads to a contradiction: in case of the point particles, the Lorentz force due to the self-interaction is infinite.

There were many attempts to overcome these difficulties. One of them consists in using the Lorentz–Dirac equation (see [1, 3, 10]) where electromagnetic field is split into retarded (singular) part and the nonsingular rest. The retarded field acts on the particle via postulated effective force and the nonsingular (finite) part enters into equation *via* the Lorentz force. Unfortunately, this approach leads to the so called runaway solutions which are unphysical.

Various remedies have been proposed to cure this disease (*cf.* [17]), most of them just based on a fine tuning of boundary conditions. Unfortunately, such a tuning excludes physically interesting problems (*i.e.* circular motion) and the question arises if one can construct a theory which does not contain unphysical solutions at all. The authors believe that to achieve the above goal we should first gain a deeper understanding of foundations of the runaway behaviour.

Andrzej Staruszkiewicz got involved in these issues since the beginning of his academic career (*cf.* [14] and [15]). His main contribution to the subject consists in an interesting proposal of a first mathematically consistent relativistic mechanics of two point particles, where one particle moves in the retarded field of the second one while the second particle moves in the advanced field of the first one, see [16].

In the present paper we perform an analysis of the stability properties of a theory proposed by one of us in papers [4] and [2]. It consists in defining an "already renormalised" four-momentum of the physical system composed of both particles and fields. We will refer to it as to "particle(s) + fields" system. Equations of motion are then derived as a consequence of a conservation law imposed on the four-momentum. We deeply believe that such an approach is a correct realization of the Einstein's programme of "deriving equations of motion from field equations" and that a similar procedure should be applied to formulate also the two-body-problem in General Relativity Theory.

In the framework of this theory we show that the physical instability is inherently contained in the renormalisation method used. More precisely: in the simplest renormalisation scheme the amount of energy contained "in the interior of the particle" decreases when the external field surrounding the particle increases. This contradicts the stability of the model. As a remedy for this drawback we propose the polarisability of the particle. Numerical analysis of such an improved model shows validity of our proposal.

For this purpose we derive the theory of polarisability. We show that the sensitivity of the particle to polarisation must is uniquely determined by the dependence of the mass of the particle upon its dipole moment. Violating this law would lead to a non-local theory (see Section 5).

In this paper we analyse the renormalised energy of the "particle + field" system on the level of statics only, but the "energetic instability" discovered here is obviously the reason for the runaway behaviour of the dynamical system as well. Indeed, the price which we pay for perturbing the particle at rest is negative and, whence, the system works like a *perpetuum mobile*. It is, therefore, very likely that also the price accelerating the particle is negative. This observation is fundamental, in our opinion, to understand the physical reasons for the runaway behaviour of the theory and in search for a remedy for this phenomenon.

The paper is organised as follows. In Section 2 the renormalisation procedure proposed by one of us in [4] (see also [2]) is presented. Then a monopole particle inside a fixed volume V is considered: we compute renormalised energy of the system and vary it with respect to particle's position. Next, we suppose that the particle assumes position corresponding to minimal value of the energy. In this way we express the energy of the system as a function of the field boundary data, imposed on ∂V. Finally, we analyse stability of the system under small changes of these data. Here, both the Dirichlet-type and the Neumann-type boundary problems are considered.

The above results are then applied to a case of a monopole particle closed in spherical box. We prove that such a system *is not* stable. Then we consider a polarisable particle. Here, the external field may generate a non-vanishing dipole momentum, which changes completely the energy balance. It turns out that for a Heaviside-like relation between the field and the dipole momentum it generates, the system is stable. This suggests a possible way to improve in the future our renormalisation method and to avoid (maybe) also dynamical instabilities, manifesting themselves in the runaway behaviour.

2. The renormalised four-momentum vector

Full description of the renormalised electrodynamics was proposed in [4] or [2]. In the present section we review briefly heuristic ideas that stand behind definition of the renormalised four-momentum of the dynamical "particle + field" system.

As a starting point of our considerations take an *extended particle* model. This means that we consider a fully relativistic, gauge-invariant, interacting "matter + electromagnetism" field theory, which is possibly highly nonlinear, but reduces in vacuum to the linear Maxwell theory if the electromagnetic field is sufficiently weak. A moving particle is described by a solution of the theory, such that the "non-linearity-region" (or the "strong-field-region") is concentrated in a tiny world tube $\mathcal{W}$ around a smooth, time-like trajectory ζ. We assume that outside of this tube the fields describing charged matter practically vanish and the electromagnetic field is sufficiently weak to be well described by the linear Maxwell theory. Let $\mathfrak{T}$ denotes the energy-momentum tensor of such a field configuration. The four momentum of the "matter + electromagnetism" system is then obtained by integration of $\mathfrak{T}$ (conserved — due to Noether Theorem) over a space-like hyperplane Σ:

$$\mathcal{P}_\lambda = \int\limits_{\Sigma} \mathfrak{T}^\mu{}_\lambda d\sigma_\mu \,. \tag{1}$$

We assume, moreover, that this fundamental theory admits a static, stable, soliton-like solution, which will be called a *extended particle at rest*. Let $\mathfrak{T}^{\text{static}}$ denote its energy-momentum tensor and let m be the *total* energy (mass) of this solution:

$$m = \int\limits_{\Sigma} \mathfrak{T}^{\text{static } \mu}{}_\lambda d\sigma_\mu \,. \tag{2}$$

Due to relativistic invariance, we have also a six parameter family of solutions obtained by acting with Poincaré transformations on the static solution. Each of these solutions may be called a "uniformly moving extended particle" because the strong-field region (interior of the particle) is concentrated around the straight line $\dot{\vec{x}}$ =const. As we boost the static solution to the four-velocity u_λ, we denote by $\mathfrak{T}^{\text{static}}(u)$ its energy-momentum tensor. Then the four-momentum of this solution equals mu_λ and we have:

$$mu_\lambda = \int\limits_{\Sigma} \mathfrak{T}^{\text{static}}(u)^\mu{}_\lambda d\sigma_\mu \,. \tag{3}$$

This leads to a trivial identity:

$$\mathcal{P}_\lambda = mu_\lambda + \int\limits_{\Sigma} \left(\mathfrak{T}^\mu{}_\lambda - \mathfrak{T}^{\text{static}}(u)^\mu{}_\lambda\right) d\sigma_\mu \,, \tag{4}$$

which becomes extremely useful in the following arrangement. Choose the straight line describing the "trajectory" of the uniformly moving extended particle in such a way that it is tangent to the approximate trajectory ζ of the generic extended particle at their intersection point with Σ. If $K(R) \subset \Sigma$ denotes the ball of radius R, which contains the strong field region of both solutions, but is small with respect to the characteristic distance of the external Maxwell fields, then we have:

$$\mathcal{P}_\lambda = mu_\lambda + \int\limits_{\Sigma - K(R)} \left(\mathfrak{T}^\mu{}_\lambda - \mathfrak{T}^{\mathrm{static}}(u)^\mu{}_\lambda\right) d\sigma_\mu + \int\limits_{K(R)} \left(\mathfrak{T}^\mu{}_\lambda - \mathfrak{T}^{\mathrm{static}}(u)^\mu{}_\lambda\right) d\sigma_\mu \,. \tag{5}$$

Our assumption about stability of the soliton-like solution means that the last integral is negligible since inside the strong field region both solutions are very close to each other. But the first integral contains only contributions from external Maxwell fields accompanying both particles. This way we have proved that the following formula:

$$\mathcal{P}_\lambda \simeq mu_\lambda + \int\limits_{\Sigma - K(R)} \left(\mathfrak{T}^\mu{}_\lambda - \mathfrak{T}^{\mathrm{static}}(u)^\mu{}_\lambda\right) d\sigma_\mu \,, \tag{6}$$

containing only external Maxwell field surrounding the particle, provides a good approximation of the total four-momentum of the "particle + field" system.

The theory proposed in [4] consists in mimicking the above formula in the point particle case. Hence, we consider solutions of Maxwell equations having a "delta-like" current corresponding to a point charge e travelling over a trajectory ζ. Such a solution is treated as an idealised description of external properties of the extended particle considered above. Denote by T the energy momentum tensor of this solution. The uniformly moving particle, whose four-velocity equals u, is represented in this picture by a boosted Coulomb field, and its energy-momentum tensor is denoted by $T^{\mathrm{static}}(u)$. If trajectories of both particles are again tangent with each other at their common point of intersection with Σ, then momentum (6) may be rewritten as:

$$\mathcal{P}_\lambda \simeq mu_\lambda + \int\limits_{\Sigma - K(R)} \left(T^\mu{}_\lambda - T^{\mathrm{static}}(u)^\mu{}_\lambda\right) d\sigma_\mu \,, \tag{7}$$

because outside of the strong-field region, $\mathfrak{T}$ reduces to T and $\mathfrak{T}^{\mathrm{static}}(u)$ reduces to $T^{\mathrm{static}}(u)$. The main observation done in [4] is that, due to cancellation of principal singularities of both T and $T(u)$, the above integration may be extended to the entire Σ. More precisely, the following quantity:

$$\mathcal{P}_\lambda := mu_\lambda + P\int\limits_{\Sigma} \left(T^\mu{}_\lambda - T^{\mathrm{static}}(u)^\mu{}_\lambda\right) d\sigma_\mu \tag{8}$$

is well defined ("P" denotes the "principal value" of the integral). According to the discussion above, we interpret this quantity as the total four-momentum of the interacting system composed of the point particle and the Maxwell field accompanying the particle. Consequently, we impose conservation of $\mathcal{P}$ as an additional condition. This implies equations of motion of the point particle as a good approximation of equations of motion of the true, extended particle.

This approach has an obvious generalisation to the system of many particles (see [4]). Also polarisable particles, carrying magnetic or electric moment (and — consequently — displaying stronger field singularity than the Coulomb field) may be treated this way (*cf.* [8]). Recently, the above approach was improved by replacing the reference Coulomb field in (8) by the Born field, matching not only particle's velocity but also its acceleration. This way the principal-value-sign "P" may be omitted in the definition because the corresponding integral converges absolutely (*cf.* [9]).

In what follows, we are going to apply definition (8) to static "particle + field" configurations only.

3. Electrostatics of a monopole particle

Consider now electrostatic field D surrounding the particle with charge e, situated at the point $\vec{r}_0$. Due to Maxwell equations, the Gauss law:

$$\nabla D = e\boldsymbol{\delta}\left(\vec{r} - \vec{r}_0\right) , \tag{9}$$

must be satisfied, where by $\boldsymbol{\delta}$ we denote Dirac delta distribution (in contrast with conventional δ, denoting variation of a function). It is, therefore, convenient to decompose the field into its singular and regular parts:

$$D = D_{\text{reg}} + D_{\text{sing}}, \tag{10}$$

where the singular part D_{sing} is simply the Coulomb field:

$$D_{\text{sing}} := \frac{e\left(\vec{r} - \vec{r}_0\right)}{4\pi\|\vec{r} - \vec{r}_0\|^3} ,$$

whereas the remaining field $D_{\text{reg}} := D - D_{\text{sing}}$ is divergenceless: $\nabla D_{\text{reg}} = 0$. Moreover, static Maxwell equations imply the existence of the scalar potential ϕ: $D = -\nabla\phi$. Hence, we have: $\Delta\phi_{\text{reg}} = 0$.

