

CHANGES IN THE GLOBAL PARAMETERS OF POLYTROPIC STARS INDUCED BY THE APPEARANCE OF THE SOFT CORE

M. BEJGER

N. Copernicus Astronomical Center
Bartycka 18, 00-716 Warsaw, Poland
`bejger@camk.edu.pl`

(Received May 23, 2005)

The effect of a soft phase core appearance in the center of polytropic star is analyzed by means of linear response theory. Approximate formulae for the changes of radius, moment of inertia and mass-energy of non-rotating configuration with arbitrary adiabatic indices are presented, followed by an example evaluation of astrophysical observables.

PACS numbers: 04.40.Dg, 97.60.Jd

1. Introduction

In spite of lack of precise Earth-based experimental data involving matter at densities higher than the nuclear saturation density the astrophysical observations of compact objects give us an unique chance of understanding the underlying physics. At present it cannot be excluded that at certain density phase transition in nuclear matter will produce a state not observed in laboratories *e.g.* pion or kaon condensation, or de-confined quarks (see [1] for review). If the phase transition happens to be of the first order, the consequences from the point of view of observations are extremely interesting. Namely, in the first order phase transition, the new phase arise only by nucleation. This means that the meta-stable core formed during the compact star evolution (*e.g.* accretion, spin-down) nucleates to a stable new-phase core. The transition is therefore accompanied by star-quake — the radius change, energy release and possible other violent phenomena.

This work is an extension of the linear response theory developed by Haensel, Zdunik and Schaeffer [2, 3] and employed to the case of first order phase transition from one pure phase (normal, N-phase) to other pure phase (super-dense, S-phase). While one demands that the transition should

conserve *locally* electric charge and the number of particles, it occurs at constant pressure, technically by so-called “Maxwell construction” [4]. In the presence of the gravitational field it is accompanied by a density jump at the boundary of phases.

Due to the works of Glendenning [5, 6] it became clear that relaxing the condition of local electrical neutrality permits for a coexistence of *both* phases within a range of pressures — a *structural* mixed-phase transition. The volume fraction occupied by the higher-density phase increases from zero at the lower pressure boundary to one at the upper pressure boundary. In realistic matter, if the surface tension and Coulomb contribution to the energy is not too large, the mixed-phase is preferred over a pure phases state.

Here we derive the formulae suitable for description of the change of the parameters (radius, moment of inertia, mass-energy) of a compact star under the transition from a meta-stable core to stable core composed of the “structural mixed-phase”. These changes are proportional to the specific powers of the newly-born core. We will neglect the rotation of the star. For simplicity, the equation of state (EOS) of matter in both N- and the mixed-phases will be approximated by polytropes. The methods used here are similar to those used in a somewhat more complicated case of realistic N-phase EOSs presented in [7].

The article is arranged as follows: Sect. 2 provides brief description of the theoretical background and methods used, Sect. 3 contains results of numerical calculations, and Sect. 4 includes conclusion and final remarks.

2. Linear response theory

We will describe theoretical background of star linear response to the appearance of the soft “mixed-phase” core. The calculation is based on expressing the change in the density profile, due to the presence to a small core, as the combination of two independent solutions of the linearly perturbed equations of stellar structure [2, 3]. The presence of a denser phase in the core changes the boundary condition at the phase transition pressure P_0 (Fig. 1) and allows to determine the numerical coefficients in the expression for the density profile change. The leading term in the perturbation of the boundary condition at the edge of the new phase results from the mass excess due to the lower stiffness, and higher density of the new phase as compared to those of the (normal, less-dense) N-phase.

We assume that at a central pressure $P_c = P_{\text{crit}}$ the nucleation of the S-phase in a super-compressed core of radius r_N , of configuration C, initiates the phase transition and formation of the “mixed-phase” core of radius r_m in a new configuration C*, as shown on Fig. 1. Transition to a mixed-phase occurring at r_m is associated with a substantial drop in the adiabatic index

