LIE ALGEBRAIC STRUCTURES IN PEGG–BARNETT QUANTIZATION FORMULATION

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The oscillator algebra of Pegg–Barnett (P–B) oscillator with a finite-dimensional number-state space is considered. It is found that such a finite-dimensional oscillator possesses an su(n) Lie algebraic structure. A so-called supersymmetric P–B oscillator is suggested, and some related topics (such as the algebraic structure and the occupation number operator of the supersymmetric P–B oscillator) are briefly discussed. In addition, as one of the applications of the P–B quantization, a potential formula for the masses of charged leptons, which agrees reasonably well with the experimental values, is constructed based on the concept of supersymmetric P–B oscillator.

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1. Introduction

It is well known that the usual mathematical model of the monomode quantized electromagnetic field is the harmonic oscillator with an infinite-dimensional number-state space, the commuting relation of which is $[a, a^\dagger] = \mathcal{I}$ with a and $a^\dagger$ being the single-mode photon annihilation and creation operators, respectively. Due to the permutation invariance for the trace of the product of two matrices (operators), *i.e.*, $\mathrm{tr}(aa^\dagger) = \mathrm{tr}(a^\dagger a)$, it follows directly that the trace of commutator is vanishing, *i.e.*, $\mathrm{tr}[a, a^\dagger] = 0$, which, however, contradicts the fact that the identity matrix $\mathcal{I}$ possesses a nonvanishing trace, namely, $\mathrm{tr}\mathcal{I} \neq 0$. This, therefore, means that there exist no such representations with finite number of generators for the Heisenberg algebra (non-semisimple Lie algebra). So, we should consider the oscillator

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algebra with finite-dimensional state spaces, which is a semisimple algebraic extension of Bosonic oscillator algebra. On the other hand, in an attempt to investigate the number-phase uncertainty relations of the maser and squeezed state in quantum optics, physicists meet, however, with difficulties arising from a fact that the classical observable phase of light *unexpectedly* has no corresponding Hermitian operator counterpart (quantum optical phase) [1–3]. In this subject, several problems we encountered are as follows: *(i)* the exponential-form operator $\exp[i\hat{\phi}]$ (with $\hat{\phi}$ being the phase operator) is not unitary; *(ii)* the number-state expectation value of Dirac's quantum relation $[\hat{\phi}, \hat{N}] = -i$ (with $\hat{N}$ being the occupation-number operator of photon fields) is even zero, *i.e.*, $\langle n|[\hat{\phi}, \hat{N}]|n\rangle = 0$; *(iii)* the number-phase uncertainty relation $\Delta N \Delta\phi \geq \frac{1}{2}$ would imply that a well-defined number state would actually have a phase uncertainty of greater than 2π [4]. In order to overcome these difficulties, Pegg and Barnett suggested an alternative, and physically indistinguishable, mathematical model of the single-mode field involving a finite but arbitrarily large state space [4], in which a phase state was defined as follows

$$|\theta\rangle = \lim_{s\to\infty}(s+1)^{-\frac{1}{2}}\sum_{n=0}^{s}\exp(in\theta)|n\rangle, \tag{1}$$

where $\{|n\rangle\}$ $(n = 0, 1, 2, ..., s)$ are the $s+1$ number states, which span an $(s+1)$-dimensional state space. This, therefore, means that the state space $\{|n\rangle\}$ with $0 \leq n \leq s$ has a finitely upper level ($|s\rangle$) and the maximum occupation number of particles is s rather than infinity. In their new quantization formulation, the dimension of number state space is allowed to tend to infinity after physically measurable results are calculated [4]. Pegg and Barnett showed that this approach and the conventional infinite state space are physically indistinguishable. However, this method has the additional advantage of being able to incorporate a well-behaved Hermitian phase operator within the formalism. The resulting number-phase commutator in the Pegg–Barnett (P–B) approach does not yet lead to any inconsistencies, but satisfies the condition for Poisson-bracket-commutator correspondence. It was shown that such an approach has several advantages over the conventional Susskind–Glogower formulation [2]. For example, the P–B phase operator is consistent with the vacuum being a state of random phase, while the Susskind–Glogower phase operator does not demonstrate such a property of vacuum [4]. The P–B formulation is useful for treating the problems of atomic coherent population trapping (CPT) and electromagnetically induced transparency (EIT) [5].