According to (8), the complete energy of this "particle + field" system contained in the entire Σ equals:

$$\mathcal{H} = m + \frac{1}{2}\int\limits_{\Sigma} \left(D^2 - D_{\text{sing}}^2\right) dv\,. \tag{11}$$

Consider now a fixed volume $V \ni \vec{r}_0$ containing the particle. Subtracting from $\mathcal{H}$ the electrostatic energy contained outside of V:

$$\mathcal{H}_{\mathbb{R}^3-V} = \frac{1}{2} \int\limits_{\mathbb{R}^3-V} D^2 dv \,, \tag{12}$$

we obtain the total energy contained in V:

$$\begin{aligned} \mathcal{H}_V \; &= \; m - \frac{1}{2} \int\limits_{\mathbb{R}^3-V} {D_{\mathrm{sing}}}^2 dv + \frac{1}{2} \int\limits_V {D_{\mathrm{reg}}}^2 dv \\ &\quad + \int\limits_V D_{\mathrm{sing}} D_{\mathrm{reg}} dv \,. \end{aligned} \tag{13}$$

Given boundary conditions, we are going to minimise the above quantity with respect to the particle's position $\vec{r}_0 \in V$. Assuming that the particle always tries to minimise the energy of the system, we can write both $\vec{r}_0$ and the total "particle+field" energy as functions of the field boundary data. Stability of the energy with respect to the boundary data on ∂V will then be studied. Before we pass to the above programme, we must specify which kind of boundary conditions on ∂V have to be controlled.

3.1. Neumann conditions

Varying the energy integral (13) with respect to the particle's position we get:

$$\begin{aligned} \delta \mathcal{H}_V = &\int\limits_V \left\{ D_{\mathrm{reg}} \cdot (\delta D_{\mathrm{reg}} + \delta D_{\mathrm{sing}}) + D_{\mathrm{sing}}\, \delta D_{\mathrm{reg}} \right\} dv \\ &- \int\limits_{\mathbb{R}^3-V} D_{\mathrm{sing}}\, \delta D_{\mathrm{reg}}\, dv \,. \end{aligned} \tag{14}$$

For Neumann conditions we put $D = -\nabla \phi$ for both the regular and the singular parts of the field, outside of the variation δ. Integrating by parts and using $\nabla D_{\mathrm{reg}} = 0$ we get:

$$\delta \mathcal{H}_V = \int\limits_V \phi_{\mathrm{reg}}\, \delta(\nabla D_{\mathrm{sing}}) dv - \int\limits_{\partial V} \left\{ \phi\, \delta D^{\perp} \right\} d\sigma \,. \tag{15}$$

But the variation of (9) gives us:

$$\delta(\nabla D_{\mathrm{sing}}) = \delta \left(e \boldsymbol{\delta} \left(\vec{r} - \vec{r}_0 \right) \right) = -e \partial_k \left(\boldsymbol{\delta} \left(\vec{r} - \vec{r}_0 \right) \right) \delta x_0^k \,, \tag{16}$$

where δx_0^k denotes a virtual displacement of the particle. Imposing Neumann conditions $D^{\perp}|\partial V = f$, where f is a fixed function, we obtain: $\delta D^{\perp} \equiv 0$ on ∂V. Hence, the surface integral vanishes. Inserting (16) into (15) we derive the following formula:

$$\delta \mathcal{H}_V = -e D_k^{\mathrm{reg}}(x_0^k)\delta x_0^k \,. \tag{17}$$

We conclude that the extremum of energy condition implies the following static equilibrium equation:

$$D_k^{\mathrm{reg}}(x_0^k) = 0 \,. \tag{18}$$

3.2. Dirichlet conditions

For Dirichlet case we put $\delta D = -\nabla \delta\phi$ for both the regular and the singular parts of the field and then integrate (14) by parts. We obtain:

$$\delta \mathcal{H}_V = \int_V \left(\nabla D_{\mathrm{sing}}\right) \delta\phi_{\mathrm{reg}} dv - \int_{\partial V} \left\{ D^{\perp}\, \delta\phi \right\} \, d\sigma \,. \tag{19}$$

Imposing Dirichlet conditions $\phi|\partial V = f$, where f is a fixed function, we obtain: $\delta\phi \equiv 0$ on ∂V and, therefore, the surface integral vanishes again. To derive the equilibrium condition (18) from the variational principle, we must perform the following Legendre transformation:

$$\begin{aligned} \int_V \left(\nabla D_{\mathrm{sing}}\right) \delta\phi_{\mathrm{reg}} dv =& \delta \int_V \left(\nabla D_{\mathrm{sing}}\right) \phi_{\mathrm{reg}} dv \\ & - \int_V \left(\delta \nabla D_{\mathrm{sing}}\right) \phi_{\mathrm{reg}} dv \,. \end{aligned} \tag{20}$$

Then we use (9) and (16). This way we obtain:

$$\delta \left(\mathcal{H}_V - e\phi_{\mathrm{reg}}(\vec{r}_0)\right) = D_k^{\mathrm{reg}}(x_0^k)\delta x_0^k \,. \tag{21}$$

Comparing (17) and (21) we observe that the equilibrium condition (18) may either be obtained from the variational principle $\delta\left(\mathcal{H}_V\right) = 0$, when the Neumann boundary data are controlled, or from the variational principle $\delta\left(\mathcal{F}_V\right) = 0$, with $\mathcal{F}_V := \mathcal{H}_V - e\phi_{\mathrm{reg}}(\vec{r}_0)$, when the Dirichlet boundary data are controlled. The quantity $\mathcal{H}_V$ is the total energy of the "particle + field" system, whereas $\mathcal{F}_V$ is an analog of the free energy in thermodynamics. We conclude that imposing Neumann condition on the boundary corresponds to the adiabatic insulation of the system, whereas imposing Dirichlet condition means that we expose it to a kind of a "thermal bath". Indeed, imposing *e.g.* condition $\phi|\partial V = 0$ we must cover the surface ∂V with a metal shell and ground it electrically. This means that we admit energy exchange of our system with the earth. Similarly as in thermodynamics, the free energy $\mathcal{F}_V$, which we optimise, contains not only the system's energy $\mathcal{H}_V$ but also the term "$-e\phi_{\mathrm{reg}}(\vec{r}_0)$" which we interpret as energy of the "boundary-condition-controlling device". Of course, from the point of view of the particle, both conditions lead to the same equation: $D^{\mathrm{reg}}(x_0^k) = 0$ because our theory is *local* and the particle interacts with its immediate neighbourhood *only*, no matter how the boundary data are controlled far away from the particle.

4. An example — monopole particle in a spherical box

In this section we shall analyse stability of a charged, monopole particle closed in a spherical box with radius R: $V = K(0, R) \subset \mathbb{R}^3$. Simplicity of the model allows us to solve explicitly the static Maxwell equations (for both the Neumann and the Dirichlet cases) and to compute renormalised energy of the system. Then we will find the *extremum* of the energy function with respect to the particle's position and check that for the Neumann case we get the minimum and for the Dirichlet case — the maximum of the energy. Assuming that the particle always minimises the energy, we will express energy function in terms of the boundary data and show that the system is unstable under small changes of these data.

The problem consists in solving equation $\Delta\phi = -e\delta(\vec{r} - \vec{r}_0)$, where $\vec{r}_0 \in K(0, R)$. In the Neumann case we impose the following condition:

$$\vec{D} \cdot \vec{n}\big|_{r=R} = \vec{E} \cdot \vec{n} + \frac{e}{4\pi R^2}\,, \tag{22}$$

where $\vec{E}$ is a fixed three dimensional vector.

In the Dirichlet case we impose the following condition:

$$\phi\big|_{r=R} = -\vec{E} \cdot \vec{n}\, R + \frac{e}{4\pi R}\,. \tag{23}$$

Because of the axial symmetry of the problem, we may restrict ourselves to the analysis of the energy functional at points $\vec{r}_0$ which are parallel to $\vec{E}$: $\vec{r}_0 \| \vec{E}$. With this simplification, we are able to find an explicit solution $\phi = \phi_{\mathrm{sing}} + \phi_{\mathrm{reg}}$, where:

$$\phi_{\mathrm{sing}} = \frac{1}{4\pi} \frac{e}{|\vec{r} - \vec{r}_0|}$$

in both Dirichlet and Neumann cases (*cf.* Appendices A and C). To write an explicit formula for ϕ_{reg} it is useful to introduce the following variable:

$$r_0 := \frac{1}{\|E\|}(\vec{E}|\vec{r}_0)\ ,$$

which runs from $-R$ to R. Under this convention we obtain:

$$\begin{aligned}\phi_{\rm reg} \;=\; & \frac{e}{4\pi}\Bigg(\frac{R}{\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta}}-\frac{1}{R}\\ & -\frac{1}{R}\ln\left|R^2-r_0r\cos\theta+\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta}\right|\Bigg)\\ & -\vec{E}\vec{r}+\frac{1}{R}\ln(2R^2) \qquad (24)\end{aligned}$$

in the Neumann case, whereas:

$$\phi_{\rm reg}=\frac{e}{4\pi}\left(\frac{1}{R}-\frac{R}{\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta}}\right)-\vec{E}\vec{r} \qquad (25)$$

in the Dirichlet case.

