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ZENO MEETS MODERN SCIENCE

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"No one has ever touched Zeno without refuting him". We will not refute Zeno in this paper. Instead we review some unexpected encounters of Zeno with modern science. The paper begins with a brief biography of Zeno of Elea followed by his famous paradoxes of motion. Reflections on continuity of space and time lead us to Banach and Tarski and to their celebrated paradox, which is in fact not a paradox at all but a strict mathematical theorem, although very counterintuitive. Quantum mechanics brings another flavour in Zeno paradoxes. Quantum Zeno and anti-Zeno effects are really paradoxical but now experimental facts. Then we discuss supertasks and bifurcated supertasks. The concept of localisation leads us to Newton and Wigner and to interesting phenomenon of quantum revivals. At last we note that the paradoxical idea of timeless universe, defended by Zeno and Parmenides at ancient times, is still alive in quantum gravity. The list of references that follows is necessarily incomplete but we hope it will assist interested reader to fill in details.

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1. Introduction

Concepts of localisation, motion and change seems so familiar to our classical intuition: everything happens in some place and everything moves from one place to another in everyday life. Nevertheless it becomes a rather thorny issue then subjected to critical analysis as witnessed long ago by Zeno's paradoxes of motion. One can superficially think that the resolution of the paradoxes was provided by calculus centuries ago by pointing out now the trivial fact that an infinite series can have a finite sum. But on the second thought we realize that this "resolution" assumes infinite divisibility of space and time and we still do not know whether the physical reality still corresponds to the continuous space and time at very small (Plankian) scales. Even in pure mathematics the infinite divisibility leads to paradoxical results like Banach–Tarski paradox which are hard to swallow despite their irrefutable mathematical correctness.

More subtle under the surface truth about Zeno's paradoxes is that even if one assumes the infinitely divisible space and time calculus does not really resolves the paradoxes but instead makes them even more paradoxical and leads to conclusion that things cannot be localised arbitrarily sharply. Of course, the latter is just what we expect from basic principles of quantum mechanics and special relativity. But it is certainly amazing to find roots of these pillars of the modern physics at Zeno's times!

2. Zeno of Elea

"No one has ever touched Zeno without refuting him, and every century thinks it worthwhile to refute him" [1]. Therefore, it seems that refuting Zeno is eternal and unchanging affair in complete accord with the Eleatic philosophy. According to this philosophy all appearances of multiplicity, change, and motion are mere illusions. Interestingly the foundation of the Eleatic philosophical school was preceded by turbulent events in drastic contrast with its teaching of the unique, eternal, and unchanging universe [2]. The school was founded by Xenophanes (born circa 570 BC), a wandering exile from his native city of Colophone in Ionia. Before finally joining the colony at Elea, he lived in Sicily and then in Catana. The Elea colony itself was founded by a group of Ionian Greeks which seize the site from the native Oenotrians. Earlier these Ionian Greeks were expelled from their native city of Phocaea by an invading Persian army. Having lost their homes, they sailed to the Corsica island and invaded it after a awful sea battle with the Carthaginians and Etruscans, just to drive once again into the sea as refugees after ten years later (in 545 BC) their rivals regained the island. We can just wonder about psychological influence of these events on the Eleatic school's belief in permanent and unalterable universe [2].

Zeno himself had experienced all treacherous vicissitudes of life. Diogenes Laertius describes him [3] as the very courageous man:

"He, wishing to put an end to the power of Nearches, the tyrant (some, however, call the tyrant Diomedon), was arrested, as we are informed by Heraclides, in his abridgment of Satyrus. And when he was examined, as to his accomplices, and as to the arms which he was taking to Lipara, he named all the friends of the tyrant as his accomplices, wishing to make him feel himself alone. And then, after he had mentioned some names, he said that he wished to whisper something privately to the tyrant; and when he came near him he bit him, and would not leave his hold till he was stabbed. And the same thing happened to Aristogiton, the tyrant slayer. But Demetrius, in his treatise on People of the same Name, says that it was his nose that he bit off.

Moreover, Antisthenes, in his Successions, says that after he had given him information against his friends, he was asked by the tyrant if there was any one else. And he replied, "Yes, you, the destruction of the city". And that he also said to the bystanders, "I marvel at your cowardice, if you submit to be slaves to the tyrant out of fear of such pains as I am now enduring". And at last he bit off his tongue and spit it at him; and the citizens immediately rushed forward, and slew the tyrant with stones. And this is the account that is given by almost every one".

Although this account of Zeno's heroic deeds and torture at the hands of the tyrant is generally considered as unreliable [4–6], Zeno after all is famous not for his brevity but for his paradoxes [7–11].

3. Zeno's paradoxes of motion

Zeno was a disciple of Parmenides, the most illustrious representative of the Eleatic philosophy. According to Parmenides, many things taken for granted, such as motion, change, and plurality, are simply illusions and the reality is in fact an absolute, unchanging oneness. Of course, nothing contradicts more to our common sense experience than this belief. It is not surprising, therefore, that Parmenides' views were ridiculed by contemporaries (and not only). In his "youthful effort" Zeno elaborated a number of paradoxes in order to defend the system of Parmenides and attack the common conceptions of things. The four most famous of these paradoxes deny the reality of motion. The Dichotomy paradox, for example, states that it is impossible to cover any distance [2]:

- *There is no motion, because that which is moved must arrive at the middle before it arrives at the end, and so on ad infinitum.*

According to Simplicius, Diogenes the Cynic after hearing this argument from Zeno's followers silently stood up and walked, so pointing out that it is a matter of the most common experience that things in fact do move [11]. This answer, very clever and effective perhaps, is unfortunately completely misleading, because it is not the apparent motion what Zeno questions but how this motion is logically possible. And the Diogenes's answer does not enlighten us at all in this respect [12]:

A bearded sage once said that there's no motion.
His silent colleague simply strolled before him, —
How could he answer better?! — all adored him!
And praised his wise reply with great devotion.
But men, this is enchanting! — let me interject,
For me, another grand occurrence comes to play:
The sun rotates around us every single day,
And yet, the headstrong Galileo was correct.

But what is paradoxical in Zeno's arguments? Let us take a closer look. He says that any movement can be subdivided into infinite number of ever decreasing steps. This is not by itself paradoxical, if we assume infinite divisibility of space and time. What is paradoxical is an ability to perform infinite number of subtasks in a finite time — to perform a supertask. Any movement seems to be a supertask according to Zeno and it is by no means obvious that it is ever possible to perform infinite number of actions in a finite time. Our intuition tells us just the contrary — that it is a clear impossibility for finite beings to manage any supertask. In the case of Dichotomy it is even not clear how the movement can begin at all because there is no first step to be taken.

Aristotle tried to resolve this situation by distinguishing potential and actual infinities [13]: "To the question whether it is possible to pass through an infinite number of units either of time or of distance we must reply that in a sense it is and in a sense it is not. If the units are actual, it is not possible; if they are potential, it is possible". But Aristotle's answer is not much better than Diogenes'. It is incomplete. In fact doubly incomplete. According to it Zeno's infinite subdivision of a motion is purely mathematical, just an action of imagination. But even if we accept Aristotle's position it is desirable to show that in mathematics we have tools to handle infinities in a logically coherent way. In fact no such tools were at Aristotle's disposal and they were only germinated after two thousand years when the notion of limit emerged, Calculus was developed and Georg Cantor created his set theory. We can say that Dichotomy is not mathematically paradoxical today. Either classical or non-standard [14, 15] analysis can supply sufficient machinery to deal with both the Dichotomy and its more famous counterpart, the Achilles and the tortoise paradox [2]:

- *The slower will never be overtaken by the quicker, for that which is pursuing must first reach the point from which that which is fleeing started, so that the slower must always be some distance ahead.*

This latter paradox is more impressively formulated in terms of two bodies but in fact it is a symmetric counterpart of the Dichotomy and has a variant involving only one moving body [16]: "To reach a given point, a body in motion must first traverse half of the distance, then half of what remains, half of this latter, and so on ad infinitum, and again the goal can never be reached". Therefore, if the Dichotomy wonders how the motion can begin as there is no first step, the Achilles makes it equally problematic the end of the motion because there is no last step. Modern mathematics partly completes Aristotle's argument and provides a coherent mathematical picture of motion. But all this mathematical developments, although very wonderful, do not answer the main question implicit in Aristotle's rebuttal of Zeno: how the real motion actually takes place and whether its present day mathematical image still corresponds to reality at the most fundamental level.

4. Zeno meets Banach and Tarski

Let us take, for example, infinite divisibility of space and time. This infinite divisibility is in fact paradoxical, even though the modern mathematics have no trouble to deal with this infinite divisibility. Let us explain what we have in mind.