In this paper we will further consider the P–B harmonic oscillator that involves a finitely large state space, and show that it possesses an su(n) Lie algebraic structures. Based on this consideration, we will generalize the P–B oscillator to a supersymmetric case. It will be demonstrated that the multiphoton interaction process, which causes the k-photon absorption and emission in each atomic transition, can model the behavior of the supersymmetric P–B oscillator.

2. The su(n) Lie algebraic structures in the P–B oscillator

The quantum harmonic oscillator possessing an infinite-dimensional number-state space (*i.e.*, the maximum occupation number s tends to infinity) can well model the Bosonic fields. Taking account of the P–B harmonic oscillator means that the non-semisimple Lie algebra should be generalized to a semisimple case, namely, the quantum commutator $[a, a^{\dagger}] = \mathcal{I}$ will be replaced with a new commuting relation $[a, a^{\dagger}] = \mathcal{A}$ ($\mathcal{A}$ will be defined in what follows). For a preliminary consideration, we first take into account the case $s = 1$, where the matrix representation of the annihilation (creation) operators and $\mathcal{A}$ of the fields are of the form (in the set of number-state base vectors)

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a^{\dagger} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

It is apparently seen that the operators a, $a^{\dagger}$ and $\mathcal{A}$ satisfy an sl(2) algebraic commuting relations. Here one can readily verify that $a = (\sigma_1 + i\sigma_2)/2$, $a^{\dagger} = (\sigma_1 - i\sigma_2)/2$ and $\mathcal{A} = \sigma_3$, where σ_i's ($i = 1, 2, 3$) are Pauli's matrices. It follows from (2) that $aa^{\dagger} + a^{\dagger}a = \mathcal{I}$. Clearly, the algebraic generators of su(2) Lie algebra can be constructed in terms of the matrices in (2). This, therefore, implies that the P–B harmonic oscillator with $s = 1$ corresponds to the fermionic field and possesses an su(2) Lie algebraic structure.

As another illustrative example, we will take into consideration the case of $s = 2$, the matrix representation of a, $a^{\dagger}$ and $\mathcal{A}$ of which are written

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad a^{\dagger} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{3}$$

Calculation of the commutators among the Lie algebraic generators of the P–B harmonic oscillator with $s = 2$ yields

$$\begin{aligned} [a, \mathcal{A}] &= 3\sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = 3\sqrt{2}\mathcal{M}, \\ [a^{\dagger}, \mathcal{A}] &= -3\sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = -3\sqrt{2}\mathcal{M}^{\dagger}, \end{aligned}$$

$$
\begin{aligned}
\left[\mathcal{M},\mathcal{M}^{\dagger}\right] &= -\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -\mathcal{K}, \\
[a,\mathcal{M}] &= -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\mathcal{F}, \quad \left[a^{\dagger},\mathcal{M}\right] = -\sqrt{2}\mathcal{K}, \\
\left[a,\mathcal{M}^{\dagger}\right] &= \sqrt{2}\mathcal{K}, \quad \left[a^{\dagger},\mathcal{M}^{\dagger}\right] = \mathcal{F}^{\dagger}, \quad [\mathcal{K},\mathcal{F}] = -\mathcal{F}, \quad [\mathcal{K},\mathcal{F}^{\dagger}] = \mathcal{F}^{\dagger}. \qquad (4)
\end{aligned}
$$