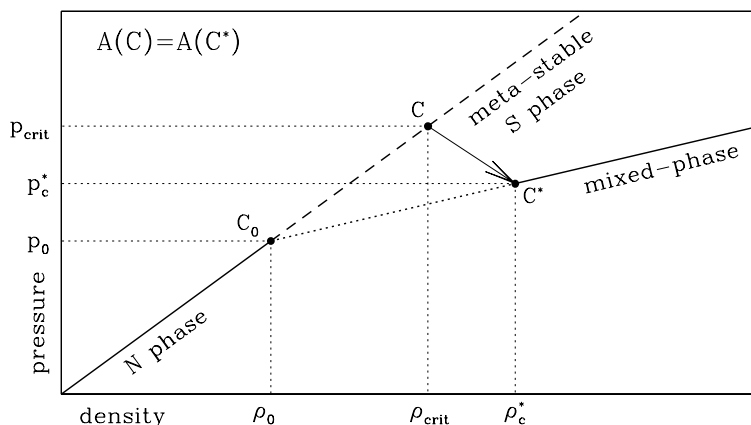


Fig. 1. Schematic plot of central pressure P_c versus central matter density ρ_c for configurations based on a pure N-phase EOS and an EOS with a mixed-phase segment. The solid line denotes stable states, the dashed line — the states which are meta-stable with respect to the transition to a mixed-phase state. For a critical central density ρ_{crit} the S-phase nucleates in the super-compressed core of configuration C, which results in the transition into a stable configuration C* with a mixed-phase core and a central density ρ_c^* . Configurations C and C* have the same baryon number A .

of matter, defined as $\gamma \equiv (n_b/P)dP/dn_b$, from γ_N to γ_m . In realistic EOSs, mixed-phase is softer than the pure one, because the increase of mean density is reached partly via conversion of a less dense N-phase into denser S-phase, and therefore requires less pressure than for a pure-phase. This remains true for any fraction of the S-phase, and leads to a discontinuity of γ at the phases boundary, ρ_0 . Realistic examples of such transitions are given by [8–10] and [11] (Table 9.1). In all those cases, near the boundary logarithm of pressure depends linearly on the logarithm of baryon density; it clearly indicates that the polytropic approximation of mixed-phase is valid in the small mixed-phase core regime. We will henceforth benefit from expansion in powers of the (tiny) core radius r_m .

As far as global properties of stars are concerned, the hydrostatic stars' equilibria corresponding to EOSs with and without the phase transition must be compared. The models are non-rotating, spherically symmetric solutions of Einstein's equations, usually called Tolman–Oppenheimer–Volkoff (TOV) equations [12, 13]. The moment of inertia was calculated using the slow-rotation approximation [14]. Hydrostatic solutions are labeled by their central density ρ_c . The configurations based on the two EOSs are identical up to $\rho_c = \rho_0$; the configuration with such central density will be denoted by C_0 and will be called a “reference configuration”.

As was demonstrated by [7], the leading changes due to a phase transition are proportional to the fifth power of r_m in the case of stellar radius R and the moment of inertia I and seventh power in the case of the gravitational mass-energy $E = Mc^2$. During calculations, we assume constant γ_m in the “mixed-phase” — in principle, the inclusion of r_m -dependent γ_m contributes to the higher order terms, but this contribution is negligible in the case of realistic EOSs (see [7] for details).

When the central density exceeds ρ_0 , the models begin to differ due to the appearance of a softer “mixed-phase” core in configurations corresponding to the mixed-phase segment in the EOS. For two stars composed of equal number of baryons A , greater than a baryon number A_0 of the reference configuration C_0 , we compare the global parameters — mass-energy, radius, and moment of inertia. Their difference corresponds to the changes in these parameters implied by the phase transition in the stellar core.

The new phase influences the boundary conditions by the presence of the prefactor $(\gamma_N/\gamma_m - 1)$, which is qualitatively the same as $(\rho_S/\rho_N - 1)$ in the case of a density jump from pure N-phase to pure S-phase [2, 3]. Moreover, the results obtained by [7] indicate that linear response effects should be proportional to the core radius power greater by two than in the case of the transition between pure-phases. One would expect that the relative change of the stellar parameter $Q = R, I, E$ takes the following form:

$$\delta\bar{Q} \equiv \frac{Q^* - Q}{Q_0} \simeq -\beta_Q \left(\frac{\gamma_N}{\gamma_m} - 1 \right) (\bar{r}_m)^l, \quad (1)$$

where $\bar{r}_m \equiv r_m/R_0$, the power $l = 5$ corresponds to the radius R and moment of inertia I , and $l = 7$ for the mass-energy $E = Mc^2$. The coefficients β_Q are then the functionals of a reference configuration, *i.e.* $\beta_Q = \beta_Q(C_0)$. The validity of the above statements will be confirmed in the next section by means of numerical calculations.