Zeno's argument shows that any spatial or temporal interval contains uncountably many points. Nevertheless a moving body manages to traverse all these points in a finite time. Let us consider any division of the interval into non-empty pairwise mutually exclusive subintervals (that is any pair of them have no common points). Then there exists at least one set N that contains one and only one point from each of the subintervals. Indeed, a moving point body enters into a given subinterval sooner or later while traversing the initial interval and will remain into this subinterval for some amount of time. We can take any point the moving body occupies during this time interval as an element of N. All this seems very natural and self-evident, and so does its natural generalisation, the axiom of choice [17]:

- *If M is any collection of pairwise mutually exclusive, non-empty sets P, there exists at least one set N that contains one and only one element from each of the sets P of the collection M.*

If one may choose an element from each of the sets P of M, the set N can evidently be formed — hence the name of the axiom.

Now this innocently looking "self-evident" axiom leads to the most paradoxical result in the mathematics, the Banach–Tarski theorem, which is so contrary to our intuition that is better known as the Banach–Tarski paradox. The most artistic presentation of this paradox can be found in the Bible [18]: "As he went ashore he saw a great throng; and he had compassion on them, and healed their sick. When it was evening, the disciples came to him and said: 'This is a lonely place, and the day is now over; send the crowds away to go into the villages and buy food for themselves'. Jesus said: 'They need not go away; you give them something to eat'. They said to him: 'We have only five loaves here and two fish'. And he said: 'Bring them here to me'. Then he ordered the crowds to sit down on the grass; and taking the five loaves and the two fish he looked up to heaven, and blessed, and broke and gave the loaves to the disciples, and the disciples gave them to the crowds. And they all ate and were satisfied. And they took up twelve baskets full of the broken pieces left over. And those who ate were about five thousand men, besides women and children".

In the more formal language the Banach–Tarski theorem states that [17]

- *in any Euclidean space of dimension $n > 2$, any two arbitrary bounded sets are equivalent by finite decomposition provided they contain interior points.*

This theorem opens, for example, the door to the following mathematical alchemy [17]: a ball of the size of orange can be divided into a finite number of pieces which can be reassembled by using merely translations and rotations to yield a solid ball whose diameter is larger than the size of the solar system. Of course, a real orange cannot be chopped in such a way because atoms and molecules constitute a limit of divisibility of any chemical substance and the pieces required in the Banach–Tarski theorem are so irregular that they are non-measurable and the concept of volume (Lebesgue measure) does not make sense for them. But does space–time itself also have a limit of divisibility? It is yet an open question.

The comprehensive discussion of the Banach–Tarski theorem is given in [19] and for an elementary approach with a full proof of the theorem see [20]. One can question whether paradoxical counter-intuitive decompositions like the ones implied by the Banach–Tarski theorem are of any use in physics. Surprisingly, there were several attempts in this direction. Pitowsky was the first (to our knowledge) to consider a certain extension of the concept of probability to non-measurable sets in connection with the Einstein–Podolsky–Rosen paradox and Bell's inequalities [21, 22]. Another examples can be traced trough [23, 24].

One cannot blame the axiom of choice as the only culprit of such paradoxical mathematical results. Even without the use of this axiom one can argue that there is some truth in the proverb that the world is small, because the results proved in [25] entirely constructively, without the axiom of choice, imply that there is a finite collection of disjoint open subsets of the sun that fill the whole sun without holes of positive radius and that nevertheless can be rearranged by rigid motions to fit inside a pea and remain disjoint.

Maybe the Banach–Tarski theorem and analogous paradoxical decompositions will appear a bit less paradoxical if we realize that the Achilles and the tortoise paradox illustrates that any two intervals contain the same number of points regardless their length. Indeed, during their race Achilles and the tortoise cover desperately different intervals. Nevertheless one can arrange a one-to-one correspondence between points of these intervals because for every point A from the Achilles' track there is only one point B on the track of tortoise which the tortoise occupied at the same instant of time when Achilles occupied A. In fact, as Cantor proved in 1877, there is a one-to-one correspondence of points on the interval [0, 1] and points in a square, or points in any n-dimensional space. Cantor himself was surprised at his own discovery and wrote to Dedekind [26] "I see it, but I don't believe it!"

5. Zeno meets quantum mechanics

Despite some paradoxical flavour, the infinite divisibility of real space–time, although unwarranted at yet, is mathematically coherent. But Zeno's paradoxes contain some other physical premises also that deserve careful consideration. The Achilles and the tortoise paradox, for example, assumes some observation procedure:

- check the positions of the contenders in the race;
- check again when Achilles reach the position the tortoise occupied at previous step;
- repeat the previous instruction until Achilles catch the tortoise (and this is an infinite loop because he never does).

Calculus teach us that the above observational process covers only finite interval of time in spite of its infinitely many steps. And during this time interval the tortoise will be indeed always ahead of Achilles. The observational procedure Zeno is offering simply does not allow us to check the contenders positions later when Achilles overtake the tortoise. So the paradox is solved? Not at all. Zeno's procedure implicitly assumes an ability to perform position measurements. Therefore, two questions remain: whether it is possible to perform infinitely frequent measurements taken for granted by Zeno, and how the race will be effected by back-reaction from these measurements. The world is quantum mechanical after all and the measurement process is rather subtle thing in quantum mechanics.

Simple arguments [27] show that something interesting is going on if the observational procedure of Zeno is considered from the quantum mechanical perspective. Let $|\Phi, 0\rangle$ denote the initial state vector of the system (Achilles and the tortoise in uniform motions). After a short time t the state vector will evolve into

$$|\Phi, t\rangle = \exp\left(-\frac{i}{\hbar}Ht\right)|\Phi, 0\rangle \approx \left(1 - \frac{i}{\hbar}Ht - \frac{1}{2\hbar^2}H^2t^2\right)|\Phi, 0\rangle \ ,$$

where H is the Hamiltonian of the system assumed to be time independent. If now the position measurements of the competitors are performed we find that the initial state have still not changed with the probability

$$|\langle\Phi, 0|\Phi, t\rangle|^2 \approx 1 - \frac{(\Delta E)^2}{\hbar^2}\, t^2\,,$$

where

$$(\Delta E)^2 = \langle\Phi, 0|H^2|\Phi, 0\rangle - \langle\Phi, 0|H|\Phi, 0\rangle^2$$

is positive (there should be some energy spread in the initial state because we assume good enough localisations for Achilles and the tortoise). If these measurements are performed n-times, at intervals t/n, there is a probability

$$\left(1-\frac{(\Delta E)^2}{\hbar^2}\frac{t^2}{n^2}\right)^n$$

that at all times the system will be found in the initial state. But this probability tends to unity when $n\to\infty$, because in this limit

$$\ln\left(1-\frac{(\Delta E)^2}{\hbar^2}\frac{t^2}{n^2}\right)^n\approx-\frac{(\Delta E)^2}{\hbar^2}\frac{t^2}{n}\to 0\,.$$

Therefore, if the observations are infinitely frequent the initial state does not change at all. Zeno was right after all: Achilles will never catch the tortoise under proposed observational scheme! This scheme implicitly assumes a continues monitoring of Achilles' position and, therefore, he will fail even to start the race.

Matters are not as simple however. Repeated measurement of a system effects its dynamics much more complex and delicate way than just slowing the evolution [28]. The above described Quantum Zeno Effect became popular after seminal paper of Misra and Sudarshan [29], although it dates back to Alan Turing (see [30, 31] and references wherein) and was known earlier as "Turing's paradox". The initial time t^2 dependence of quantum mechanical evolution, from which the Quantum Zeno Effect follows most simply, is quite general though not universal. The experimental difficulty resides in the fact that the t^2 dependence takes place usually at very short times for natural unstable systems. For example, the "Zeno" time of the 2P–1S transition of the hydrogen atom is estimated to be approximately 3.6×10^{-15} s [32]. Nevertheless modern experimental techniques enable to prepare artificial unstable systems with long enough Zeno time. In beautiful experiment [33] ultra-cold sodium atoms were trapped in a periodic optical potential created by a accelerated standing wave of light. Classically atoms can be trapped inside the potential wells but they will escape by quantum tunnelling. The number that remain is measured as a function of duration of the tunnelling. The results are shown in the Fig. 1. Hollow squares in this figure show the probability of survival in the accelerated potential as a function of duration of the tunnelling acceleration. The solid line represents what is expected according to quantum mechanics and we see a very good agreement with the experimental data. The Zeno time for this unstable system is of the order of about μs and during this short time period the survival probability exhibits a t^2 drop. For longer times we see a gradual transition from the t^2 dependence to linear t dependence, which corresponds