4.1. Stability

In both cases, the renormalised energy can be computed explicitly. Denoting $E:=\|\vec{E}\|$ we obtain the following result:

$$\begin{aligned}\mathcal{H}_{\mathcal{N}} \;=\; & m+\frac{1}{2}\Bigg(\frac{e^2}{4\pi}\left(\frac{R}{R^2-r_0^2}-\frac{1}{R}\ln\left|1-\frac{r_0^2}{R^2}\right|-\frac{2}{R}\right)\\ & +\frac{4}{3}\pi R^3E^2-2eEr_0\Bigg)\,, \qquad (26)\end{aligned}$$

in the Neumann case (*cf.* Appendix B) and:

$$\mathcal{H}_{\mathcal{D}}=m+\frac{1}{2}\left(\frac{4}{3}\pi R^3E^2-\frac{e^2}{4\pi}\frac{R}{R^2-r_0^2}\right)\,, \qquad (27)$$

in the Dirichlet case (*cf.* Appendix C). Finally, we compute the electric "free energy" $\mathcal{F}=\mathcal{H}-e\phi_{\rm reg}(\vec{r}_0)$ in the Dirichlet case:

$$\mathcal{F}=m+\frac{1}{2}\left(\frac{e^2}{4\pi}\frac{R}{R^2-r_0^2}+2eEr_0+\frac{4}{3}\pi R^3E^2-\frac{e^2}{4\pi}\frac{2}{R}\right)\,. \qquad (28)$$

We see that the equilibrium condition in the Neumann case reads:

$$D_{\rm reg}\big|_{\vec{r}=\vec{r}_0}=0\Leftrightarrow\left(eE-\frac{e^2}{4\pi}\frac{r_0}{R(R^2-r_0^2)}\right)=0\,, \qquad (29)$$

whereas in the Dirichlet case it reads:

$$
\begin{aligned}
eD_{\mathrm{reg}}\big|_{\vec{r}=\vec{r}_0} &= 0 \Longleftrightarrow \\
\frac{e^2}{4\pi}\frac{Rr_0}{(R^2-r_0^2)^2}+eE &= \frac{\partial}{\partial r_0}\mathcal{F}=0\,.
\end{aligned} \tag{30}
$$

We express the energy in terms of the following, standardised variables:

$$
x=\frac{r_0}{R}\in]-1,1[, \qquad q=\frac{4\pi R^2}{e}E\,. \tag{31}
$$

Denoting:

$$
\mathcal{H}'=(\mathcal{H}-m)\frac{8\pi R}{e^2}\,, \tag{32}
$$

we obtain:

$$
\mathcal{H}'_{\mathcal{N}}=\frac{1}{1-x^2}-\ln|1-x^2|-2qx+\frac{1}{3}q^2-2\,, \tag{33}
$$

$$
\mathcal{H}'_{\mathcal{D}}=\frac{1}{3}q^2-\frac{1}{1-x^2}\,. \tag{34}
$$

Observe that for $q=0$ both energies may be expanded as follows (*cf.* Fig. 1):

$$
\mathcal{H}'_{\mathcal{N}} = -1+2x^2+O(x^4)\,, \tag{35}
$$

$$
\mathcal{H}'_{\mathcal{D}} = -1-x^2+O(x^4)\,. \tag{36}
$$

This implies that only in the Neumann case the equilibrium point ($x=0$) is also a minimum of the energy. In the Dirichlet case the energy has a local maximum at the equilibrium point. As may be easily seen, this happens also for any value of E. Hence, for the Dirichlet case the free energy $\mathcal{F}$ should be used, for which local extremum is also minimum. In what follows we shall use the true (local) energy and consequently, we restrict ourselves to the Neumann case only.

4.2. Neumann conditions

In terms of the standardised variables, the equilibrium condition (29) reads:

$$
q=\frac{x(2-x^2)}{(1-x^2)^2}\,. \tag{37}
$$

For small values of q this enables us to express equilibrium position in terms of the boundary data:

$$
x\approx\frac{q}{2}\,. \tag{38}
$$

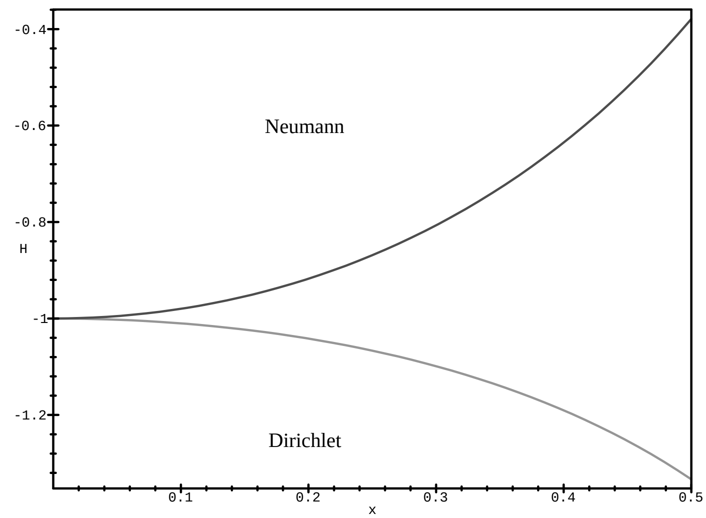

Fig. 1. Graph of renormalised energy *vs* particle's position and $q = 0$ for $\mathcal{H}'_{\mathcal{N}}$ and $\mathcal{H}'_{\mathcal{D}}$.

The same result could be obtained from the following expansion:

$$\mathcal{H}'_{\mathcal{N}}(x,q) \;=\; -1 + \frac{1}{3}q^2 - 2qx + 2x^2 + O(x^4)\,, \tag{39}$$

$$\partial_x \mathcal{H}'_{\mathcal{N}}(x,q) \;=\; 0 \Rightarrow x \approx \frac{q}{2}\,. \tag{40}$$

Knowing relation between position of the particle and the boundary data we express the energy in terms of boundary data *only*:

$$\mathcal{H}'_{\mathcal{N}}(x,q)|_{x=\frac{q}{2}} = -1 - \frac{1}{6}q^2 + O(q^3)\,. \tag{41}$$

Observe that for *increasing* values of q, the energy of the system *decreases* (*cf.* figure 2)! The system "particle + field" turns out to be *unstable* — even small fluctuations of the external field q can decrease its total energy. This means that the particle behaves like a *perpetuum mobile*, providing a source of energy at no costs. In our opinion this unphysical feature of the model, manifestly seen in its static behaviour, could possibly be a source of its dynamical instability, *i.e.* the existence of "runaway" solutions of Dirac equation. As a remedy, described in the sequel, we propose to equip the particle with an additional mechanism which, *via* electric polarisability, will restore its static stability.

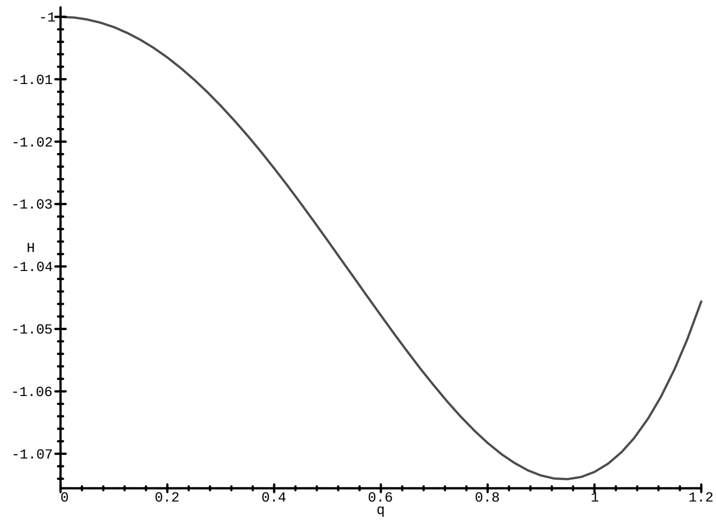

Fig. 2. Graph of renormalised energy *vs* boundary field q for $\mathcal{H}'_{\mathcal{N}}(x(q), q)$.

5. Polarisable particle

We assume that the particle may get a non-vanishing electric dipole moment due to interaction with the neighbouring field. We prove in the sequel that, under a suitable choice of the polarisability properties of the particle, the resulting "particle + field" system becomes statically stable.

For a polarised particle, formula (13) for the total energy remains valid but the field singularity is now deeper than in (9), namely:

$$\nabla D = \nabla D_{\text{sing}} = e\boldsymbol{\delta}(\vec{r} - \vec{r}_0) - p^k \partial_k \boldsymbol{\delta}(\vec{r} - \vec{r}_0)\,, \tag{42}$$

where p^k is a dipole moment. We assume that p^k has been generated by the surrounding electric field D according to some law $p = p(D_{\text{reg}}(\vec{r}_0))$, describing the sensitivity of the particle. Moreover, we admit the dependence of the coefficient m in (8) (and, consequently, in (13)) upon polarisation. It will be shown in the sequel that insisting in having m constant we are not able to make the model physically consistent. Moreover, it will be shown that the electric sensitivity is *uniquely* implied by the dependence $m = m(p)$.

5.1. Variational principle

Variation of the renormalised energy (13) with respect to the particle's position contains now the non-vanishing term δm. Similar calculations as for the scalar particle lead, in case of the Neumann boundary conditions, to formula:

$$\delta\mathcal{H}_V \;=\; \delta m + \int\limits_V \phi_{\rm reg}\delta(\nabla D_{\rm sing})dv - \int\limits_{\partial V} \left\{\phi\,\delta D^{\perp}\right\} d\sigma\,, \tag{43}$$

and, in case of the Dirichlet conditions, to:

$$\begin{aligned}\delta\mathcal{H}_V &= \delta m + \int\limits_V (\nabla D_{\rm sing})\delta\phi_{\rm reg}dv - \int\limits_{\partial V} \left\{D^{\perp}\delta\phi\right\} d\sigma \\ &= \delta m + \delta \int\limits_V (\nabla D_{\rm sing})\phi_{\rm reg}dv - \int\limits_V \phi_{\rm reg}\delta(\nabla D_{\rm sing})dv - \int\limits_{\partial V} \left\{D^{\perp}\delta\phi\right\} d\sigma\,. \end{aligned}\tag{44}$$

According to (42), the new version of formula (16) reads:

$$\begin{aligned}\delta(\nabla D_{\rm sing}) \;=\; &-\left(e\partial_k\boldsymbol{\delta}(\vec r-\vec r_0) - p^j\partial_j\partial_k\boldsymbol{\delta}(\vec r-\vec r_0)\right)\delta x_0^k \\ &-\left(\partial_k\boldsymbol{\delta}(\vec r-\vec r_0)\right)\delta p^k\,. \end{aligned}\tag{45}$$

Plugging (45) into (43) we see that the total energy variation splits into the sum of two pieces: the work due to virtual displacement of the particle and the remaining work, due to variation of m and p:

$$\begin{aligned}\delta\mathcal{H}_V \;=\; &\underbrace{-\left(eD_{\rm reg} + p^k\partial_k D_{\rm reg}\right)\big|_{\vec r=\vec r_0}\delta\vec r_0}_{\mathcal{A}} \\ &+\underbrace{\delta m - D_{\rm reg}\big|_{\vec r=\vec r_0}\delta p}_{\mathcal{B}}\,. \end{aligned}\tag{46}$$

The second part $\mathcal{B}$ is obviously *nonlocal* — both the mass m and the moment p depend upon the value of $D_{\rm reg}(\vec r_0)$. This quantity must be obtained from the field equation: $\Delta\phi_{\rm reg} = 0$, with boundary value depending upon the particle's position. The only way to save locality of the model is to force the term $\mathcal{B}$ to vanish *identically* by imposing the following constraint:

$$\delta m = D_{\rm reg}(\vec r_0)\delta p\,. \tag{47}$$

Denoting by $m_0 = m(0)$ the mass of the unpolarised particle and by $f(p)$ the additional polarisation energy:

$$m(p) = m_0 + f(p)\,, \tag{48}$$

formula (47) may be written as:

$$D_k^{\rm reg}(\vec r_0) = \frac{\partial f(p)}{\partial p^k}\,. \tag{49}$$

We see that the polarisation energy f must play a role of the generating function for the polarisability relation, otherwise the model would not be local. Indeed, suppose that $\mathcal{B}$ does not vanish and the particle's equilibrium condition implies vanishing of the whole right hand side of (46). To decide whether or not its actual position is acceptable as an equilibrium position, the particle must know not only the field in its immediate neighbourhood, but also the shape of V and the field boundary data on ∂V. Such a behaviour would be physically non-acceptable.