It is readily verified that the algebraic generators $a, a^{\dagger}, \mathcal{A}, \mathcal{M}, \mathcal{M}^{\dagger}, \mathcal{K}, \mathcal{F}, \mathcal{F}^{\dagger}$ form an sl(3) algebra. The eight Gell-Mann matrices can therefore be constructed in terms of them, *i.e.*,

$$
\begin{aligned}
&\lambda_1 = a + a^{\dagger} + \sqrt{2}(\mathcal{M} + \mathcal{M}^{\dagger})\,, \quad \lambda_2 = i[a^{\dagger} - a + \sqrt{2}(\mathcal{M}^{\dagger} - \mathcal{M})]\,, \\
&\lambda_3 = \mathcal{A} + 2\mathcal{K}, \quad \lambda_4 = \mathcal{F} + \mathcal{F}^{\dagger}\,, \quad \lambda_5 = i(\mathcal{F}^{\dagger} - \mathcal{F})\,, \\
&\lambda_6 = -(\mathcal{M} + \mathcal{M}^{\dagger})\,, \quad \lambda_7 = -i(\mathcal{M}^{\dagger} - \mathcal{M})\,, \quad \lambda_8 = \frac{1}{\sqrt{3}}\lambda_8\,. \qquad (5)
\end{aligned}
$$

It is thus demonstrated that the P–B harmonic oscillator with $s = 2$ possesses an su(3) Lie algebraic structure.

In what follows, we will study the algebraic structures of P–B harmonic oscillators with arbitrary occupation numbers. For the P–B oscillator with a finite but arbitrarily large state space of $s + 1$ dimensions, the matrix representation (in the set of number-state base vectors) of the operators a, $a^{\dagger}$ and $\mathcal{A}$ takes the following form

$$
a_{mn} = \sqrt{n}\delta_{m,n-1}, \quad a^{\dagger}_{mn} = \sqrt{n+1}\delta_{m,n+1}, \quad \mathcal{A}_{mn} = \delta_{mn} - (s+1)\delta_{ms}\delta_{ns}, \qquad (6)
$$

where the subscript m, n (which run from 0 to s only) denote the matrix row-column indices. The remaining generators $\mathcal{M}, \mathcal{M}^{\dagger}, \mathcal{K}, \mathcal{F}, \mathcal{F}^{\dagger}, \ldots$ can be obtained as follows

$$
\begin{aligned}
&[a,\mathcal{A}]_{mn} = (s+1)\sqrt{s}(-\delta_{m+1,s}\delta_{ns}) = (s+1)\sqrt{s}\mathcal{M}_{mn}\,, \\
&\left[a^{\dagger},\mathcal{A}\right]_{mn} = -(s+1)\sqrt{s}(-\delta_{ms}\delta_{n+1,s}) = -(s+1)\sqrt{s}\mathcal{M}^{\dagger}_{mn}\,, \\
&\left[\mathcal{M},\mathcal{M}^{\dagger}\right]_{mn} = -(\delta_{ms}\delta_{ns} - \delta_{m+1,s}\delta_{n+1,s}) = -\mathcal{K}_{mn}\,, \\
&[\mathcal{A},\mathcal{M}] = (1+s)\mathcal{M}, \quad \left[\mathcal{A},\mathcal{M}^{\dagger}\right] = -(1+s)\mathcal{M}^{\dagger}\,, \\
&[a,\mathcal{M}]_{mn} = -\sqrt{s-1}\delta_{m+1,s-1}\delta_{ns} = -\sqrt{s-1}\mathcal{F}_{mn}\,, \\
&\left[a^{\dagger},\mathcal{M}^{\dagger}\right]_{mn} = \sqrt{s-1}\delta_{ms}\delta_{n+1,s-1} = \sqrt{s-1}\mathcal{F}^{\dagger}_{mn}\,, \\
&[\mathcal{K},\mathcal{F}] = -\mathcal{F}, \quad \left[\mathcal{K},\mathcal{F}^{\dagger}\right] = \mathcal{F}^{\dagger}, \quad [\mathcal{M},\mathcal{K}] = 2\mathcal{M}, \quad \left[\mathcal{M}^{\dagger},\mathcal{K}\right] = -2\mathcal{M}^{\dagger}\,, \ldots \qquad (7)
\end{aligned}
$$