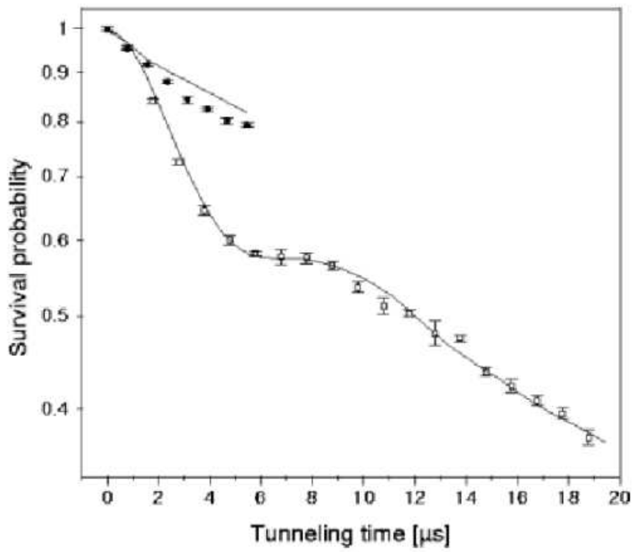

Fig. 1. Observation of the Quantum Zeno Effect in the experiment [33].

to the usual exponential decay law. Such behaviour can be simply understood in the framework of time-dependent perturbation theory [34]. The probability of decay of some excited state $|i\rangle$ under the influence of time independent small perturbation V is given by the formula

$$Q(t) = \frac{1}{\hbar^2} \int\limits_{-\infty}^{\infty} |\langle i|V|k\rangle|^2 \rho_k \sin^2\left(\frac{(E_k - E_i)t}{2\hbar}\right)\left(\frac{2\hbar}{E_k - E_i}\right)^2 dE_k\,, \quad (1)$$

where $|i\rangle$, $|k\rangle$ are eigenstates of unperturbed Hamiltonian and ρ_k is the density of states $|k\rangle$. For very short times one has clearly a t^2 dependence:

$$Q(t) \approx \frac{1}{\hbar^2}\left(\int\limits_{-\infty}^{\infty} |\langle i|V|k\rangle|^2 \rho_k dE_k\right) t^2.$$

For longer times, when

$$\frac{(E_k - E_i)t}{2\hbar}$$

is not small, one cannot replace the sine function by the first term of its Taylor expansion. However, we can expect that only states with small $|E_k - E_i|$ contribute significantly in the integral, because if

$$z = \frac{(E_k - E_i)t}{2\hbar} \gg 1$$

the integrand oscillates quickly. But then it can be assumed that $|\langle i|V|k\rangle|^2$ and ρ_k are constant and by using

$$\int\limits_{-\infty}^{\infty} \frac{\sin^2 z}{z^2} dz = \pi,$$

one obtains linear t dependence

$$Q(t) \approx \frac{2\pi}{\hbar} |\langle i|V|k\rangle|^2 \rho_k t\,.$$

In the experiment [33] the number of atoms remaining in the potential well after some time of tunnelling was measured by suddenly interrupting the tunnelling by a period of reduced acceleration. For the reduced acceleration tunnelling was negligible and the atoms increased their velocity without being lost out of the well. Therefore, the remaining atoms and the ones having tunnelled out up to the point of interruption become separated in velocity space enabling the experimenters to distinguish them. This measurement of the number of remaining atoms projects the system in a new initial state when the acceleration is switched back and the system returns to its unstable state. The evolution must, therefore, start again with the non-exponential initial segment and one expects the Zeno impeding of the evolution under frequent measurements.

Fig. 1 really shows the Zeno effect in a rather dramatic way. The solid circles in this figure correspond to the measurement of the survival probability when after each tunnelling segment of 1 μs an interruption of 50 μs duration was inserted. One clearly sees a much slower decay trend compared to the measurements without frequent interruptions (hollow squares). Some disagreement with the theoretical expectation depicted by the solid line is attributed by the authors of [33] to the under-estimate of the actual tunnelling time, so that in reality the decay might be slowed down even at higher degree.

Uninterrupted decay curve shows damped oscillatory transition region between initial period of slow decay and the exponential decay at longer times. At that a steep drop in the survival probability is observed immediately after the Zeno time. It is expected, therefore, that the decay will be not slowed down but accelerated if frequent interruptions take place during the steep drop period of evolution forcing the system to repeat the initial period of fast decay again and again after every measurement. This so called Quantum Anti-Zeno effect [35–37] is experimentally demonstrated in Fig. 2. The solid circles in this figure show the evolution of the unstable system when after every 5 μs of tunnelling the decay was interrupted by a slow acceleration segment.

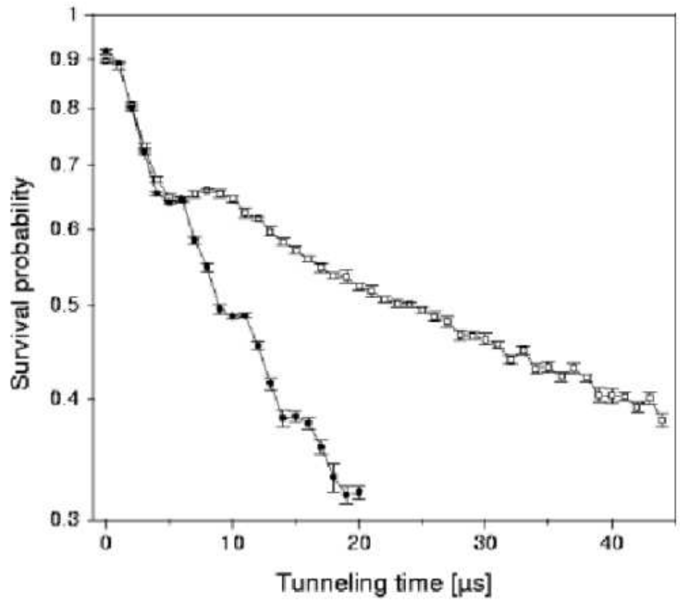

Fig. 2. Observation of the Quantum Anti-Zeno Effect in the experiment [33].

6. Supertasks

The Zeno and Anti-Zeno effects in quantum theory are, of course, very interesting phenomena and even some practical application of the Zeno effect in quantum computing is foreseeable [38]. But in the context of Zeno paradoxes we are more interested in the limit of infinitely frequent measurements with complete inhibition of evolution. Although such limit leads to interesting mathematics [39], its physical realisability is dubious. The Calculus argument that it is possible for infinite sum to converge to a finite number is not sufficient to ensure a possibility to perform a supertask. This becomes obvious if we somewhat modify Zeno's arguments to stress the role supertasks play in them. The resulting paradox, Zeno Zigzag, goes as follows [2]. A light ray is bouncing between an infinite sequence of mirrors as illustrated in the Fig. 3. The sizes of mirrors and their separations decrease by a factor of two on each step. The total length of the photon's zigzag path is finite (because the geometric series $1 + 1/2 + 1/4 + \dots$ converges), as well as the envelope box size around the mirrors. Therefore, the photon is expected to perform infinite number of reflections in finite time and emerge on the other side of our mirror box. But the absence of the last mirror the photon hit is obviously troublesome now: there is no logical way for the photon to decide at what direction to emerge from the box.

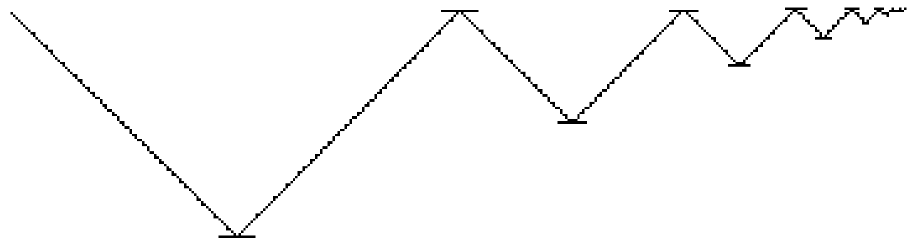

Fig. 3. The Zeno Zigzag.

Of course, in reality it is impossible to realize the Zeno Zigzag for a number of reasons. One cannot make arbitrarily small "mirrors", for example, because sharp localisation leads to a significant momentum spread according to uncertainty relations and then the relativity makes possible a pair production which will smear the "mirror" position. The wave-like behaviour of the photon (or any other particle) will anyway make impossible to maintain definite direction of the reflected photon if the mirror size is less than the photon wavelength.

The question, however, naturally arises whether supertasks are logically impossible irrespective of the nature of physical reality which may restrict their practical realization. To support the opinion that the very notion of completing an infinite sequence of acts in a finite time is logically contradictory, Thomson suggested the following supertask [40]. A lamp is switched ON and OFF more and more rapidly so that at the end of the two minutes a supertask of infinite switching of the lamp is over. The question now is whether the lamp is in the ON state or in the OFF state after this two minutes. Clearly the lamp must be in one of these states but both seem equally impossible. The lamp cannot be in the ON state because we never turned it on without immediately turning it off. But the lamp cannot be in the OFF state either because we never turned it off without at once turning it on. It seems impossible to answer the question and avoid a contradiction.