Inverting the generating formula (49), we may find the dependence $p = p(D_{\mathrm{reg}}(\vec{r}_0))$, which is uniquely implied by the "equation of state" (48). It may be rewritten as follows:

$$p^k = -\frac{\partial h(D)}{\partial D_k}\,, \tag{50}$$

where

$$h(D) = f - D_k p^k \tag{51}$$

describes the Legendre transformation between "canonically conjugate" quantities p and D and may be interpreted as the "free polarisation energy", whereas the "true energy" (48) may be rewritten as:

$$m(D) = m_0 + D_k p^k + h(D)\,. \tag{52}$$

Hence, (46) implies:

$$\delta\mathcal{H}_V = -\left(eD_{\mathrm{reg}} + p^k\partial_k D_{\mathrm{reg}}\right)|_{\vec{r}=\vec{r}_0}\delta\vec{r}_0\,, \tag{53}$$

and the equilibrium condition becomes a local equation:

$$\left(eD_{\mathrm{reg}} + p^k\partial_k D_{\mathrm{reg}}\right)|_{\vec{r}=\vec{r}_0} = 0\,. \tag{54}$$

A similar procedure works in the Dirichlet case as well. Applying the state equation to (44) we obtain:

$$\delta\mathcal{F}_V = \left(eD_{\mathrm{reg}} + p^k\partial_k D_{\mathrm{reg}}\right)|_{\vec{r}=\vec{r}_0}\delta\vec{r}_0\,, \tag{55}$$

where the "free energy" $\mathcal{F}_V$ is given as:

$$\begin{aligned}\mathcal{F}_V \;&:=\; \mathcal{H}_V - \int_V (\nabla D_{\mathrm{sing}})\phi_{\mathrm{reg}} - 2f \\ &:=\; \mathcal{H}_V - e\phi_{\mathrm{reg}}(\vec{r}_0) + D_{\mathrm{reg}}\big|_{\vec{r}=\vec{r}_0}\cdot p - 2f\,.\end{aligned} \tag{56}$$

Equilibrium condition $\delta\mathcal{F}_V = 0$ reduces to the same, local equation (54).

6. An example — polarisable particle in a spherical box

Let us come back to the simple model described in Section 4 on page 83. For the polarisable particle we must solve the field equation:

$$\Delta\phi = -e\delta(\vec{r}-\vec{r}_0) + \vec{p}\cdot\nabla(\delta(\vec{r}-\vec{r}_0))\,, \tag{57}$$

where $\vec{r}_0 \in K(0,R)$, with either Neumann (22) or Dirichlet condition (23). We want to compute renormalised total energy of the "particle + field" system and to prove that for a suitable state equation (48) our model becomes stable.

Splitting the solution ϕ into two parts:

$$\phi = \phi^{\mathrm{mon}} + \phi^{\mathrm{dip}}\,, \tag{58}$$

where by ϕ^{mon} we denote the solution of the monopole problem, found earlier (*cf.* Section (4), page 84), we reduce the problem to equation:

$$\Delta\phi^{\mathrm{dip}} = \vec{p}\cdot\nabla\left(\delta(\vec{r}-\vec{r}_0)\right)\,, \tag{59}$$

with homogeneous boundary conditions: $\vec{D}^{\mathrm{dip}}\cdot\vec{n}\big|_{r=R} = 0$ in the Neumann case and $\phi^{\mathrm{dip}}\big|_{r=R} = 0$ in the Dirichlet case. Choosing the axis $\boldsymbol{e}_z$ parallel to $\vec{E}$ and passing to spherical coordinates (r,θ,φ) we obtain for $\vec{r}_0 = (r_0,0,0)$ and $\vec{p} = p\boldsymbol{e}_z + p_x\boldsymbol{e}_x$ (see Appendix D on page 101):

$$\phi^{\mathrm{dip}} = \phi^{\mathrm{dip}}_{\mathrm{sing}} + \phi^{\mathrm{dip}}_{\mathrm{reg}}\,, \tag{60}$$

where:

$$\phi^{\mathrm{dip}}_{\mathrm{sing}} = \frac{1}{4\pi}\frac{\vec{p}\cdot(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}, \tag{61}$$

$$\begin{aligned}\phi^{\mathrm{dip}}_{\mathrm{reg}} &= \frac{p}{4\pi}\left(\frac{R^3\left(R^2 - rr_0\cos\theta\right)}{r_0\left(R^4 + (r_0r)^2 - 2rr_0R^2\cos\theta\right)^{\frac{3}{2}}} - \frac{1}{r_0R}\right)\\ &+ \frac{p_x}{4\pi}\left(\frac{rR^3\sin\theta\cos\varphi}{\left(R^4 + {r_0}^2r^2 - 2r_0rR^2\cos\theta\right)^{\frac{3}{2}}}\right.\\ &\left. - \frac{\cos\varphi(R^2\cos\theta - r_0r)}{Rr_0\sin\theta\sqrt{R^4 + {r_0}^2r^2 - 2r_0rR^2\cos\theta}} + \frac{\cos\theta\cos\varphi}{Rr_0\sin\theta}\right).\end{aligned} \tag{62}$$

As we already noticed in the monopole case, axial symmetry of the problem implies that minimum of the energy is assumed at the point $\vec{r}_0$ which is parallel to $\vec{E}$. The same argument implies that we have $p_x = 0$ in this configuration. We are going to limit our analysis to such configurations only.

6.1. Stability

We compute the total, renormalised energy of the system as a sum of two parts:

$$\mathcal{H} = \mathcal{H}^{\mathrm{mon}} + \mathcal{H}^{\mathrm{dip}}\,, \tag{63}$$

where $\mathcal{H}^{\mathrm{mon}}$ denotes the energy of the monopole field obtained earlier ((26), page 84), and $\mathcal{H}^{\mathrm{dip}}$ denotes the remaining part, containing energy of the dipole field and the interaction energy. The latter term is computed in Appendix E (page 104). The final result for the Neumann case, written in terms of standardised variables, reads:

$$\begin{aligned}\mathcal{H}_{\mathcal{N}}{}'(x,q,p) \;&=\; \frac{1}{1-x^2} - \ln|1-x^2| - 2qx + \frac{1}{3}q^2 - 2 \\ &\quad + \frac{2}{3}\left(\frac{p}{eR}\frac{x(2-x^2)}{(1-x^2)^2} - \frac{p^2}{e^2R^2}\frac{1}{(1-x^2)^3} - \frac{p}{eR}\,q\right)\,. \end{aligned}\tag{64}$$

Now, stability of the system depends upon the polarisability of the particle, *i.e.* upon the choice of the "function of state" f (*cf.* (48) on page 88). At the moment we have no general criterion which would guarantee stability. However, it is easy to show that for:

$$f(\vec{p}) = -\frac{c^2}{3}\|\vec{p}\|^3 \Longrightarrow D_{\mathrm{reg}} = -c^2\|\vec{p}\|\vec{p}, \quad c>0 \tag{65}$$

our system is stable. Indeed, using (24) and (62) we obtain the following equation for the value of the dipole moment p:

$$\begin{aligned}-c^2p^2\mathrm{sgn}(p) &= D_{\mathrm{reg}}\Big|_{\vec{r}=\vec{r}_0} = -\nabla\left(\phi^{\mathrm{mon}}_{\mathrm{reg}} + \phi^{\mathrm{dip}}_{\mathrm{reg}}\right) \\ &= \frac{1}{4\pi}\left(\frac{eq}{R^2} - \frac{2p}{R^3(1-x^2)^3} - \frac{ex(2-x^2)}{R^2(1-x^2)^2}\right)\,. \end{aligned}\tag{66}$$

Denoting $4\pi ec^2R^4 = C$ and $\widetilde{p} = \frac{p}{eR}$, we get equation for $\widetilde{p}$:

$$-C\widetilde{p}^2\mathrm{sgn}(\widetilde{p}) = \left(q - \frac{2\widetilde{p}}{(1-x^2)^3} - \frac{x(2-x^2)}{(1-x^2)^2}\right)\,. \tag{67}$$

For small x, we use Taylor expansion of the right-hand side. Consequently, we have:

$$-C\widetilde{p}^2\mathrm{sgn}(\widetilde{p}) \approx q - 2\widetilde{p} - 2x - 6\widetilde{p}x^2 - 3x^3. \tag{68}$$

For $\widetilde{p}>0$ there are two solutions of this equation for small x and q:

$$\widetilde{p}_1 \;\approx\; \frac{1}{C}\left(1+\sqrt{1-qC} + \frac{xC}{\sqrt{1-qC}}\right)\,, \tag{69}$$

$$\widetilde{p}_2 \;\approx\; \frac{1}{C}\left(1-\sqrt{1-qC} - \frac{xC}{\sqrt{1-qC}}\right)\,. \tag{70}$$

For $\widetilde{p} < 0$ there is only one solution for small x and q:

$$\widetilde{p}_3 \approx -\frac{1}{C}\left(1+\sqrt{1+qC}-\frac{xC}{\sqrt{1+qC}}\right). \tag{71}$$

Inserting the above solutions into the energy function (64) we define for $i = 1, 2, 3$:

$$\mathcal{H}'_i(x,q) = \mathcal{H}_{\mathcal{N}}{}'(x,q,eR\widetilde{p}_i)\,.$$

It turns out that $\mathcal{H}'_2$ does not admit any minimum with respect to x (*i.e.* a stable "field + particle" configuration). For the remaining two cases we use Taylor expansion for small x:

$$\begin{aligned}\mathcal{H}'_1 \;\approx\; & -1+\frac{1}{3}q^2-\frac{4}{3C^2}+\frac{2}{3\sqrt{1-qC}}\left(-\frac{2}{C^2}+\frac{q}{C}+q^2\right)\\ & -2q\left(1+\frac{1}{\sqrt{1-qC}}\right)x+2\left(1-\frac{2}{C^2}+\frac{q}{C}-\frac{1}{3}\frac{1}{1-qC}\right.\\ & \left.+\frac{2}{\sqrt{1-qC}}\left(\frac{1}{3}-\frac{1}{C^2}+\frac{q}{C}\right)\right)x^2. \end{aligned}\tag{72}$$

$$\begin{aligned}\mathcal{H}'_3 \;\approx\; & -1+\frac{1}{3}q^2-\frac{4}{3C^2}+\frac{2}{3\sqrt{1+qC}}\left(-\frac{2}{C^2}-\frac{q}{C}+q^2\right)\\ & -2q\left(1+\frac{1}{\sqrt{1+qC}}\right)x+2\left(1-\frac{2}{C^2}-\frac{q}{C}-\frac{1}{3}\frac{1}{1+qC}\right.\\ & \left.+\frac{2}{\sqrt{1+qC}}\left(\frac{1}{3}-\frac{1}{C^2}-\frac{q}{C}\right)\right)x^2\,. \end{aligned}\tag{73}$$