where $0 \leq m, n \leq s$. For the case of $s = 2$, it has been shown above that Hermitian operators (such as the eight Gell-Mann matrices) can be constructed in terms of $a, a^{\dagger}, \mathcal{A}, \mathcal{M}, \mathcal{M}^{\dagger}, \mathcal{K}, \mathcal{F}, \mathcal{F}^{\dagger}$. Likewise, here the Hermitian operators (generators) of Lie algebra can also be obtained via the linear combination of the above generators (7). If $\mathcal{G}$ represents the linear combination of the Hermitian operators, and consequently $\mathcal{G} = \mathcal{G}^{\dagger}$, then the exponential-form group element operator $U = \exp(i\mathcal{G})$ is unitary. Besides, since a, $a^{\dagger}$ and $\mathcal{A}$ are traceless, all the generators derived by the commutators in (7) (and hence $\mathcal{G}$) are also traceless due to the cyclic invariance for the trace of matrices product. Thus the determinant of the group element U is unity, *i.e.*, $\det U = 1$, because of $\det U = \exp[\mathrm{tr}(i\mathcal{G})]$. Since it is known that such a group element U that satisfies simultaneously the above two conditions is the group element of the su(n) Lie group, the high-dimensional Gell-Mann matrices, which closes the corresponding su(n) Lie algebraic commutation relations among themselves, can also be constructed in terms of the generators $a, a^{\dagger}, \mathcal{A}, \mathcal{M}, \mathcal{M}^{\dagger}, \mathcal{K}, \mathcal{F}, \mathcal{F}^{\dagger}, \ldots$ presented above. It is thus concluded that the P–B harmonic oscillator with a maximum occupation number s has an $(s+1)$-dimensional number-state space and possesses an su$(s+1)$ Lie algebraic structure.

Consideration of the case of $s \to \infty$ is of typical interest. Apparently, it is seen that when s approaches infinity, $\mathcal{A}$ will tend to an identity matrix $\mathcal{I}$, and all the remaining generators (except a and $a^{\dagger}$), the off-diagonal matrix elements of which approach zero, are hence reduced to $\mathcal{O}$. This, therefore, means that the P–B harmonic oscillator with an infinite-dimensional state space corresponds just to the oscillator of a Bosonic field.

3. Supersymmetric P–B oscillator and its potential applications

In the preceding section, we extend the non-semisimple algebra of harmonic oscillator with an infinite-dimensional state space to a semisimple algebraic case, which can characterize the algebraic structures of the P–B oscillator. In what follows we will consider a generalization of P–B oscillator, *i.e.*, the so-called supersymmetric P–B oscillator, which may possess some physically interesting significance.

3.1. Supersymmetric algebra and its physical realization

For this aim, first we take into account a set of algebraic generators $(N, N^{'}, Q, Q^{\dagger})$, which possesses a supersymmetric Lie algebraic properties, *i.e.*,

$$
\begin{aligned}
& Q^2 = (Q^\dagger)^2 = 0, \quad \left[Q, Q^\dagger\right] = N'\sigma_3\,, \quad \left[N, N'\right] = 0\,, \quad [N, Q] = -Q\,, \\
& \left[N, Q^\dagger\right] = Q^\dagger\,, \quad \left\{Q, Q^\dagger\right\} = N'\,, \quad \{Q, \sigma_3\} = \left\{Q^\dagger, \sigma_3\right\} = 0\,, \\
& [Q, \sigma_3] = -2Q\,, \quad \left[Q^\dagger, \sigma_3\right] = 2Q^\dagger\,, \quad \left(Q^\dagger - Q\right)^2 = -N'\,,
\end{aligned} \tag{8}
$$