The Thomson lamp argument is seductive but fallacious [41, 42]. Surprisingly there is a coherent answer to the Thomson's question without any contradiction. To come to this answer, it is instructive to consider another supertask [42] which is not paradoxical in any obvious way. A ball bounces on a hard surface so that on each rebound it loses $(1-k)$-fraction of its velocity prior to the bounce, where $0<k<1$. The ball will perform infinitely many bounces in a finite time because, in classical mechanics, the time between bounces is directly proportional to the initial velocity of the ball and the geometric series $1,\ k,\ k^2,\ k^3,\ldots$ has finite sum $1/(1-k)$. Now let us use this bouncing ball as a switching mechanism for the Thomson's lamp. Then it is immediately obvious that depending on the organisation of the circuit the lamp can be in either state (ON or OFF) after the supertask is completed, see Fig. 4.

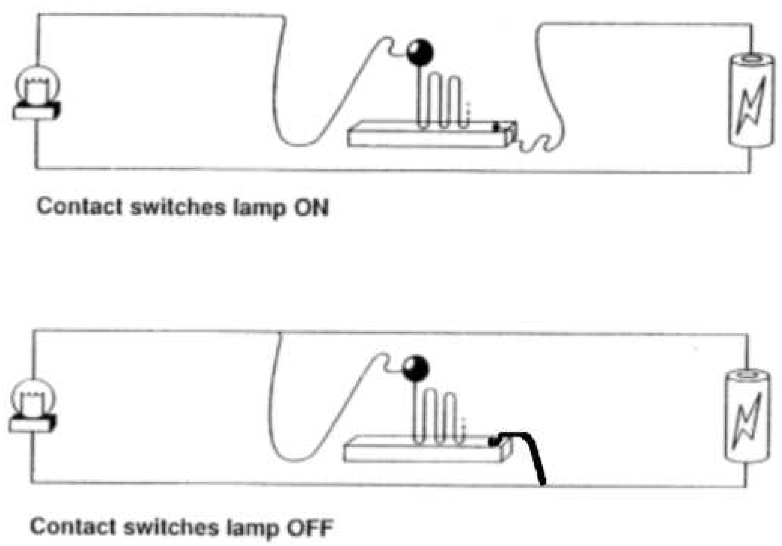

Fig. 4. Alternative switching mechanisms for Thomson's lamp from [42].

Logic once again demonstrates its flexibility. Note that even such a weird notion as the lamp being in a superposition of the ON and OFF states makes perfect sense in quantum mechanics. Although, as was indicated above, we can make the Thomson's lamp logically consistent without any such weirdness. But other surprises with supertasks are lurking ahead.

In [43] Pérez Laraudogoitia constructed a beautifully simple supertask which demonstrates some weird things even in the context of classical mechanics. Fig. 5 shows an infinite set of identical particles arranged in a straight line. The distance between the particles and their sizes decrease so that the whole system occupies an interval of unit length. Some other particle of the same mass approaches the system from the right with unit velocity. In elastic collision with identical particles the velocities are exchanged after the collision. Therefore, a wave of elastic collisions goes through the system in unit time. And what then? Any particle of the system and the projectile particle comes to rest after colliding its left closest neighbour. Therefore, all particles are at rest after the collision supertask is over and we are left with paradoxical conclusion [43] that the total initial energy (and momentum) of the system of particles can disappear by means of an infinitely denumerable number of elastic collisions, in each one of which the energy (and momentum) is conserved!

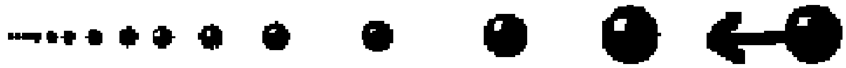

Fig. 5. Pérez Laraudogoitia's supertask.

If you fill uneasy about this energy-momentum non-conservation, here is the same story in more entertaining incarnation [44].

Suppose you have some amount of one dollar bills and the Devil approaches you in a nefarious underground bar. He says that he has a hobby of collecting one dollar bills of particular serial numbers and it happened that you do have one such bill. So he is offering you a bargain: he will give you ten one dollar bills for this particular one. Should you accept the bargain? Why not, it seems so profitable. You agreed and the bargain is done. After half an hour the Devil appears again with the similar offer. Then after a quarter-hour and so on. The time interval between his appearances decreases by a factor of two each time. The amount of your money increases quickly. After an hour infinite number of bargains are done and how much money will you have? You would have a lot if the Devil wanted you to succeed. But he tries to perish your soul not to save it and during the bargains is indeed very selective about bill serial numbers: he always takes the bill with smallest serial number you posses and instead gives the bills with serial numbers greater than any your bill's at that moment. Under such arrangement any bill you posses will end its path into Devil's pocket. Indeed, for any particular bill at your disposal at some instant you will have only finite amount of bills with less serial numbers and you never will get additional bills with smaller serial numbers. Therefore, sooner or later this particular bill will become the bill with smallest serial number among your bills and hence subject of the exchange. We come again to the strange conclusion that in spite of your money's continuous growth during the transaction you will have no bill at all left after the infinite transaction is over!

Of course, this example does not enlighten us much about how the initial energy disappears in the Pérez Laraudogoitia's supertask. However, it clearly shows a somewhat infernal flavour the supertasks have. And not only the energy-momentum conservation is on stake. Classical dynamics is time reversal invariant. Therefore, the following process, which is the time reversal of the Pérez Laraudogoitia's supertask, is also possible [43]: a spontaneous self-excitation can propagate through the infinite system of balls at rest causing the first ball to be ejected with some nonzero velocity. The system displays not only the energy-momentum non-conservation but also indeterminism [43, 45, 46].

The locus of the difficulty is the same as in the Zeno paradoxes: there is no last member in a sequence of acts (collisions) and, therefore, there is no last ball to carry off the velocity [47]. The supertask illustrate the indeterminism of classical Newtonian or even relativistic dynamics as far as infinite localisation of bodies is admissible. In quantum mechanics balls cannot be simultaneously at rest and infinitely localised thanks to uncertainty relations. Therefore, Pérez Laraudogoitia's supertask will not persist in quantum theory. However, it was shown [48] that a (nonrelativistic) quantum mechanical supertask can be envisaged in which the deterministic time development of the wave function is lost and spontaneous self-excitation of the ground state is allowed. Yet pathologies disappear if one demands normalizability of the state vector. In this sense quantum mechanical supertasks are better behaved than their classical counterparts [48].

Surprisingly and contrary to common wisdom, classical Newtonian physics is more hostile and unfriendly to determinism than either quantum mechanics or special relativity [49]. Another example of esoteric behaviour of seemingly benign Newtonian system was given by Xia [50] while solving the century-old intriguing problem of non-collisional singularities. Xia's construction constitutes in fact a supertask and involves only five bodies interacting via familiar Newtonian inverse square force law. The essentials of this supertask can be explained as follows [51]. Imagine a system of two equal masses M moving in the $x-y$ plane under Newton's law of attraction with center of mass at rest, and the third mass $m \ll M$ (so that it does not disturb the motion of the first two) on the z-axis which goes through the center of mass of the planar binary system. It is clear from the symmetry that the total force of gravitational attraction acting on the mass m has only z-component and, therefore, the third mass will experience a one dimensional motion along the z-axis. When the mass m passes through and is lightly above the binary plane, we can arrange extremely powerful downward pull on m if the highly elliptical binary has its closest approach at that moment. In fact, assuming an ideal case of point masses, the downward force acting on m can be made arbitrarily strong by just adjusting the separating distances among the bodies. As a result the third mass is jolted down with high velocity while the binary starts separating and loses significantly its braking effect on m.

The mirror replica of the binary, see Fig. 6, is placed at large distance (not to disturb the first binary) further down to prevent the mass m from being expelled to infinity. In case of proper timing, the second binary will break m's downward motion and will thrust it upwards with even higher velocity.

Xia was able to show that there exists a Cantor set of the initial conditions allowing to repeat this behaviour infinitely often in a finite time. As a result the four of five bodies from the Xia's construction will escape to spatial infinity in a finite time, while the fifth will oscillate back and forth among the other four with ever increasing speed.

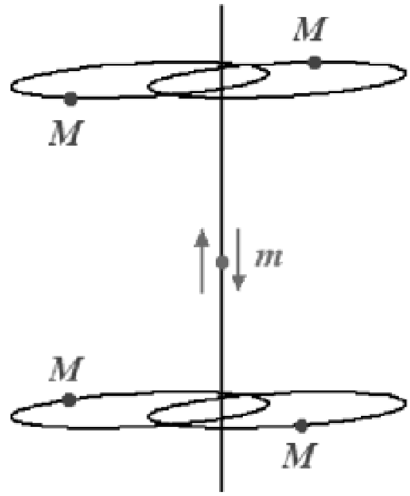

Fig. 6. Xia's five body supertask.

Xia's supertask does not violate energy conservation, the bodies drawing out their energies from the infinitely deep $1/r$ potential well. However, it implies indeterminacy in idealised Newtonian world. The time reverse of this supertask is an example of "space invaders" [49], particles appearing from spatial infinity in a surprise attack. Note that [49] "the prospects for determinism brighten considerably when we leave classical space–times for Minkowski space–time, the spacetime setting for special relativistic theories". Because with no superluminal propagation there are no space invaders.