Minimising both energies with respect to x we obtain:

$$\begin{aligned}x_1(q) \approx & \frac{3C^2}{32(C^2-3)}\left(8q+\frac{2C(C^2-9)}{C^2-3}q^2\right.\\ & \left.+\frac{C^2(2C^4-15C^2+45)}{(C^2-3)^2}q^3\right), \end{aligned}\tag{74}$$

$$\begin{aligned}x_3(q) \approx & \frac{3C^2}{32(C^2-3)}\left(8q-\frac{2C(C^2-9)}{C^2-3}q^2\right.\\ & \left.+\frac{C^2(2C^4-15C^2+45)}{(C^2-3)^2}q^3\right). \end{aligned}\tag{75}$$

Plugging $x_i(q)$ into the energy we get for small q:

$$\begin{aligned}\mathcal{H}'{}_1(q) \;&\approx\; -1-\frac{8}{3C^2}+\frac{15+4C^2}{6(3-C^2)}q^2\\ &\quad+\frac{(18+42C^2-7C^4)C}{12(3-C^2)^2}q^3\,,\end{aligned} \tag{76}$$

$$\begin{aligned}\mathcal{H}'{}_3(q) \;&\approx\; -1-\frac{8}{3C^2}+\frac{15+4C^2}{6(3-C^2)}q^2\\ &\quad-\frac{(18+42C^2-7C^4)C}{12(3-C^2)^2}q^3\,.\end{aligned} \tag{77}$$

We see that for $C \in]0,\sqrt{3}[$ the q^2 term is positive. This means that the system "particle + field" does not behave any longer like a *perpetuum mobile*: to deform its original configuration, corresponding to $q=0$, the boundary-condition controlling device must perform a positive work. Hence, the system is *stable* under small changes of q (see figure 3).

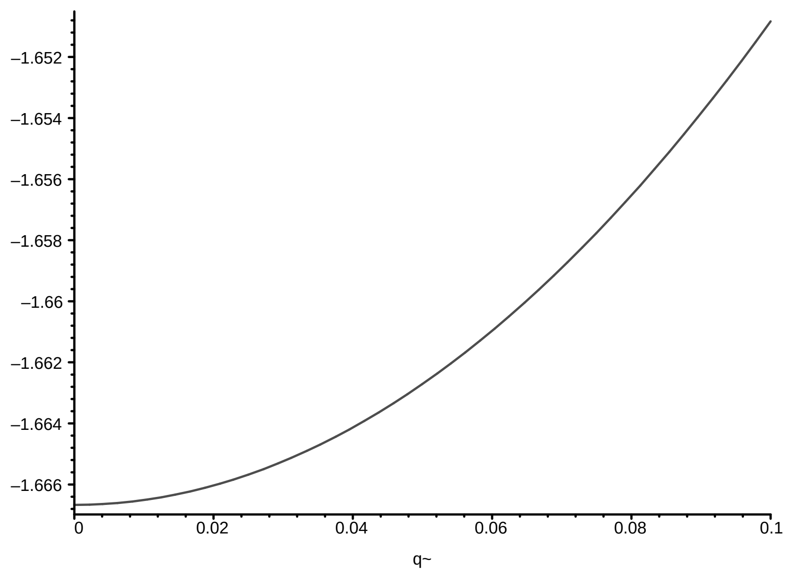

Fig. 3. Graph of $\mathcal{H}'(q)$ — renormalised energy *vs* boundary field for dipole particle, $C=1$.

6.2. Conclusions

We have shown that the polarisability of the particle, described by a suitable "state function" f (*e.g.* by (65)), may be a good remedy for the static instability of the renormalised electrodynamics of point particles. Whether or not this will cure also the dynamical instability, *i.e.* the existence of "runaway" solutions, is another question which we would like to study in the nearest future.

At the moment the bifurcation phenomenon occurring near the ground state $q = 0$ is worthwhile to study. Observe that the point $\vec{r}_0 = 0$, corresponding to $q = 0$ and described by the purely monopole field, *is not* stable. This configuration corresponds to a local maximum of the energy and belongs to the unstable branch of stationary points, described by the function $\mathcal{H}'_2$.

Appendix A

Neumann solution for "particle + field system"

We are looking for a solution of the Poisson equation $\Delta\phi = -e\delta(\vec{r}-\vec{r}_0)$ with boundary condition (22), where $\|\vec{r}_0\| < R$ and $\vec{r}_0 \| \vec{E}$. Denote:

$$\phi = \phi_{\text{sing}} + \overbrace{\phi^0_{\text{reg}} - \vec{E}\vec{r}}^{\phi_{\text{reg}}}, \tag{A.1}$$

where $\phi_{\text{sing}} = \frac{1}{4\pi}\frac{e}{|\vec{r}-\vec{r}_0|}$, $\Delta\phi^0_{\text{reg}} = 0$ and:

$$\vec{D}^0_{\text{reg}} \cdot \vec{n}\big|_{r=R} = \vec{n} \cdot \frac{1}{4\pi}\nabla\left(\frac{e}{|\vec{r}-\vec{r}_0|}\right)\Bigg|_{r=R} + \frac{e}{4\pi R^2}\,. \tag{A.2}$$

To find ϕ^0_{reg}, we use the following formula (*cf.* [11], p. 83):

$$\frac{1}{\sqrt{r^2 + {r_0}^2 - 2\,r r_0 \cos\theta}} - \frac{1}{r} = \sum_{n=1}^{\infty} P_n(\cos\theta)\frac{{r_0}^n}{r^{n+1}} \tag{A.3}$$

(θ is the angle between $\vec{r}$ and $\vec{E}$) valid for $-r \le r_0 \le r$, together with the following *Ansatz*:

$$\phi^0_{\text{reg}} = \sum_{n=1}^{\infty} c_n r^n P_n(\cos\theta)\,. \tag{A.4}$$

Write boundary condition as:

$$\frac{\partial}{\partial r}\phi^0_{\text{reg}}\Bigg|_{r=R} = \frac{e}{4\pi}\frac{\partial}{\partial r}\left(\frac{1}{r} - \frac{1}{|\vec{r}-\vec{r}_0|}\right)\Bigg|_{r=R}, \tag{A.5}$$

and substitute (A.3) and (A.4) to (A.5). This way we get the solution given as a series:

$$\phi^0_{\mathrm{reg}} = \frac{eR}{4\pi}\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)\frac{(rr_0)^n}{(R^2)^{n+1}}P_n(\cos\theta)\,. \tag{A.6}$$

Observe that (A.3) gives, after rescaling, the first component of (A.6). The second one will be obtained from the following:

Lemma A.1 For $\|r_0\| \geq r$ we have

$$\sum_{n=1}^{\infty}\frac{1}{n}\frac{{r_0}^n}{r^{n+1}}P_n(\cos\theta) = -\frac{1}{r}\ln\left|\frac{1}{2}\left(1-\frac{r_0}{r}\cos\theta+\sqrt{1+\left(\frac{r_0}{r}\right)^2-2\frac{r_0}{r}\cos\theta}\right)\right|.$$

Proof: Substituting t for r_0 in (A.3):

$$\begin{aligned}
&\int_0^{r_0}\left(\sum_{n=1}^{\infty}\frac{t^{n-1}}{r^{n+1}}P_n(\cos\theta)\right)dt \\
&= \int_0^{r_0}\left(\frac{1}{t\sqrt{r^2+t^2-2rt\cos\theta}}-\frac{1}{tr}\right)dt \\
&\Leftrightarrow \sum_{n=1}^{\infty}\frac{1}{n}\frac{{r_0}^n}{r^{n+1}}P_n(\cos\theta) \\
&= -\frac{1}{r}\left(\ln\left|\frac{r}{t}-\cos\theta+\frac{1}{t}\sqrt{r^2+t^2-2rt\cos\theta}\right|+\ln t\right)\Bigg|_0^{r_0} \\
&= -\frac{1}{r}\ln\left|\frac{1}{2}\left(1-\frac{r_0}{r}\cos\theta+\sqrt{1+\left(\frac{r_0}{r}\right)^2-2\frac{r_0}{r}\cos\theta}\right)\right|.
\end{aligned} \tag{A.7}$$

Plugging R^2 instead of r and rr_0 instead of r in Lemma (A.1) yields:

$$\begin{aligned}
\phi^0_{\mathrm{reg}} = \frac{e}{4\pi}\Bigg(&\frac{R}{\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta}}-\frac{1}{R}+\frac{1}{R}\ln(2R^2) \\
&-\frac{1}{R}\ln\left|R^2-r_0r\cos\theta+\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta}\right|\Bigg).
\end{aligned} \tag{A.8}$$

Figure 4 shows the directions of the field $D-E = D_{\mathrm{sing}} + D^0_{\mathrm{reg}} + \frac{e}{4\pi}\nabla\frac{1}{r}$. Observe that the field is tangent to the boundary of $K(0,R)$.

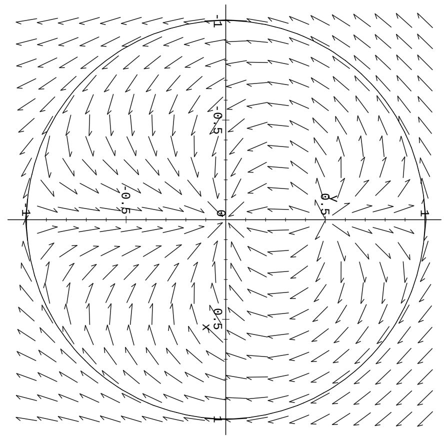

Fig. 4. Directions of the field D–E for $R = 1$, $r_0 = 0.5$.