where $\{\}$ denotes the anticommuting bracket. Such a Lie algebra (8) can be physically realized by a two-level multiphoton Jaynes–Cummings model, the Hamiltonian of which is of the form (in the rotating wave approximation) [6–8]

$$
H = \omega a^\dagger a + \frac{\omega_0}{2}\sigma_3 + g(a^\dagger)^k \sigma_- + g^* a^k \sigma_+, \tag{9}
$$

where $a^\dagger$ and a stand for the creation and annihilation operators for the electromagnetic field, respectively, and obey the commutation relation $\left[a, a^\dagger\right] = 1$; $\sigma_\pm$ and σ_3 denote the two-level atomic operators, which satisfy the commutation relation $[\sigma_3, \sigma_\pm] = \pm 2\sigma_\pm$, $[\sigma_+, \sigma_-] = \sigma_3$; g (g^*) is the coupling coefficient and k the total photon number in each atomic transition process; ω_0 and ω represent the atomic transition frequency and the photon mode frequency, respectively. By the aid of both the commutation relations (8) and the following expressions (10) [9–11]

$$
\begin{aligned}
N &= a^\dagger a + \frac{k-1}{2}\sigma_3 + \frac{1}{2} = \begin{pmatrix} a^\dagger a + \frac{k}{2} & 0 \\ 0 & a a^\dagger - \frac{k}{2} \end{pmatrix}, \\
N' &= \begin{pmatrix} \frac{a^k (a^\dagger)^k}{k!} & 0 \\ 0 & \frac{(a^\dagger)^k a^k}{k!} \end{pmatrix}, \\
Q^\dagger &= \frac{1}{\sqrt{k!}}(a^\dagger)^k \sigma_- = \begin{pmatrix} 0 & 0 \\ \frac{(a^\dagger)^k}{\sqrt{k!}} & 0 \end{pmatrix}, \\
Q &= \frac{1}{\sqrt{k!}} a^k \sigma_+ = \begin{pmatrix} 0 & \frac{a^k}{\sqrt{k!}} \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{10}
$$

Hamiltonian (9) of the two-level multiphoton Jaynes–Cummings model can be rewritten as

$$
H = \omega N + \frac{\omega - \delta}{2}\sigma_3 + g Q^\dagger + g^* Q - \frac{\omega}{2} \tag{11}
$$

with the frequency detuning $\delta = k\omega - \omega_0$. In the following, δ is taken to be zero. It can be verified that under this condition the interaction in the multiphoton system can model the supersymmetric P–B oscillator.

3.2. Modelling the supersymmetric P–B oscillator

With the help of the relations $\frac{1}{k!}a^k(a^\dagger)^k\left|m\right\rangle = \frac{(m+k)!}{m!k!}\left|m\right\rangle$ and $\frac{1}{k!}(a^\dagger)^k a^k\left|m+k\right\rangle = \frac{(m+k)!}{m!k!}\left|m+k\right\rangle$, one can arrive at

$$N^{'}\begin{pmatrix} \left|m\right\rangle \\ \left|m+k\right\rangle \end{pmatrix} = C^m_{m+k}\begin{pmatrix} \left|m\right\rangle \\ \left|m+k\right\rangle \end{pmatrix} \tag{12}$$

with the combination coefficient

$$C^m_{m+k} = \frac{(m+k)!}{m!k!}. \tag{13}$$

One can, therefore, obtain the following supersymmetric quasialgebra $(N, Q, Q^\dagger, \sigma_3)$ in a sub-Hilbert-space corresponding to the particular eigenvalue C^m_{m+k} of the Lewis–Riesenfeld invariant operator $N^{'}$ [11] by replacing the generator $N^{'}$ with C^m_{m+k} in the commutation relations in (8), namely,

$$\left[Q, Q^\dagger\right] = C^m_{m+k}\sigma_3\,, \quad \left\{Q, Q^\dagger\right\} = C^m_{m+k}\,, \quad \left(Q^\dagger - Q\right)^2 = -C^m_{m+k}\,. \tag{14}$$

Based on such a quasialgebra in the sub-Hilbert-space, one can propose a concept of the supersymmetric P–B oscillator. The algebraic generators of such a generalized P–B oscillator agree with the commuting relation (14), where $Q^\dagger$ and Q can be regarded as the creation and annihilation operators, respectively, and the eigenvalue C^m_{m+k} of $N^{'}$ may be considered the particle occupation number of the supersymmetric P–B oscillator in a certain number state.