3. Results

In order to obtain quantitative results the polytropic EOSs for the N- and mixed-phases will be used. The main reason why we have chosen the polytropic EOSs are precision of numerical calculation and simplicity during the exploration of the parameter space. Discussion of the polytropic EOSs and their application to relativistic stellar structure calculations was presented by [15]. Details needed for the calculations are given in the Appendix. Section 2 described the form of the expressions for the changes of stellar parameters obtained in the linear approximation. According to Eq. (1), the coefficients split into two factors, first one depending on the

mixed-phase via γ_s , and the second one β_Q being a functional of the reference configuration C_0 . As it is a complicated task to write an analytical form of the functional $\beta_Q(C_0)$ we will restrict ourselves to the derivation of approximate, but accurate fitting formulae, based on the results of precise numerical calculations.

The β_I, β_R and β_E coefficients obtained in the limit of vanishing mixed-phase core $r_m \rightarrow 0$ as a function of the mass of the sample reference configuration $M(C_0)$ are plotted in Fig. 2. Whole range of masses of the reference configuration is presented on account of our ignorance about the density at which the phase transition takes place. Results were obtained for specific choices of γ_N and γ_m . The values of γ_N and K_N were chosen in such a way, that the EOS of the N-phase produced massive ($M \gtrsim M_\odot$) neutron-star models similar to those obtained for realistic stiff EOS of dense matter. In particular, this EOS yields $M_{\max} = 2.37 M_\odot$ and $R_{M_{\max}} = 12.52$ km. Let us stress, however, that while this EOS is a reasonable representation of the EOS of matter with $\rho > 2\rho_0$, it is completely unrealistic at sub-nuclear densities and for masses much smaller than one solar mass.

One notices a characteristic behavior of β_R and β_I . Despite the fact that we do not know the precise density of onset of the new phase, this information is not substantial to predict stars response. The coefficients are almost constant for a wide range of masses for β_I and β_R . The behavior of β_E is different: for $M \lesssim 0.8 M_{\max}$ this coefficient is proportional to $M(C_0)$.

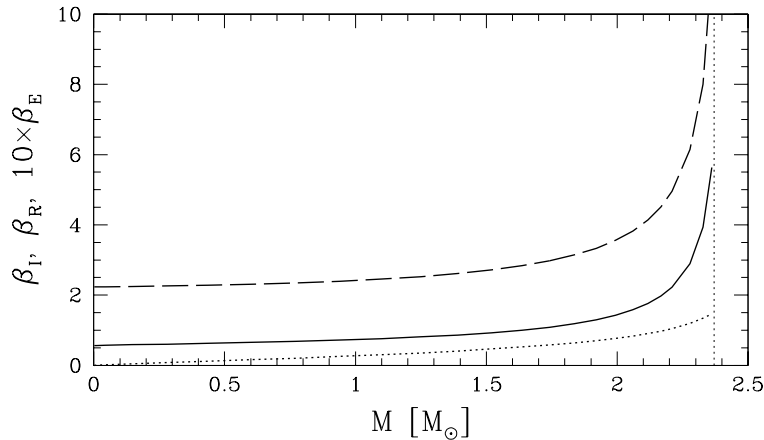


Fig. 2. The response coefficients $\beta_I, \beta_R, \beta_E$ (dashed, solid and dotted line, the value of β_E was multiplied by a factor of 10) plotted against the stellar mass for a sample polytropic EOS, $\gamma_N = 2.5$, $\gamma_m = 1.5$, $K_N = 0.025$ (see the Appendix for details). The vertical dotted line denotes the maximal mass of the star, $2.37 M_\odot$.