Appendix B

Renormalised energy for Neumann solutions

To compute integral (14):

$$\mathcal{H} = m - \frac{1}{2} \int\limits_{\mathbb{R}^3 - V} D^2_{\text{sing}} dv + \frac{1}{2} \int\limits_{V} D^2_{\text{reg}} dv + \int\limits_{V} D_{\text{sing}} D_{\text{reg}} dv \tag{B.1}$$

observe that:

$$-\frac{1}{2} \int\limits_{\mathbb{R}^3 - V} D^2_{\text{sing}} dv = \frac{1}{2} \int\limits_{\partial\mathbb{R}^3 - \partial V} \phi_{\text{sing}} D^{\perp}{}_{\text{sing}}\, d\sigma\,, \tag{B.2}$$

$$\frac{1}{2} \int\limits_{V} D^2_{\text{reg}} dv = -\frac{1}{2} \int\limits_{\partial V - \partial\mathbb{R}^3} \phi_{\text{reg}} D^{\perp}{}_{\text{reg}}\, d\sigma\,. \tag{B.3}$$

Integrals containing products of singular and regular fields are understood in the sense of distributions (*cf.* [13], p. 748). Denoting $k_\epsilon := K(\vec{r}_0, \epsilon)$ we obtain:

$$\begin{aligned}\int\limits_V D_{\rm sing} D_{\rm reg} dv \; &= \; \lim_{\epsilon\to 0} \int\limits_{V-k_\epsilon} D_{\rm sing} D_{\rm reg} dv \\ &= \; -\lim_{\epsilon\to 0} \frac{1}{2} \int\limits_{V-k_\epsilon} \nabla\left(\phi_{\rm sing} D_{\rm reg} + \phi_{\rm reg} D_{\rm sing}\right) dv \\ &= \; -\lim_{\epsilon\to 0} \frac{1}{2} \int\limits_{\partial V-\partial k_\epsilon} \left(\phi_{\rm sing} D^{\perp}{}_{\rm reg} + \phi_{\rm reg} D^{\perp}{}_{\rm sing}\right) d\sigma \,. \end{aligned} \tag{B.4}$$

Hence, for $V = K_R := K(0,R)$ we have:

$$\begin{aligned}\mathcal{H} \; &= \; m - \frac{1}{2} \int\limits_{\partial K_R} \phi D^{\perp}\, d\sigma \\ &\quad + \lim_{\epsilon\to 0} \frac{1}{2} \int\limits_{\partial k_\epsilon} \left(\phi_{\rm reg} D^{\perp}_{\rm sing} + \phi_{\rm sing} D^{\perp}_{\rm reg}\right) d\sigma \,. \end{aligned} \tag{B.5}$$

The formula is true for both the monopole and the dipole singularity of $D_{\rm sing}$. Here, we consider the monopole (Coulomb) singularity. In this case the function $\phi_{\rm sing}$ multiplied by ϵ^2 (coming from the surface measure $d\sigma$) vanishes for $\epsilon \to 0$. Hence, we have:

$$\mathcal{H} = m - \frac{1}{2} \int\limits_{\partial K_R} \phi D^{\perp}\, d\sigma + \lim_{\epsilon\to 0} \frac{1}{2} \int\limits_{\partial k_\epsilon} \phi_{\rm reg} D^{\perp}_{\rm sing}\, d\sigma \,. \tag{B.6}$$

To compute the integral over ∂k_ϵ, we use spherical coordinates $(\epsilon, \beta, \varphi)$ centred at $\vec{r}_0$. Parameters r and $\cos\theta$ present in $\phi_{\rm reg}$ may be expressed as follows:

$$r^2 \; = \; r_0^2 + \epsilon^2 - 2\epsilon r_0 \cos\beta, \quad r\cos\theta = r_0 - \epsilon\cos\beta \,, \tag{B.7}$$

$$\lim_{\epsilon\to 0} r^2 = r_0^2, \qquad \lim_{\epsilon\to 0} r\cos\theta = r_0 \,. \tag{B.8}$$

Then:

$$\begin{aligned}&\lim_{\epsilon\to 0} \frac{1}{2} \int\limits_{\partial k_\epsilon} \phi_{\rm reg} D^{\perp}_{\rm sing}\, d\sigma = \lim_{\epsilon\to 0} \frac{e}{2} \frac{2\pi}{4\pi} \int\limits_0^{\pi} \frac{1}{\epsilon^2} \phi_{\rm reg}(r_0, \epsilon, \beta) \sin\beta \epsilon^2\, d\beta \\ &= \frac{e}{4} \phi_{\rm reg}(r_0, r = r_0, \theta = 0) \int\limits_0^{\pi} \sin\beta d\beta = \frac{1}{2} e \phi_{\rm reg}\big|_{\vec{r}=\vec{r}_0} \,. \end{aligned} \tag{B.9}$$

Consequently:

$$\mathcal{H} = m - \frac{1}{2} \int\limits_{\partial K_R} \phi D^{\perp}\, d\sigma + \frac{1}{2} e\phi_{\mathrm{reg}}\big|_{\vec{r}=\vec{r}_0} . \tag{B.10}$$

Knowing ϕ we can compute $\mathcal{H}$:

$$D^{\perp}|_{r=R} = \frac{e}{4\pi}\frac{1}{R^2} + E\cos\theta, \tag{B.11}$$

$$\begin{aligned}\phi|_{r=R} &= -ER\cos\theta \\ &\quad + \frac{e}{4\pi}\left(\frac{2}{\sqrt{R^2 + {r_0}^2 - 2r_0 R\cos\theta}} - \frac{1}{R} + \frac{1}{R}\ln(2R)\right. \\ &\quad \left. - \frac{1}{R}\ln\left| R - r_0\cos\theta + \sqrt{R^2 + {r_0}^2 - 2r_0R\cos\theta}\right|\right),\end{aligned} \tag{B.12}$$

$$\begin{aligned}&e\phi_{\mathrm{reg}}\big|_{\vec{r}=\vec{r}_0} \\ &= -eEr_0 + \frac{e^2}{4\pi}\left(\frac{R}{R^2 - r_0^2} - \frac{1}{R} - \frac{1}{R}\ln\left|1 - \frac{r_0^2}{R^2}\right|\right) .\end{aligned} \tag{B.13}$$

Note that:

$$\int\limits_{K_R} \frac{1}{\sqrt{R^2 + {r_0}^2 - 2r_0R\cos\theta}} d\sigma = 4\pi R\,, \tag{B.14}$$

$$\int\limits_{K_R} E\cos\theta d\sigma = 0\,, \tag{B.15}$$

$$\begin{aligned}&\int\limits_{K_R} \ln\left| R - r_0\cos\theta + \sqrt{R^2 + {r_0}^2 - 2r_0R\cos\theta}\right| d\sigma \\ &= 4\pi R^2 \ln 2R\,,\end{aligned}$$

where we used two integrals 2.736 from [12]. Then:

$$\frac{e}{4\pi R^2}\int\limits_{K_R} \phi d\sigma = \frac{e^2}{4\pi R} . \tag{B.16}$$

Moreover:

$$E^2 R \int_{K_R} \cos^2\theta d\sigma = \frac{4}{3}\pi R^3 E^2, \tag{B.17}$$

$$\int_{K_R} \frac{E\cos\theta}{\sqrt{R^2 + {r_0}^2 - 2r_0 R\cos\theta}} d\sigma = \frac{4}{3}\pi E r_0, \tag{B.18}$$

$$\begin{aligned}&\int_{K_R} \ln\left| R - r_0\cos\theta + \sqrt{R^2 + {r_0}^2 - 2r_0 R\cos\theta}\right| \\ &\times E\cos\theta d\sigma = -\frac{4}{3}\pi r_0 R,\end{aligned} \tag{B.19}$$

where we used four integrals 2.736 from [12]. Then:

$$\begin{aligned} E\int_{K_R} \phi\cos\theta d\sigma &= -\frac{4}{3}\pi R^3 E^2 + \frac{eE}{4\pi}\left(\frac{8}{3}\pi r_0 + \frac{4}{3}\pi r_0\right) \\ &= -\frac{4}{3}\pi R^3 E^2 + eEr_0 \,. \end{aligned} \tag{B.20}$$

The final result is the sum of (B.13), (B.16) and (B.20) with coefficient $\frac{1}{2}$:

$$\begin{aligned} \mathcal{H} &= m + \frac{1}{2}\left(\frac{e^2}{4\pi}\left(\frac{R}{R^2 - r_0^2} - \frac{1}{R}\ln\left|1 - \frac{r_0^2}{R^2}\right| - \frac{2}{R}\right)\right. \\ &\quad \left. + \frac{4}{3}\pi R^3 E^2 - 2eEr_0\right) . \end{aligned} \tag{B.21}$$

Appendix C

Dirichlet solution and the corresponding energy

To find a solution of the Poisson equation $\Delta\phi = -e\delta(\vec{r} - \vec{r}_0)$ with boundary conditions (23), where $\|\vec{r}_0\| < R$ and $\vec{r}_0 \| \vec{E}$, we denote: $\phi = \phi_{\text{sing}} + \phi^0_{\text{reg}} - \vec{E}\vec{r}$, where $\phi_{\text{sing}} = \frac{1}{4\pi}\frac{e}{|\vec{r}-\vec{r}_0|}$, $\Delta\phi^0_{\text{reg}} = 0$ and:

$$\phi^0_{\text{reg}}\big|_{r=R} = -\frac{1}{4\pi}\left(\frac{e}{|\vec{r} - \vec{r}_0|}\right)\Bigg|_{r=R} + \frac{e}{4\pi R} \,. \tag{C.1}$$

Again, we use *Ansatz* (A.4) as we did in Appendix A, page 94, and expand also boundary conditions:

$$\phi^0_{\text{reg}}\Bigg|_{r=R} = \frac{e}{4\pi}\left(\frac{1}{r} - \frac{1}{|\vec{r} - \vec{r}_0|}\right)\Bigg|_{r=R} , \tag{C.2}$$

in series of Legendre polynomials. After substituting (A.3) and (A.4) to (C.2) we obtain:

$$\phi^0_{\mathrm{reg}} = -\frac{eR}{4\pi}\sum_{n=1}^{\infty}\frac{(rr_0)^n}{(R^2)^{n+1}}P_n(\cos\theta)\,. \tag{C.3}$$

After rescaling (A.3) we get:

$$\phi^0_{\mathrm{reg}} = \frac{e}{4\pi}\left(\frac{1}{R} - \frac{R}{\sqrt{R^4 + {r_0}^2r^2 - 2r_0rR^2\cos\theta}}\right)\,. \tag{C.4}$$

Singular part of the electric field has the Coulomb singularity at $\vec{r}_0$. Hence, formula (B.10) is valid. However, we have:

$$D^{\perp}|_{r=R} = \frac{e}{4\pi R}\frac{R^2 - r_0^2}{(R^2 + {r_0}^2 - 2r_0R\cos\theta)^{\frac{3}{2}}} + E\cos\theta, \tag{C.5}$$

$$\phi|_{r=R} = -ER\cos\theta + \frac{e}{4\pi}\frac{1}{R}, \tag{C.6}$$

$$e\phi_{\mathrm{reg}}\big|_{\vec{r}=\vec{r}_0} = -eEr_0 + \frac{e^2}{4\pi}\left(\frac{1}{R} - \frac{R}{R^2 - r_0^2}\right)\,. \tag{C.7}$$