In the following, we consider the problem of eigenvalue equation of the supersymmetric multiphoton system. Since $\left|m\right\rangle\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\left|m+k\right\rangle\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ form a complete set of base vectors in the sub-Hilbert-space corresponding to the particular eigenvalue C^m_{m+k} of the generator $N^{'}$, the eigenstate $\left|\Psi_m\right\rangle$ of the Hamiltonian (11) can be written as a liner combination of these base vectors, *i.e.*,

$$\left|\Psi_m\right\rangle = \begin{pmatrix} c_+\left|m\right\rangle \\ c_-\left|m+k\right\rangle \end{pmatrix}, \tag{15}$$

where $c_\pm$ denotes the time-independent coefficients. By using the relation (the eigenvalue equation of the free Hamiltonian)

$$\omega\left[\left(N - \frac{1}{2}\right) + \frac{1}{2}\sigma_3\right]\begin{pmatrix} \left|m\right\rangle \\ \left|m+k\right\rangle \end{pmatrix} = \omega\left(m + \frac{k}{2}\right)\begin{pmatrix} \left|m\right\rangle \\ \left|m+k\right\rangle \end{pmatrix}, \tag{16}$$

where both N and σ_3 are the diagonal matrices, one can obtain the eigenvalue equation of the Hamiltonian (11) as follows

$$H|\Psi_m\rangle = \left[\omega\left(m + \frac{k}{2}\right) + \epsilon_g\right]|\Psi_m\rangle\,. \tag{17}$$

As the squared of the interaction Hamiltonian $gQ^\dagger + g^*Q$ is $g^*g\left(QQ^\dagger + Q^\dagger Q\right)$, where the relation $(Q^\dagger)^2 = Q^2 = 0$ has been substituted, the eigenvalue equation (17) can be rewritten as

$$\left(QQ^\dagger + Q^\dagger Q\right)|\Psi_m\rangle = \frac{\epsilon_g^2}{g^*g}|\Psi_m\rangle\,, \tag{18}$$

which can be viewed as an eigenvalue equation for the occupation number operator of the supersymmetric P–B oscillator. Thus we have shown that the multiphoton process can model the behavior of the supersymmetric P–B oscillator. Since the eigenvalue of $QQ^\dagger + Q^\dagger Q$ is C^m_{m+k}, the parameter ϵ_g^2 in Eq. (18) is

$$\epsilon_g^2 = g^*gC^m_{m+k}\,. \tag{19}$$

3.3. Mass spectrum of charged leptons

The extension of P–B formulation to the supersymmetric case may be physically interesting. Here, we will consider a potential application of the supersymmetric P–B oscillator to the mass spectrum of charged leptons. Based on the experimental values for the masses of charged leptons (*i.e.*, electron, μ and τ) and the concept of the above supersymmetric P–B oscillator, we may construct a mass formula for the charged leptons, *i.e.*,

$$m_n = C_3^n\left(\frac{1}{2}\right)^{n^2}\left(\frac{1}{\alpha}\right)^n m_e \tag{20}$$