Leaving aside interesting philosophical questions raised by Xia's supertask, it seems otherwise completely artificial and far from reality. Curiously, some ideas involved in this construction can be used by mankind, admittedly in the very long run, to transfer orbital energy from Jupiter to the Earth by a suitable intermediate minor space body, causing the Earth's orbit to expand and avoid an excessive heating from enlarging and brightening Sun at the last stage of its main sequence life [52].

7. Bifurcated supertasks

Let us return to Zeno. Hermann Weyl pointed out [53] another possibility mankind can benefit from Zeno supertask: "if the segment of length 1 really consists of infinitely many subsegments of length $1/2, 1/4, 1/8, \ldots$, as of "chopped-off" wholes, then it is incompatible with the character of the infinite as the "incompletable" that Achilles should have been able to traverse them all. If one admits this possibility, then there is no reason why a machine should not be capable of completing an infinite sequence of distinct acts of decision within a finite amount of time; say, by supplying the first result after 1/2 minute, the second after another 1/4 minute, the third 1/8 minute later than the second, *etc.* In this way it would be possible, provided the receptive power of the brain would function similarly, to achieve a traversal of all natural numbers and thereby a sure yes-or-no decision regarding any existential question about natural numbers!"

Relativity brings additional flavour in discussion of Weyl's infinity machines. One can imagine, for example, bifurcated supertasks [42, 54]. To check the Goldbach's conjecture whether every even number greater than two can be written as the sum of two primes, the Master organises two space missions. In one of them a computer, the Servant, is sent with constant acceleration which examines all even numbers one case after the other. In another space mission the Master himself contrives to accelerate in such a way to keep the Servant within his causal horizon. It is possible to arrange Master's acceleration so that the Servant disappears from his causal shadow only after spending infinite amount of time on the Goldbach's conjecture, while in the Master's frame the amount of time passed remains finite. If the Servant finds a counterexample to the Goldbach's conjecture it sends a message to the Master and the latter will know that the conjecture is false. If no message is received, however, the Master will know in a finite time that Goldbach was wright.

This looks too good to be true and indeed the realization of such Pitowsky [55] infinite machine is suspicious for a number of reasons [42, 54]. The Servant moves with constant acceleration during infinite proper time. Therefore, it needs an infinite fuel supply. Moreover, the Master's acceleration increases without limit and eventually he would be crushed to death by artificial gravity. There is a conceptual problem also. At no point on his world line does the Master have a causal access to all events on the Servant's world line. This means there is no Moment of Truth, if the Goldbach's conjecture is true, at which the Master attained that knowledge. Emergence of the knowledge about validity of the Goldbach's conjecture will be as mysterious in Pitowsky bifurcated supertask as is the disappearance of energy in Pérez Laraudogoitia's supertask.

But general relativity allows to improve the above construction by admitting the so called Malament–Hogarth space–times [54, 56]. In such a spacetime there is a point (the Malament–Hogarth point) at Master's world line such that entire infinite history of the Servant's world line is contained in this point's causal shadow. From that event on the Master will be enlightened about validity of that particular number theoretic problem through the Servant's infinite labours.

Some Malament–Hogarth space–times seem quite reasonable physically. Among them are such well known space–times as the anti-de Sitter spacetime, Reissner–Nordström spacetime and Kerr–Newman spacetime [57], the latter being the natural outcome of the late-time evolution of a collapsed rotating star. Therefore, it is not excluded that "if the Creator had a taste for the bizarre we might find that we are inhabiting one of them" [42]. Even if this proves to be the case, the practical realization of infinite computation is not at all guaranteed. Physics beyond the classical general relativity, for example proton decay and other issues related to the long term fate of the universe [58], can prohibit the infinite calculations. There is still another reason that can make a practical realization of bifurcated supertasks dubious [55]: the Servant might have infinite time to accomplish its labours but not enough computation space. the argument goes as follows [59, 60].

Any system which performs computation as an irreversible process will dissipate energy. Many-to-one logical operations such as AND or ERASE are not reversible and require dissipation of at least $kT \ln 2$ energy per bit of information lost when performed in a computer at temperature T. While one-to-one logical operations such as NOT are reversible and in theory can be performed without dissipation [61].

That erasure of one bit information has an energy cost $kT \ln 2$ (the Landauer's principle) can be demonstrated by Maxwell's demon paradox originally due to Leo Szilard [62, 63].

A schematic view of the Szilard's demon engine is shown in the Fig. 7. Initially the entire volume of a cylinder is available to the one-atom working gas (step (a)). At step (b) the demon measures the position of the atom and if it is found in the right half of the cylinder inserts a piston. At step (c) the one-atom gas expands isothermally by extracting necessary heat from the reservoir and lifts the load. At step (d) the piston is removed and the system is returned to its initial state ready to repeat the cycle.

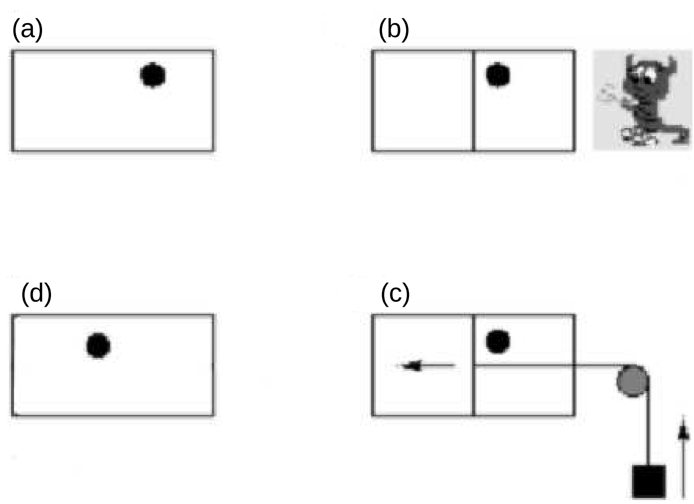

Fig. 7. Szilard's demon engine.

It seems the Szilard's demon engine defeats the second law of thermodynamics: the heat bath, which has transferred energy to the gas, has lowered its entropy while the engine has not changed its entropy because it returned to the initial state. But this reasoning is fallacious because it misses an important point: the demon has not returned to his initial state. He still possesses one bit information about left–right position of the atom he recorded. The system truly to return to its initial state it is necessary to erase this information from the demon's memory. According to the Landauer's principle, one has to pay $kT\ln 2$ energy cost for this erasure. On the other hand work done during isothermal expansion equals

$$\int\limits_{V/2}^{V} pdV = \int\limits_{V/2}^{V} \frac{m}{\mu}\,RT\frac{dV}{V} = \frac{m}{\mu}\,RT\ln 2\,,$$

and normalised to one atom this is just $(R/N_A)T\ln 2 = kT\ln 2$. Therefore, all extracted work is paid back for erasure of the demon's memory and the net effect of the circle is zero. The second law defeats the demon.

In principle all computations could be performed using reversible logical operations and hence without energy cost [61]. Interestingly enough, some important stages of biomolecular information processing, such as transcription of DNA to RNA, appear to be accomplished by reversible chemical reactions [64]. Real computers, however, are subject to thermal fluctuations that cause errors. To perform reliable computations, therefore, some error-correcting codes must be used to detect inevitable errors and reject them to the environment at the cost of energy dissipation [61].

The Servant computer from bifurcated supertask needs to consume energy from surrounding universe to perform its task. Different energy mining strategies were considered in [59] and it was shown that none of them allows to gather infinite amount of energy even in infinite time irrespective assumed cosmological model. To perform infinite number of computations with finite available energy the Servant should be able to continuously decrease its operating temperature to reduce the energy cost of computations. But the Servant is not completely free to choose its temperature: the waste heat produced while performing computations must be radiated away to avoid overheating. But physical laws place limits on the rate at which the waste heat can be radiated. Assuming that the electromagnetic dipole radiation by electrons is the most efficient way to get rid of the heat, Dyson argued [65] that there is a lower limit on the operating temperature

$$T > \frac{Q}{N_e}10^{-12}K\,,$$

where Q is a measure of the complexity of the computing device ($Q \sim 10^{23}$ for humans) and N_e is the number of electrons in the computer. Since Q/N_e cannot be made arbitrarily small, it seems infinite computations are impossible.

But where is a way out [65]: hibernation. The Servant can compute intermittently while continuing to radiate waste heat into space during its periods of hibernation. This strategy will allow to operate below the Dyson limiting temperature. But eventually the wavelength of thermal radiation will become very large compared to the characteristic size of the computer, the thermal energies will be small compared to the characteristic quantised energy levels of the system and radiation will be suppressed by an exponential factor compared to the estimates of Dyson [59]. Therefore, the Servant must increase its size as time goes by. It also needs more and more memory to store digital codes of ever increasing even and prime numbers. This is another reason it to increase in size. But as was mentioned above there is only finite amount of energy, and hence material, available. Therefore, it appears the servant will not be able to accomplish its infinite labours.