This implies:

$$2\pi R^2\int_0^{\pi}\frac{e^2}{(4\pi R)^2}\frac{(R^2 - r_0^2)\sin\theta\,d\theta}{(R^2 + {r_0}^2 - 2r_0R\cos\theta)^{\frac{3}{2}}} = \frac{e^2}{4\pi R}\,, \tag{C.8}$$

$$-\,2\pi R^2\int_0^{\pi}ER\cos\theta\,\frac{e}{4\pi R}\frac{(R^2 - r_0^2)\sin\theta\,d\theta}{(R^2 + {r_0}^2 - 2r_0R\cos\theta)^{\frac{3}{2}}} = -eEr_0\,, \tag{C.9}$$

$$-\,2\pi R^2\int_0^{\pi}E^2R\cos^2\theta\,\sin\theta\,d\theta = -\frac{4}{3}\pi R^3E^2\,, \tag{C.10}$$

$$\int_0^{\pi}\cos\theta\,\sin\theta\,d\theta = 0\,. \tag{C.11}$$

Consequently, we obtain:

$$\mathcal{H} = m + \frac{1}{2}\left(\frac{4}{3}\pi R^3E^2 - \frac{e^2}{4\pi}\frac{R}{R^2 - r_0^2}\right)\,, \tag{C.12}$$

or, in standardised variables (31),

$$\mathcal{H}'_{\mathcal{D}} = \frac{1}{3}q^2 - \frac{1}{1 - x^2}\,. \tag{C.13}$$

Appendix D

Dipole particle in a spherical box

We must solve equation $\Delta\phi^{\mathrm{dip}} = \vec{p}\cdot\nabla\left(\delta(\vec{r}-\vec{r}_0)\right)$ with boundary conditions $\vec{D}^{\mathrm{dip}}\cdot\vec{n}\big|_{r=R} = 0$. Denoting $\phi = \phi^{\mathrm{dip}}_{\mathrm{sing}} + \phi^{\mathrm{dip}}_{\mathrm{reg}}$, where

$$\phi^{\mathrm{dip}}_{\mathrm{sing}} = \frac{1}{4\pi}\frac{\vec{p}\cdot(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}\,, \tag{D.1}$$

we get Laplace equation $\Delta\phi^{\mathrm{dip}}_{\mathrm{reg}} = 0$ with boundary condition:

$$\vec{D}^{\mathrm{dip}}_{\mathrm{reg}}\cdot\vec{n}\big|_{r=R} = \vec{n}\cdot\frac{1}{4\pi}\nabla\left(\frac{\vec{p}\cdot(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}\right)\bigg|_{r=R}. \tag{D.2}$$

For any pair of vectors $\vec{r}_0$ i $\vec{p}$ we choose coordinates in which $\vec{r}_0$ is parallel to the z-axis $\boldsymbol{e}_z$ and polarisation vector assumes the form $\vec{p} = p\boldsymbol{e}_z + p_x\boldsymbol{e}_x$. The final solution will be the sum of two harmonic functions fulfilling boundary condition (D.2), calculated *separately* for $p\boldsymbol{e}_z$ and $p_x\boldsymbol{e}_x$.

Observe that, for $\phi^{\mathrm{mon}}_{\mathrm{reg}}(\vec{r}_0,\vec{r})$ being a solution of Laplace equation, also the function $\frac{\vec{p}}{e}\cdot\nabla_{\vec{r}_0}\phi^{\mathrm{mon}}_{\mathrm{reg}}$ is harmonic. Moreover, if $\phi^{\mathrm{mon}}_{\mathrm{reg}}$ fulfils conditions ((A.5) condition from page 94):

$$\frac{\partial}{\partial r}\phi^{\mathrm{mon}}_{\mathrm{reg}}(\vec{r},\vec{r}_0)\bigg|_{r=R} = \frac{e}{4\pi}\frac{\partial}{\partial r}\left(\frac{1}{r}-\frac{1}{|\vec{r}-\vec{r}_0|}\right)\bigg|_{r=R}, \tag{D.3}$$

then, after differentiation with respect to $\vec{r}_0$ we obtain:

$$-\frac{\partial}{\partial r}\left(\frac{\vec{p}}{e}\cdot\nabla_{\vec{r}_0}\phi^{\mathrm{mon}}_{\mathrm{reg}}\right)\bigg|_{r=R} = \frac{1}{4\pi}\frac{\partial}{\partial r}\left(\vec{p}\cdot\nabla_{\vec{r}_0}\frac{1}{|\vec{r}-\vec{r}_0|}\right)\bigg|_{r=R}. \tag{D.4}$$

Hence, the function $\frac{\vec{p}}{e}\cdot\nabla_{\vec{r}_0}\phi^{\mathrm{mon}}_{\mathrm{reg}}$ satisfies boundary conditions (D.2). We conclude that:

$$\phi^{\mathrm{dip}}_{\mathrm{reg}} = \frac{1}{4\pi e}\left(\vec{p}\cdot\nabla_{\vec{r}_0}\right)\phi^{\mathrm{mon}}_{\mathrm{reg}}\,, \tag{D.5}$$

(*cf.* [11], p.14). Applying (D.5) for $\vec{p} = p\boldsymbol{e}_z + p_x\boldsymbol{e}_x$ allows us to solve the problem separately for p parallel and orthogonal to $\vec{r}_0$.

Solution for $\vec{p} \parallel \vec{r}_0$

To obtain the parallel part we differentiate monopole solution ((A.8), Appendix A) along the $\boldsymbol{e}_z$-axis:

$$\begin{aligned}
\phi^{\mathrm{dip}}_{\mathrm{reg}} &= \frac{p}{e}\frac{\partial}{\partial r_0}\phi^{\mathrm{mon}}_{\mathrm{reg}} \\
&\times \frac{p}{4\pi}\frac{\partial}{\partial r_0}\left(\frac{R}{\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta}}-\frac{1}{R}+\frac{1}{R}\ln(2R^2)\right. \\
&\left.-\frac{1}{R}\ln\left|R^2-r_0r\cos\theta+\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta}\right|\right) \\
&= \frac{p}{4\pi}\left(-\frac{R(r_0r^2-rR^2\cos\theta)}{(R^4+{r_0}^2r^2-2r_0rR^2\cos\theta)^{\frac{3}{2}}}\right. \\
&-\frac{1}{R\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta}} \\
&\left.\times\frac{-r\cos\theta\left(\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta}+R^2\right)+r^2r_0}{R^2-rr_0\cos\theta+\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta}}\right).
\end{aligned} \tag{D.6}$$

But:

$$\left(R^2-rr_0\cos\theta+\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta}\right) \tag{D.7}$$

$$\begin{aligned}
&\times\left(R^2-rr_0\cos\theta-\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta}\right) \\
&= -(r_0r)^2\sin^2\theta,
\end{aligned} \tag{D.8}$$

$$\begin{aligned}
&\left(-r\cos\theta\left(\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta}+R^2\right)+r^2r_0\right) \\
&\times\left(R^2-rr_0\cos\theta-\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta}\right) \\
&= -r_0r^2\sin^2\theta\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta} \\
&+r\cos\theta(-r_0rR^2\cos\theta)+R^2r^2r_0 \\
&= r^2r_0\sin^2\theta\left(R^2-\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta}\right).
\end{aligned} \tag{D.9}$$

So:

$$\begin{aligned}
\phi^{\mathrm{dip}}_{\mathrm{reg}} &= \frac{p}{4\pi}\left(-\frac{R(r_0r^2-rR^2\cos\theta)}{(R^4+{r_0}^2r^2-2r_0rR^2\cos\theta)^{\frac{3}{2}}}\right. \\
&\left.+\frac{r^2r_0\sin^2\theta\,(R^2-\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta})}{R(rr_0)^2\sin^2\theta\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta}}\right) \\
&= \frac{1}{4\pi}\left(\frac{p\,R^3\left(R^2-rr_0\cos\theta\right)}{r_0\left(R^4+(r_0r)^2-2rr_0R^2\cos\theta\right)^{\frac{3}{2}}}-\frac{p}{r_0R}\right).
\end{aligned} \tag{D.10}$$

Figure 5 shows the directions of the field D^{dip}. Observe that the field is tangent to the boundary of $K(0, R)$.

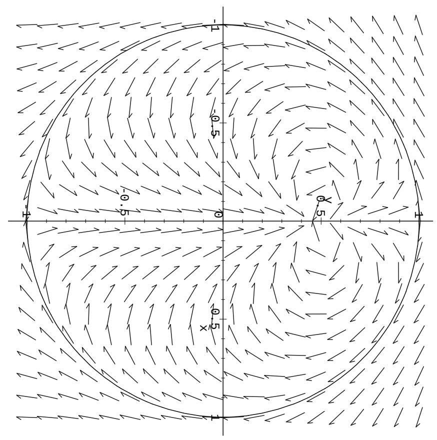

Fig. 5. Directions of the field $D^{\text{dip}}_{\text{sing}} + D^{\text{dip}}_{\text{reg}}$, $R = 1$, $r_0 = 0.5$, $p = 1$.