The “plateau” values of the coefficients β_I and β_R do not depend on the pressure coefficient K_N , but on γ_N only. The formulae which describe accurately those response parameters read

$$\beta_R(C_0) \simeq \frac{0.015 \times \gamma_N^{9.4}}{(\gamma_N^{1.13} - 1)^{8.2}}, \quad \beta_I(C_0) \simeq \frac{0.12 \times \gamma_N^{9.1}}{(\gamma_N^{1.22} - 1)^{7.5}}, \quad (2)$$

for β_R and β_I , respectively.

The case of the mass-energy parameter β_E is different, as it can be seen in Fig. 2. As we mentioned before, sufficiently far from the maximal mass, it is proportional to $M(C_0)$. We can approximate the value of $\beta_E(C_0)$ as follows:

$$\beta_E(C_0) \simeq \frac{0.085}{\gamma_N^{2.56} K_N^{0.5(\gamma_N-1)}} \times \left(\frac{M}{M_\odot} \right). \quad (3)$$

It has to be mentioned that the fitting formulae from Eqs. (2), (3) have been checked against the adiabatic indices in the range $5/3$ to 3.5 . The fitted expressions are fairly accurate within a wide range of masses, to within few per-cent compared to the exact numerical calculations. The response coefficients underestimate the magnitude of the linear response for configurations near the maximum allowable mass, $M_{\max}$. The increase of β_Q for $M(C_0) \rightarrow M_{\max}$ is due to a “softening” of the reference configuration by the effects of General Relativity (similar to the case presented in [2, 3]). Thus the linear approximation and, what follows, the approximate expressions presented above cease to be valid in this region.

It should be also expected, as far as the realistic EOS of the crust is concerned, that the assumption of constant γ in the crust is an oversimplification of the problem. Therefore, the region of small masses (smaller than $0.5 M_\odot$) is affected by this unrealistic type of crust. In the case of realistic NSs, the behavior of the β_Q coefficients near the star’s minimum mass should, due to the softening of matter, be generally similar to those near the maximal mass (such calculations for the case of realistic EOSs are presented in [7]).

The fitting formulae for β_Q in the region of validity of the linear-response approximation allow us to compute the change of the interesting stellar parameters for every pair of the parameters γ_N and γ_m . For example, the $1.4 M_\odot$ C_0 configuration from Fig. 2 has radius $R_0 = 15.18$ km and moment of inertia $I_0 = 2.27 \times 10^{45} \text{ g} \cdot \text{cm}^2$. The response coefficients are then equal $\beta_R = 0.62$, $\beta_I = 2.23$ and $\beta_E = 0.03$. The appearance of $r_m = 1$ km core with $\gamma_m = 1.5$ adiabatic index inside the $\gamma_N = 2.5$ polytrope star changes the radius by ~ 1 cm, a number which can be related to the change of radius during macro-glitches in pulsar timing. A relative change of the moment of inertia ($\Delta I/I = -\Delta\Omega/\Omega$) implies speed-up of the order of 10^{-6} ,

again a value common to pulsar macro-glitches. The released energy equals 3×10^{44} erg. If the core radius is 4 km (still well described by the linear response theory) the change in radius is 8 m, an impressive number even in view of the size of terrestrial earthquakes. The value of $\Delta\Omega/\Omega$ will be equal approximately 10^{-3} , three orders of magnitude larger than in biggest macro-glitches. Finally, the creation of a 4 km “mixed-phase” core produces $\sim 5 \times 10^{48}$ erg of energy.

4. Conclusions

The appearance of soft core was studied by means of the linear response theory to the star parameters. Simple model of a polytropic star with the “mixed-phase” core provides a set of well-approximated formulae, usable for estimation of the change of stellar parameters (radius, moment of inertia, emitted energy) for a given “stiffness” *i.e.* the adiabatic index γ of a star, the newly-born core, and its *a priori* unknown radius. The fitted formulae are precise to within a few per-cent. Resulting parameter changes are of the order of observed astrophysical phenomena, which gives hope for future observations of phase transitions inside real neutron stars. Inclusion of a varying adiabatic index certainly modifies the values of the response coefficients, presented in [7], but the comparison with the results shown here proves that the difference is not dramatic.

I warmly thank prof. P. Haensel and dr L. Zdunik for reading a preliminary version of the manuscript and for valuable comments. This work was partially supported by the Polish State Committee for Scientific Research (KBN) grant No. 1P03D-008-27 and by the Astrophysics Poland–France (Astro-PF) program.