8. Zeno meets Newton and Wigner

Our discussion of Zeno's paradoxes indicates a rather subtle role the concept of localisation plays in description of motion. It is not surprising, therefore, that the notion of localisation in relativistic quantum mechanics was intensively examined. Many concepts of localisation have been proposed but we focus on the one known as the Newton–Wigner localisation [66, 67]. Initially this concept of localisation and the corresponding position operators were suggested in the context of the single particle relativistic quantum mechanics. But later their result was reformulated in quantum field theory also, where the concept of local observables is the central concept leading to numerous troubles (infinities) which are swept under carpet with great artistic skill by renormalisation. Many troubles related to localisation in relativistic quantum field theory have their formal root in the Reeh–Schlieder theorem which in the non-formal artistic formulation of Hans Halvorson [68] states that the vacuum is seething with activity at the local level: any local event has the nonzero probability to occur in the vacuum state for the standard localisation scheme. The Newton–Wigner localisation scheme avoids some consequences of the Reeh–Schlieder theorem and leads to a mathematical structure which seems more comfortable for our a priori physical intuition about localisation [69]. But the story is not yet over. The suggested generalisations of the Newton–Wigner localisation are still not immune against the full strength of the Reeh–Schlieder theorem and have their own counter-intuitive features [68, 70].

Let us sketch the derivation of the Newton–Wigner position operator for Dirac particles [71]. The Newton–Wigner operator $\hat{Q}^k$ can be defined as the operator whose eigenvalues are the most localised wave-packets formed from only positive-energy solutions of the Dirac equation. Let $\psi^{(s)}_{(\vec{y})}(\vec{x})$ be such a wave-function describing an electron with spin projection s localised at the point $\vec{y}$ at the time $t=0$. Defining the scalar product by

$$(\psi,\phi)=\int\psi^+(\vec{x})\phi(\vec{x})\,d\vec{x}=\int\psi^+(\vec{p})\phi(\vec{p})\,d\vec{p}\,,$$

the natural normalisation condition for these states is

$$\left(\psi^{(s)}_{(\vec{y})},\psi^{(r)}_{(\vec{z})}\right)=\delta_{rs}\delta(\vec{z}-\vec{y})\,. \tag{2}$$

Translational invariance implies

$$\psi^{(s)}_{(\vec{y})}(\vec{x})=\psi^{(s)}_{(\vec{y}+\vec{a})}(\vec{x}+\vec{a})\,,$$

or in the momentum space

$$\psi^{(s)}_{(\vec{y})}(\vec{p})=e^{i\vec{p}\cdot\vec{a}}\psi^{(s)}_{(\vec{y}+\vec{a})}(\vec{p})\,, \tag{3}$$

where momentum space wave-functions are defined through

$$\psi(\vec{p})=\frac{1}{(2\pi)^{3/2}}\int\psi(\vec{x})e^{-i\vec{p}\cdot\vec{x}}\,d\vec{x}\,.$$

Equations (2) and (3) imply

$$\psi^{(s)+}_{(\vec{y})}(\vec{p})\,\psi^{(r)}_{(\vec{y})}(\vec{p})=\frac{\delta_{rs}}{(2\pi)^3}\,. \tag{4}$$

On the other hand

$$\psi^{(s)}_{(\vec{y})}(\vec{p})=f_{(\vec{y})}(\vec{p})\,u(\vec{p},s)\,, \tag{5}$$

where $u(\vec{p},1/2)$ and $u(\vec{p},-1/2)$ are two independent positive-energy solutions of the Dirac equation, which can be taken in the form

$$u(\vec{p},s)=\Lambda_+(\vec{p})\,u(\vec{0},s)\,,$$

where the rest state four-component spinors are

$$u(\vec{0},1/2)=(1,0,0,0)^T\,,\qquad u(\vec{0},-1/2)=(0,1,0,0)^T\,,$$

and $\Lambda_+(\vec{p})$ is a positive-energy projection operator for a Dirac particle of mass m:

$$\begin{aligned}\Lambda_+(\vec{p}) &= \frac{1}{2}\left(1+\frac{\vec{\alpha}\cdot\vec{p}+\beta m}{p_0}\right) = \frac{(\hat{p}+m)\gamma_0}{2p_0}\,,\\ p_0 &= \sqrt{\vec{p}^2+m^2}\,.\end{aligned}$$

The normalisation of $u(\vec{p},s)$ is

$$u^+(\vec{p},s)u(\vec{p},s') = u^+(\vec{0},s)\Lambda_+(\vec{p})u(\vec{0},s') = \frac{p_0+m}{2p_0}\delta_{ss'}\,. \tag{6}$$

From equations (4)–(6) we get

$$|f_{(\vec{y})}(\vec{p})|^2 = (2\pi)^{-3}\frac{2p_0}{p_0+m}\,.$$

We fix the phase of $f_{(\vec{y})}(\vec{p})$ by assuming

$$f^*_{(\vec{0})}(\vec{p}) = f_{(\vec{0})}(\vec{p})\,.$$

Then (3) indicates that the momentum space wave-function for the electron localised at $\vec{y}$ at the time $t=0$ has the form

$$\psi^{(s)}_{(\vec{y})}(\vec{p}) = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{2p_0}{p_0+m}}e^{-i\vec{p}\cdot\vec{y}}\;u(\vec{p},s). \tag{7}$$

Now we construct the Newton–Wigner position operator $\hat{Q}^k$ for which (7) is the eigenfunction with eigenvalue y^k:

$$\hat{Q}^k\psi^{(s)}_{(\vec{y})}(\vec{p}) = y^k\psi^{(s)}_{(\vec{y})}(\vec{p})\,.$$

Assuming that $\psi^{(s)}_{(\vec{y})}$ eigenfunctions form a complete system for positive-energy solutions, we get for any positive-energy wave-function $\psi(\vec{p})$

$$\hat{Q}^k\psi(\vec{p}) = \hat{Q}^k\sum_s\int d\vec{y}\left(\psi^{(s)}_{(\vec{y})},\psi\right)\psi^{(s)}_{(\vec{y})}(\vec{p}) = \sum_s\int d\vec{y}y^k\left(\psi^{(s)}_{(\vec{y})},\psi\right)\psi^{(s)}_{(\vec{y})}(\vec{p})\,.$$

Substituting (7) and performing y-integration we get

$$\hat{Q}^k\psi(\vec{p}) = \int d\vec{q}\frac{2\sqrt{q_0p_0}}{(q_0+m)(p_0+m)}\sum_s u(\vec{p},s)\,u^+(\vec{q},s)\left(-i\frac{\partial}{\partial q^k}\delta(\vec{q}-\vec{p})\right)\psi(\vec{q}).$$

But

$$\sum_s u(\vec{p},s)\,u^+(\vec{q},s) \;=\; \Lambda_+(\vec{p})\left(\sum_s u(\vec{0},s)\,u^+(\vec{0},s)\right)\Lambda_+(\vec{q})$$
$$=\; \frac{1}{2}\Lambda_+(\vec{p})(1+\gamma_0)\Lambda_+(\vec{q})\,.$$

Therefore,

$$\hat{Q}^k\psi(\vec{p}) = \Lambda_+(\vec{p})(1+\gamma_0)\sqrt{\frac{p_0}{p_0+m}}\left(i\frac{\partial}{\partial p^k}\right)\sqrt{\frac{p_0}{p_0+m}}\Lambda_+(\vec{p})\psi(\vec{p})\,.$$

From this equation it is clear that

$$\hat{Q}^k = \Lambda_+(\vec{p})(1+\gamma_0)\sqrt{\frac{p_0}{p_0+m}}\left(i\frac{\partial}{\partial p^k}\right)\sqrt{\frac{p_0}{p_0+m}}\Lambda_+(\vec{p})\,. \tag{8}$$

Newton–Wigner position operator simplifies in the Foldy–Wouthuesen representation. In this representation its eigenfunctions are obtained via unitary transformation [72–74]:

$$\phi^{(s)}_{(\vec{y})}(\vec{p}) = e^{iW}\psi^{(s)}_{(\vec{y})}(\vec{p})\,,$$

where the Foldy–Wouthuesen unitary operator is

$$e^{iW} = \sqrt{\frac{2p_0}{p_0+m}}\left\{\frac{1}{2}(1+\gamma_0)\Lambda_+(\vec{p})+\frac{1}{2}(1-\gamma_0)\Lambda_-(\vec{p})\right\}\,.$$

Using the identity

$$\frac{1}{2}(1+\gamma_0)\Lambda_+(\vec{p})\,u(\vec{0},s) = \frac{1}{2}(1+\gamma_0)\Lambda_+(\vec{p})\frac{1}{2}(1+\gamma_0)\,u(\vec{0},s) = \frac{p_0+m}{2p_0}u(\vec{0},s)\,,$$

we get

$$\phi^{(s)}_{(\vec{y})}(\vec{p}) = \frac{1}{(2\pi)^{3/2}}e^{-i\vec{p}\cdot\vec{y}}u(\vec{0},s)\,.$$