Solutions for $\vec{p} \perp \vec{r_0}$

For $\vec{p} = p_x \boldsymbol{e}_x$ we get:

$$\phi^{\text{dip}}_{\text{reg}} = p_x \frac{\boldsymbol{e}_x}{e} \cdot \nabla_{\vec{r}_0} \phi^{\text{mon}}_{\text{reg}} = \frac{p_x}{e} \frac{\partial}{\partial x_0} \phi^{\text{mon}}_{\text{reg}} \,. \tag{D.11}$$

The easiest way to calculate this derivative is to use spherical coordinates $\vec{r}_0 = (r_0, \theta_0, \varphi_0)$. Then:

$$x_0 = r_0 \sin\theta_0 \sin\varphi_0 \,, \quad y_0 = r_0 \sin\theta_0 \cos\varphi_0 \,, \quad z_0 \quad = r_0 \cos\theta_0 \,, \tag{D.12}$$

and:

$$\frac{\partial}{\partial x_0} = \sin\theta_0 \cos\varphi_0 \frac{\partial}{\partial r_0} + \cos\theta_0 \cos\varphi_0 \frac{1}{r_0} \frac{\partial}{\partial \theta_0} - \frac{\sin\varphi_0}{r_0 \sin\theta_0} \frac{\partial}{\partial \varphi_0} \,. \tag{D.13}$$

But for $\vec{r}_0 \| \boldsymbol{e}_z$ this procedure is singular because $\sin\theta_0 = 0$. To overcome this difficulty we first calculate the result for $\vec{r}_0 \nparallel \mathbf{e}_z$ and then pass to the limit $\theta_0 \to 0$ and $\varphi_0 \to 0$. For this purpose we must be able to differentiate the function $\cos\gamma$, where γ is the angle between $\vec{r}$ and $\vec{r}_0$, *i.e.*:

$$\vec{r} \cdot \vec{r}_0 = r r_0 \cos\gamma \,, \tag{D.14}$$

or, equivalently:

$$\cos\gamma = \cos\theta \cos\theta_0 + \sin\theta \sin\theta_0 \cos(\varphi - \varphi_0) \,. \tag{D.15}$$

Hence, (D.13) gives us:

$$\frac{\partial}{\partial x_0}\cos\gamma = \left(\frac{1}{r_0}\cos\theta\cos\varphi_0\left(-\cos\theta\sin\theta_0+\sin\theta\cos\theta_0\cos(\varphi-\varphi_0)\right)\right.$$
$$\left.-\frac{1}{r_0}\sin\varphi_0\sin\theta\sin(\varphi-\varphi_0)\right)\underset{\varphi_0\to 0}{\overset{\theta_0\to 0}{\longrightarrow}}\frac{1}{r_0}\sin\theta\cos\varphi\,. \tag{D.16}$$

This method allows us to calculate effectively the derivative of the monopole field from Appendix A (p. 94) along x_0. The final result reads:

$$\phi^{\mathrm{dip}}_{\mathrm{reg}} = \frac{p_x}{4\pi}\left(\frac{rR^3\sin\theta\cos\varphi}{(R^4+{r_0}^2r^2-2r_0rR^2\cos\theta)^{\frac{3}{2}}}\right.$$
$$\left.-\frac{\cos\varphi(R^2\cos\theta-r_0r)}{Rr_0\sin\theta\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta}}+\frac{\cos\theta\cos\varphi}{Rr_0\sin\theta}\right). \tag{D.17}$$

We stress that the above function is regular at $\theta=0$ due to cancellations between the second and the third term.

Appendix E

Renormalised energy of a dipole particle

To calculate $\mathcal{H}^{\mathrm{dip}}$ we use results of Appendix B. It turns out that in formula (B.5), only the following non-vanishing terms were not taken into account in $\mathcal{H}^{\mathrm{mon}}$:

$$\mathcal{H}^{\mathrm{dip}} = -\frac{1}{2}\int\limits_{\partial K_R}\phi^{\mathrm{dip}}D^{\perp}\,d\sigma + \lim_{\epsilon\to 0}\frac{1}{2}\int\limits_{\partial k_\epsilon}\left(\phi^{\mathrm{dip}}_{\mathrm{reg}}D^{\perp\,mon}_{\mathrm{sing}}+\phi^{\mathrm{dip}}_{\mathrm{sing}}D^{\perp}_{\mathrm{reg}}\right)d\sigma\,, \tag{E.1}$$

where:

$$\phi^{\mathrm{dip}}_{\mathrm{reg}} = \frac{1}{4\pi}\left(\frac{p\,R^3\left(R^2-rr_0\cos\theta\right)}{r_0\left(R^4+(r_0r)^2-2rr_0R^2\cos\theta\right)^{\frac{3}{2}}}-\frac{p}{r_0R}\right), \tag{E.2}$$

$$\phi^{\mathrm{dip}}_{\mathrm{sing}} = \frac{1}{4\pi}\frac{p(r\cos\theta-r_0)}{(r^2+r_0^2-2rr_0\cos\theta)^{\frac{3}{2}}}\,, \tag{E.3}$$

$$\phi^{\mathrm{mon}}_{\mathrm{reg}} = \frac{e}{4\pi}\left(\frac{R}{\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta}}-\frac{1}{R}\right.$$
$$\left.-\frac{1}{R}\ln\left|R^2-r_0r\cos\theta+\sqrt{R^4+{r_0}^2r^2-2r_0rR^2\cos\theta}\right|\right)$$
$$-Er\cos\theta+\frac{1}{R}\ln(2R^2)\,, \tag{E.4}$$

$$\phi^{\mathrm{dip}} = \phi^{\mathrm{dip}}_{\mathrm{reg}}+\phi^{\mathrm{dip}}_{\mathrm{sing}}\,. \tag{E.5}$$

Moreover, we have:

$$D^{\perp}\big|_{\partial K_R} = \frac{1}{4\pi}\frac{e}{R^2} + E\cos\theta\,, \tag{E.6}$$

$$D^{\perp}_{\mathrm{reg}}\big|_{\partial k_\epsilon} = -\frac{\partial}{\partial\epsilon}\left(\phi^{\mathrm{mon}}_{\mathrm{reg}} + \phi^{\mathrm{dip}}_{\mathrm{reg}}\right)\,. \tag{E.7}$$

To compute the integral over ∂K_R we note that:

$$\phi^{\mathrm{dip}}\Big|_{r=R} = \phi^{\mathrm{dip}}_{\mathrm{reg}} + \phi^{\mathrm{dip}}_{\mathrm{sing}} = \frac{p}{4\pi r_0}\left(\frac{R^2 - r_0^2}{(R^2 + r_0^2 - 2Rr_0\cos\theta)^{\frac{3}{2}}} - \frac{1}{R}\right)\,, \tag{E.8}$$

whereas $D^{\perp}$ is expressed by (E.6). Moreover:

$$\frac{1}{4\pi}\frac{e}{R^2}2\pi\int_0^\pi \phi^{\mathrm{dip}}\sin\theta\,d\theta = 0\,. \tag{E.9}$$

So:

$$-\frac{1}{2}2\pi\,E\int_0^\pi \phi^{\mathrm{dip}}\cos\theta\sin\theta\,d\theta = -\frac{1}{2}p\,E\,.$$

To find the limit:

$$\lim_{\epsilon\to 0}\frac{1}{2}\int_{\partial k_\epsilon}\left(\phi^{\mathrm{dip}}_{\mathrm{sing}}D^{\perp}_{\mathrm{reg}} + D^{\mathrm{mon}}_{\mathrm{sing}}\phi^{\mathrm{dip}}_{\mathrm{reg}}\right)\,d\sigma\,, \tag{E.10}$$

we analyse behaviour of fields (E.2)–(E.7) for $\epsilon \to 0$. All these terms have at most the ϵ^{-2}-singularity. Therefore, they are continuous and bounded when multiplied by ϵ^2. Thus, we can interchange the limit and the integration operations.

We follow our procedure described in Appendix B, page 96. Using (B.7) and (B.8) we obtain in terms of the standardised variable $x = \frac{r_0}{R}$:

$$\lim_{\epsilon\to 0}\left(\epsilon^2\,\phi^{\mathrm{dip}}_{\mathrm{sing}}\right) = -\frac{p}{4\pi}\cos\beta\,, \tag{E.11}$$

$$\begin{aligned}
&\lim_{\epsilon\to 0}\left(-\frac{\partial}{\partial\epsilon}\phi^{\mathrm{mon}}_{\mathrm{reg}}\right)\\
&=\frac{e}{4\pi}\left(\frac{r_0 R}{(R^2-r_0^2)^2}+\frac{r_0}{(R^2-r_0^2)^2 R}\right)\cos\beta - E\cos\beta\\
&=\left(\frac{e}{4\pi}\frac{1}{R^2}\frac{x(2-x^2)}{(1-x^2)^2}-E\right)\cos\beta = -\cos\beta\, D^{\mathrm{mon}}_{\mathrm{reg}}\Big|_{\vec{r}=\vec{r_0}}, \qquad \text{(E.12)}\\
&\lim_{\epsilon\to 0}\left(-\frac{\partial}{\partial\epsilon}\phi^{\mathrm{dip}}_{\mathrm{reg}}\right)=\frac{1}{4\pi}\frac{2pR^3}{(R^2-r_0^2)^3}\cos\beta\\
&=\frac{1}{4\pi}\frac{2p}{R^3(1-x^2)^3}\cos\beta = -\cos\beta\, D^{\mathrm{dip}}_{\mathrm{reg}}\Big|_{\vec{r}=\vec{r_0}}, \qquad \text{(E.13)}\\
&\lim_{\epsilon\to 0}\left(\epsilon^2 D^{\mathrm{mon}}_{\mathrm{sing}}\right)=\frac{e}{4\pi}, \qquad \text{(E.14)}\\
&\lim_{\epsilon\to 0}\left(\phi^{\mathrm{dip}}_{\mathrm{reg}}\right)=\frac{1}{4\pi}\frac{pr_0(2R^2-r_0^2)}{R(R^2-r_0^2)^2}\\
&=\frac{p}{4\pi}\frac{x(2-x^2)}{R^2(1-x^2)^2}=-\frac{p}{e}D^{\mathrm{mon}}_{\mathrm{reg}}\Big|_{\vec{r}=\vec{r_0}}, \qquad \text{(E.15)}\\
&\int_0^\pi \cos^2\beta\sin\beta\, d\beta=\frac{2}{3}. \qquad \text{(E.16)}
\end{aligned}$$

Then:

$$\begin{aligned}
&\frac{1}{2}\int_{\partial k_\epsilon}\left(\phi^{\mathrm{dip}}_{\mathrm{sing}}D^{\perp}_{\mathrm{reg}}+D^{\mathrm{mon}}_{\mathrm{sing}}\phi^{\mathrm{dip}}_{\mathrm{reg}}\right)d\sigma=\frac{1}{2}\left(\frac{4\pi}{3}\frac{p}{4\pi}\left(D^{\mathrm{mon}}_{\mathrm{reg}}\Big|_{\vec{r}=\vec{r_0}}+D^{\mathrm{dip}}_{\mathrm{reg}}\Big|_{\vec{r}=\vec{r_0}}\right)\right.\\
&\left.-\frac{e}{4\pi}4\pi\frac{p}{e}D^{\mathrm{mon}}_{\mathrm{reg}}\Big|_{\vec{r}=\vec{r_0}}\right)\\
&=\frac{1}{2}\left(\frac{pe}{4\pi}\frac{1}{R^2}\frac{2}{3}\frac{x(2-x^2)}{(1-x^2)^2}-\frac{1}{4\pi}\frac{1}{3}\frac{2p^2}{R^3(1-x^2)^3}+\frac{1}{3}pE\right). \qquad \text{(E.17)}
\end{aligned}$$

Using $q=\frac{4\pi R^2}{e}E$ and (32) we obtain:

$$\mathcal{H}'^{\ \mathrm{dip}}:=\frac{8\pi R}{e^2}\mathcal{H}^{\mathrm{dip}}=\frac{2}{3}\left(\frac{p}{eR}\frac{x(2-x^2)}{(1-x^2)^2}-\frac{p^2}{e^2R^2}\frac{1}{(1-x^2)^3}-\frac{p}{eR}q\right). \qquad \text{(E.18)}$$

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