with m_e and α being the electron mass and the electromagnetic fine structure constant, respectively. The integer n in (20) denotes the leptonic generation label (generation quantum number) standing for the various generations of charged leptons, *i.e.*, the electron (e), muon (μ) and tau (τ) particles correspond to $n = 0, 1, 2$, respectively. Note that here we have assumed that the Hamiltonian of the supersymmetric P–B oscillator may be written in the form $H = \frac{1}{2}\left\{Q, Q^\dagger\right\}\Omega$ by analogy with the Hamiltonian ($H = \frac{1}{2}\left\{a, a^\dagger\right\}\omega$) of the Bosonic oscillator, and that such a supersymmetric P–B oscillator might have close relation to the charged lepton mass spectrum, say, the rest mass of charged leptons may be proportional to the eigenvalues (C^m_{m+k}) of the operator $\left\{Q, Q^\dagger\right\}$. Thus, by using the experimental data for charged lepton masses and introducing the generation quantum number n and fine structure constant α, we can construct the mass formula (20). The present mass formula for the cases of $n = 0, 1, 2$ agrees to the experimental results about one part in 10^2: specifically, the relative precession of formula (20) for muon is -0.59% and for tau $+1.3\%$ (here, α is taken to be $1/137.036$. The relative precession may be more ideal for the tau particle if the running coupling coefficient is taken into account). It follows from the charged leptons mass formula (20) that the maximum leptonic generation label n can be taken to be three and the total generation number of charged leptons may therefore be four, and hence there might exist a fourth charged lepton, which (should such exist) is, however, unknown both theoretically and experimentally up to now.

The occurrence of the leptonic fermion chain $(e,\ \mu,\ \tau,\ \ldots)$ is a novel phenomenon for which we have so far not any theories to interpret the origin of the generation of leptons. Regarding this subject (*i.e.*, generation phenomenon), the fundamental problems may be as follows: why does the generation phenomenon exist? What causes the generation number of fermions to be not up to three [12]? Studying the mass formula for leptons may provide clue to the physicists on how the fundamental mechanism involved works in the above-mentioned problems. For this aim, historically, several authors probed the lepton and quark mass spectra [13–15]. The previous charged lepton mass formulae may have two disadvantages: *(i)* they could not agree with experimental results very well; *(ii)* the generation number of fermions in these mass formulae were often infinite or they could not account for the finite-generation-number phenomenon of fermions. It is, however, of physical interest that the finite-generation-number phenomenon may be in some sense explained by the charged lepton mass spectrum (20) in the present paper.

Although several experimental evidences have shown that the total number of the generations of fermion chain is three [12], many researchers has so far tried to explore the possibility of the existence of the fourth generation [16–26]. For example, since the latest electroweak precision data allows the existence of additional chiral generations in the standard model [17], Arik *et al.* studied the influence of extra generations on the production of the standard model Higgs boson at hadron colliders [16, 17]. Some authors considered the exotic interactions involving fourth-generation quarks and leptons which cannot be confused experimentally with those of the standard model, or suggested a completely different interaction model for the extra-generation fermions [22–31]. These studies (extension of the standard electroweak gauge model to include a fourth generation of fermions) may provide a possible test of fourth generation and would probably give a signal of new physics [22–26, 30, 31].

4. Concluding remarks

We considered the su(n) Lie algebraic structures for the P–B quantization formulation and generalized the P–B oscillator to the supersymmetric case. It was shown that the fermionic and bosonic fields are two special cases of P–B oscillator, the corresponding dimensions of state spaces of which are two and infinity, respectively. A property that the multiphoton system, which possesses a supersymmetric Lie algebraic structure, can model the behavior of the supersymmetric P–B oscillator was demonstrated in the present paper. The potential application of the supersymmetric P–B oscillator algebra to the mass spectrum of charged leptons was briefly suggested. As far as the generalized P–B oscillator is concerned, the algebraic commutation relation (14) may clue physicists on the mathematical mechanism and physical meanings of the above mass spectrum of charged leptons. Even though at present it is well known that various experimental evidences show that there are only three generations of fundamental particles, the detection of potentially new generations of particles is still of physical interest. We hope that the consideration of algebraic structures of P–B oscillator presented here may open up new opportunities for investigating the generation quantum number (and hence the generation structure) of particles as well as other related topics such as fractional statistics, anyon [32, 33] and cyclic representation of quantum algebra (group) [34].

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