These are Newton–Wigner position operator eigenstates in the Foldy–Wouthuesen representation. The corresponding position operator can be derived from them in the above described manner. The result is (in the momentum space)

$$\hat{\vec{Q}}_{(W)} = \frac{1}{2}(1+\gamma_0)i\frac{\partial}{\partial\vec{p}}\,,$$

or in configuration space

$$\hat{\vec{Q}}_{(W)} = \frac{1}{2}(1+\gamma_0)\vec{x} = \frac{1}{2}(1+\gamma_0)\vec{x}\,\frac{1}{2}(1+\gamma_0)\,.$$

Therefore, the Newton–Wigner position operator appears to be just the positive-energy projection of the Foldy–Wouthuesen "mean position operator" [72–74] and hence (8) is equivalent to

$$\hat{\vec{Q}} = \Lambda_+(\vec{p}) \left\{ i\frac{\partial}{\partial \vec{p}} + i\frac{\beta\vec{\alpha}}{2p_0} - \frac{i\beta(\vec{\alpha}\cdot\vec{p})\vec{p} + (\vec{\sigma}\times\vec{p})p_0}{2p_0^2(p_0+m)} \right\} \Lambda_+(\vec{p})\,, \tag{9}$$

where $\vec{\sigma} = (1/2i)\vec{\alpha}\times\vec{\alpha}$.

Let us compare time evolutions of the conventional (Dirac) and Newton–Wigner position operators in the Heisenberg picture [75, 76]. The Dirac position $\vec{x}(t) = e^{i\hat{H}t}\vec{x}e^{-i\hat{H}t}$ satisfies the Heisenberg equation of motion

$$\frac{d\vec{x}(t)}{dt} = i[\hat{H}, \vec{x}(t)] = e^{i\hat{H}t}\vec{\alpha}e^{-i\hat{H}t} = \vec{\alpha}(t)\,,$$

where we have used $\hat{H} = \vec{\alpha}\cdot\hat{\vec{p}} + \beta m$ and the canonical commutation relations (we are using $\hbar = c = 1$ convention throughout this section)

$$[x_i, \hat{p}_j] = i\delta_{ij}\,.$$

On the other hand by using

$$[\alpha_i, \alpha_j] = 2(\delta_{ij} - \alpha_j\alpha_i) = 2(\alpha_i\alpha_j - \delta_{ij})\,,$$
$$[\beta, \vec{\alpha}] = 2\beta\vec{\alpha} = -2\vec{\alpha}\beta\,,$$

we get

$$\frac{d\vec{\alpha}(t)}{dt} = i[\hat{H}, \vec{\alpha}(t)] = ie^{i\hat{H}t}[\hat{H}, \vec{\alpha}]e^{-i\hat{H}t} = 2i[\hat{\vec{p}} - \vec{\alpha}(t)\hat{H}] = 2i[\hat{H}\vec{\alpha}(t) - \hat{\vec{p}}]\,.$$

Differentiating once more, we obtain

$$\frac{d}{dt}\dot{\vec{\alpha}}(t) = -2i\dot{\vec{\alpha}}(t)\hat{H} = 2i\hat{H}\dot{\vec{\alpha}}(t)\,. \tag{10}$$

Therefore,

$$\dot{\vec{\alpha}}(t)\hat{H} = -\hat{H}\dot{\vec{\alpha}}(t)$$

and the solution of (10) is

$$\dot{\vec{\alpha}}(t) \equiv \frac{d\vec{\alpha}(t)}{dt} = \dot{\vec{\alpha}}(0)e^{-2i\hat{H}t} = 2i(\hat{\vec{p}} - \vec{\alpha}\hat{H})e^{-2i\hat{H}t}\,. \tag{11}$$

One can integrate (11) by using

$$\int\limits_0^t e^{-2i\hat{H}\tau}d\tau = \frac{i}{2}\hat{H}^{-1}\left(e^{-2i\hat{H}t} - 1\right)\,,$$

and the result is

$$\vec{\alpha}(t) = \hat{\vec{p}}\hat{H}^{-1} + (\vec{\alpha} - \hat{\vec{p}}\hat{H}^{-1})e^{-2i\hat{H}t}.$$

Inserting this into

$$\frac{d\vec{x}(t)}{dt} = \vec{\alpha}(t)$$

and integrating the resulting equation, we get the Dirac position operator in the Heisenberg picture

$$\vec{x}(t) = \vec{x} + \hat{\vec{p}}\hat{H}^{-1}t + \frac{i}{2}\left(\vec{\alpha} - \hat{\vec{p}}\hat{H}^{-1}\right)\hat{H}^{-1}\left(e^{-2i\hat{H}t} - 1\right). \tag{12}$$

This equation shows that a free electron surprisingly performs a very complicated oscillatory motion. The first two terms are just what is expected, because

$$\vec{p}H^{-1} = \frac{\vec{p}}{p_0}\frac{H}{p_0} = \frac{\vec{p}}{p_0}\left(\Lambda_+(\vec{p}) - \Lambda_-(\vec{p})\right)$$

is essentially the conventional relativistic velocity for positive-energy wave functions. But the last term in (12) is at odds with our classical intuition. According to it the Dirac electron executes rapid oscillatory motion, which Schrödinger called "Zitterbewegung". As $\vec{\alpha}(t)\cdot\vec{\alpha}(t) = 1$, the instantaneous value of electron's velocity during this trembling motion is always 1 (that is the velocity of light). The Zitterbewegung is a result of an interference between positive and negative frequency Fourier components of the particle wave-packet. The way the Zitterbewegung shows itself depends on the character of the particle wave-packet [75]. In the case of plane wave (not localised particles) one has a steady-state violent oscillations with amplitude $\sim 1/m$ and angular frequency $\sim 2m$ ($\hbar/mc$ and $2mc^2/\hbar$ if $\hbar$ and c are restored; for the electron $\hbar/mc = 3.85\times 10^{-3}$ Å and $2mc^2/\hbar = 1.55\times 10^{21}$ Hz).

For the Newton–Wigner position operator we will have

$$\frac{d\hat{\vec{Q}}(t)}{dt} = i[\hat{H},\hat{\vec{Q}}(t)] = ie^{-iW}e^{i\hat{H}_{(W)}t}[\hat{H}_{(W)},\hat{\vec{Q}}_{(W)}]e^{-i\hat{H}_{(W)}t}e^{iW}.$$

But in the Foldy–Wouthuesen representation

$$\hat{H}_{(W)} = \beta p_0,\ \ \hat{\vec{Q}}_{(W)} = \frac{1}{2}(1+\beta)i\frac{\partial}{\partial\vec{p}}$$

and

$$[\hat{H}_{(W)},\hat{\vec{Q}}_{(W)}] = -\frac{i}{2}(1+\beta)\frac{\vec{p}}{p_0}.$$

Therefore,

$$\frac{d\hat{\vec{Q}}(t)}{dt} = ie^{-iW}\frac{1}{2}(1+\beta)\frac{\vec{p}}{p_0}\,e^{iW} = \frac{\vec{p}}{p_0}\Lambda_+(\vec{p})$$

and

$$\hat{\vec{Q}}(t) = \hat{\vec{Q}} + \frac{\vec{p}}{p_0}\Lambda_+(\vec{p})t\,, \tag{13}$$

where $\hat{\vec{Q}} \equiv \hat{\vec{Q}}(0)$ is the initial value of the Newton–Wigner position operator given by (9). As we see, the Zitterbewegung is absent in the time development of the Heisenberg picture Newton–Wigner position operator. This is not surprising because only positive frequencies are present in the Fourier components of wave packets that are localised in the Newton–Wigner sense. However, this absence of trembling has a price: Newton–Wigner wave-packets cannot be localised sharper than $1/m$. To see this, let us consider the Newton–Wigner eigenfunction (7) in configuration space.

$$\psi^{(s)}_{(\vec{y})}(\vec{x}) = \frac{1}{(2\pi)^3}\int e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\sqrt{\frac{2p_0}{p_0+m}}\Lambda_+(\vec{p})\,u(\vec{0},s)\,d\vec{p}\,.$$

By using

$$\begin{aligned} u(\vec{0},s) &= \frac{1}{2}(1+\gamma_0)u(\vec{0},s),\, \Lambda_+(\vec{p})\frac{1}{2}(1+\gamma_0) \\ &= \frac{1}{2p_0}\begin{pmatrix} p_0+m & 0 \\ \vec{p}\cdot\vec{\sigma} & 0 \end{pmatrix}, \end{aligned} \tag{14}$$

we get

$$\psi^{(s)}_{(\vec{y})}(\vec{x}) = \frac{1}{(2\pi)^3}\int e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\frac{1}{\sqrt{2p_0(p_0+m)}}\begin{pmatrix} p_0+m & 0 \\ \vec{p}\cdot\vec{\sigma} & 0 \end{pmatrix}u(\vec{0},s)\,d\vec{p}\,,$$

or

$$\psi^{(s)}_{(\vec{y})}(\vec{x}) = \begin{pmatrix} A(\vec{z};m) & 0 \\ -2i\vec{\sigma}\cdot\frac{\partial^2 A(\vec{z};m)}{\partial\vec{z}\partial m} & 0 \end{pmatrix}u(\vec{0},s)\,,$$

where $\vec{z} = \vec{x}-\vec{y}$ and

$$A(\vec{z};m) = \frac{1}{(2\pi)^3}\int e^{i\vec{p}\cdot\vec{z}}\sqrt{\frac{p_0+m}{2p_0}}\,d\vec{p}\,.$$