Appendix

Relativistic polytropes

The relativistic polytrope is defined as the power-law dependence between pressure P and baryon number density n_b (see, *e.g.* [15]):

$$P(n_b) = K n_b^\gamma, \quad (4)$$

where γ is often called adiabatic index, and K is the pressure coefficient¹.

Assume that matter is strongly degenerate, so that $T = 0$ approximation is valid. For simplicity, we consider one type of baryons in the outer N-phase,

¹ The coefficient K is expressed in $\hat{\rho}c^2/\hat{n}^\gamma$ units, where $\hat{\rho} := 1.66 \times 10^{14}$ g/cm³, and $\hat{n} := 0.1$ fm⁻³.

with the rest mass $m_N = 1.66 \times 10^{-24}$ g. The total mass-energy density $\mathcal{E}$ of particles of rest mass m_N in N-phase is related to their baryon number density n_b by the First Law of Thermodynamics

$$\mathcal{E}(n_b) = \frac{K_N}{\gamma_N - 1} n_b^{\gamma_N} + m_N c^2 n_b. \quad (5)$$

Baryon chemical potential match to the change of the energy of matter, at constant P and $T = 0$, due to an increase of the baryon number by one. This implies

$$\mu(n_b) = \frac{d\mathcal{E}}{dn_b} = \frac{P + \mathcal{E}}{n_b} = \frac{K_N \gamma_N}{\gamma_N - 1} n_b^{\gamma_N - 1} + m_N c^2. \quad (6)$$

At zero pressure (the surface of the star) the chemical potential μ is equivalent to particle rest energy.

The connection between two polytropes (the one corresponding to the core being softer than the outer part, $\gamma_m < \gamma_N$, see Fig. 1 and discussion) aims for approximating the transition to the mixed-phase, known from many realistic calculations [8–11]. Thermodynamic equilibrium implies that the choice of parameters K_m , γ_m , and the *mean mass* m_m should assure the continuity of pressure and baryon chemical potential are continuous along the transition point n_0 , that is

$$K_m = K_N n_0^{\gamma_N - \gamma_m}, \quad m_m = m_N - \frac{(\gamma_N - \gamma_m) P(n_0)}{(\gamma_N - 1)(\gamma_m - 1) n_0 c^2}. \quad (7)$$

REFERENCES

- [1] F. Weber, *Pulsars as Astrophysical Laboratories for Nuclear and Particle Physics*, IoP Publishing, Bristol, Philadelphia 1999.
- [2] J.L. Zdunik, P. Haensel, R. Schaeffer, *Astron. Astrophys.* **172**, 95 (1987).
- [3] P. Haensel, J.L. Zdunik, R. Schaeffer, *Astron. Astrophys.* **160**, 251 (1986).
- [4] L.D. Landau, E.M. Lifshitz, *Statistical Physics*, Part I, Pergamon, Oxford 1993.
- [5] N.K. Glendenning, *Nucl. Phys. Proc. Suppl.* **B24**, 110 (1991).
- [6] N.K. Glendenning, *Phys. Rev.* **D46**, 1274 (1992).
- [7] M. Bejger, P. Haensel, J.L. Zdunik, *Mon. Not. Roy. Astron. Soc.* **359**, 699 (2005).
- [8] N.K. Glendenning, J. Schaffner-Bielich, *Phys. Rev. Lett.* **81**, 4564 (1998).
- [9] N.K. Glendenning, J. Schaffner-Bielich, *Phys. Rev.* **C60**, 025803 (1999).

- [10] J.A. Pons *et al.*, *Phys. Rev.* **C62**, 035803 (2000).
- [11] N.K. Glendenning, *Compact Stars: Nuclear Physics, Particle Physics, and General Relativity*, Springer, New York 1997, p. 390.
- [12] R.C. Tolman, *Phys. Rev.* **55**, 364 (1939).
- [13] R.C. Oppenheimer, G.M. Volkoff, *Phys. Rev.* **55**, 374 (1939).
- [14] J.B. Hartle, *Astrophys. J.* **150**, 1005 (1967).
- [15] R.F. Tooper, *Astrophys. J.* **142**, 1541 (1965).