For evaluation of the latter function, it is convenient to decompose

$$\sqrt{1+\frac{m}{p_0}} = 1 + \sum_{n=1}^{\infty} a_n\left(\frac{m}{p_0}\right)^n\,.$$

Then

$$A(\vec{z};m) = \frac{\delta(\vec{z})}{\sqrt{2}} + \frac{\sqrt{2}}{(2\pi)^2}\sum_{n=1}^{\infty} a_n \int\limits_0^{\infty} p\left(\frac{m}{p_0}\right)^n \frac{\sin(pz)}{z}\, dp\,, \qquad (15)$$

where $p = |\vec{p}|$ and $z = |\vec{z}|$. But

$$I = m^n \int\limits_0^{\infty} \frac{p}{p_0^n}\frac{\sin(pz)}{z} dp = -\frac{m^n}{z}\frac{d}{dz}\int\limits_0^{\infty} \frac{\cos(pz)}{(p^2+m^2)^{\frac{n}{2}}}\, dp\,.$$

This latter integral can be evaluated by using [77]

$$K_\nu(mz) = \frac{\Gamma(\nu+\frac{1}{2})(2m)^\nu}{\sqrt{\pi} z^\nu}\int\limits_0^{\infty} \frac{\cos(pz)}{(p^2+m^2)^{\nu+\frac{1}{2}}}\, dp$$

and

$$\frac{1}{z}\frac{d}{dz}\left[z^\nu K_\nu(z)\right] = -z^{\nu-1}K_{\nu-1}(z)\,,$$

where $K_\nu(z)$ is the Macdonald function. The result is

$$I = \frac{\sqrt{\pi} m^3}{\Gamma(\frac{n}{2}) 2^{\frac{n-1}{2}}}(mz)^{\frac{n-3}{2}} K_{\frac{n-3}{2}}(mz)\,.$$

Substituting this into (15), we get

$$A(\vec{z};m) = \frac{1}{\sqrt{2}}\left[\delta(\vec{z}) + \frac{\sqrt{\pi} m^3}{(2\pi)^2}\sum_{n=1}^{\infty}\frac{a_n}{\Gamma(\frac{n}{2})}\left(\frac{mz}{2}\right)^{\frac{n-3}{2}} K_{\frac{n-3}{2}}(mz)\right].$$

But for large arguments [77]

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z}\left\{1 + \frac{4\nu^2-1}{8z}\right\}.$$

Therefore, the function $A(z;m)$ decays for large $|\vec{x}-\vec{y}|$ as $e^{-m|\vec{x}-\vec{y}|}$ indicating that the localisation provided by the $\psi^{(s)}_{(\vec{y})}(\vec{x})$ wave-function is no better than $1/m$.

It seems, therefore, that the quantum mechanics and relativity create conceptual problems for Zeno's prescription to observe Achilles and the tortoise race (check the tortoise position when Achilles reach the position tortoise occupied at previous step). If we use the Dirac position for this goal, we will face Zitterbewegung, and if we use the Newton–Wigner one, the contenders will be not sharply localised. The notion of object's localisation, which seems so benignly obvious in classical mechanics, undergoes a radical change when distances are smaller than the Compton wavelength of the object. At first sight the notion of localisation at Compton scales appears as purely academic concept because even for electron the Compton wavelength is very small and the Zitterbewegung frequency very high, far beyond the present day experimental accessibility. But this situation can be changed in near future thanks to narrow-gap semiconductors [78].

In such semiconductors the dispersion relation between the energy and the wavenumber for electrons is analogous to that of relativistic electrons in vacuum and as a result effective relativity arises with maximum velocity of about 10^8 cm/sec instead of the velocity of light [78, 79]. Such systems can be used to model various relativistic phenomena the Zitterbewegung included. The amplitude of the corresponding trembling motion in semiconductors can be quite large, as much as 64 Å for InSb [78], and can be experimentally observed using high-resolution scanning-probe microscopy imaging techniques [80].

9. Quantum revivals of Zeno

Achilles is localised not only at the initial instant but remains so during the whole race. Maybe for Zeno this facet of classical world was not paradoxical at all but it appears not so trivial at modern times, after the quantum revolution. In quantum mechanics wave-packets spread as time goes by and it is not after all obvious how the classical reality with its definiteness arises from the weird quantum world. It is assumed, usually, that classical behaviour of macroscopic objects, like Achilles and tortoise, is something obvious and always guaranteed. But this is not correct. For example, a cryogenic bar gravity-wave detector must be treated as a quantum harmonic oscillator even though it may weigh several tons [81]. Superconductivity and superfluidity provide another examples of quantum behaviour at macroscopic scales.

To illustrate surprises that a quantum particle on a racetrack can offer, let us consider a wave packet of an electron in a hydrogen atom constructed from a superposition of highly excited stationary states centred at a large principal quantum number $\bar{n} = 320$ [82]. Initially the wave packet is well-localised near a point on the electron's classical circular orbit. Then it propagates around the orbit in accordance with the correspondence principle. During the propagation the wave packet spreads along the orbit and after a dozen classical periods $T_{\rm Kepler}$ appears as nearly uniformly distributed around the orbit (Fig. 8).

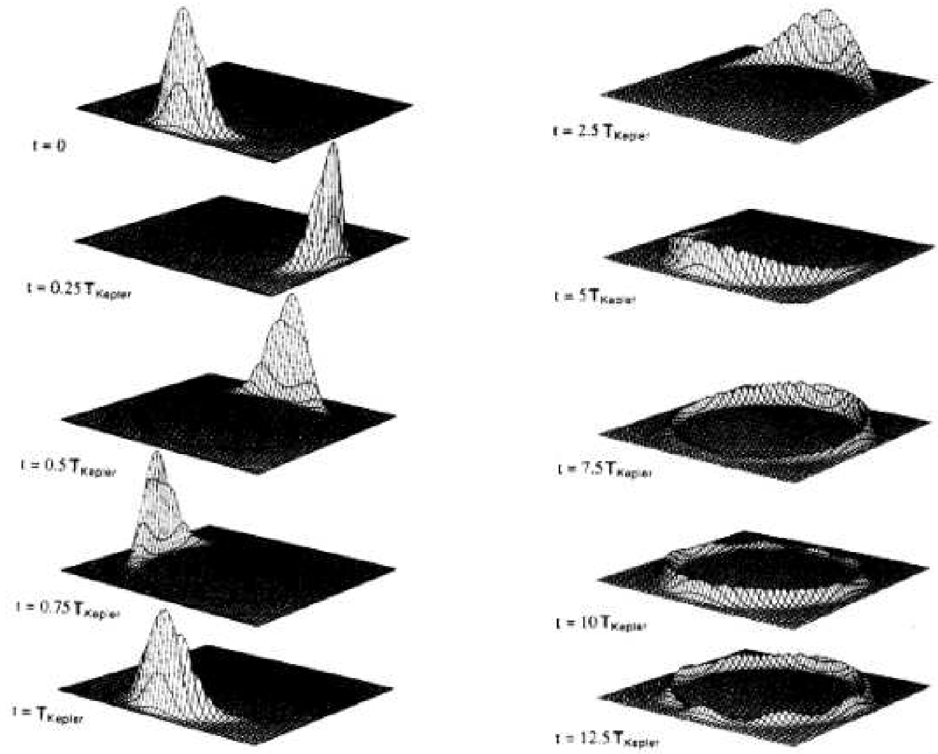

Fig. 8. Circular-orbit wave packet at initial stages of its time evolution [82].

If we wait long enough something extraordinary will happen: after some time $T_{\rm rev}$ the wave packet contracts and reconstructs its initial form (Fig. 9). This resurrection of the wave packet from the dead is called a quantum revival and it is closely related to the Talbot effect in optics [83]. In many circumstances the revivals are almost perfect and repeat as time goes on. At times $(k_1/k_2)T_{\rm rev}$, where k_1 and k_2 are two mutually prime numbers, fractional revivals happen and the wave packet consists of several high-correlated smaller clones of the original packet.

To explain the origin of quantum revivals, let us consider a particle of mass m in an infinite square well of width L [84, 85]. The energy spectrum of the system is given by

$$E_n = \frac{1}{2m}\left(\frac{\pi\hbar}{L}\right)^2 n^2.$$

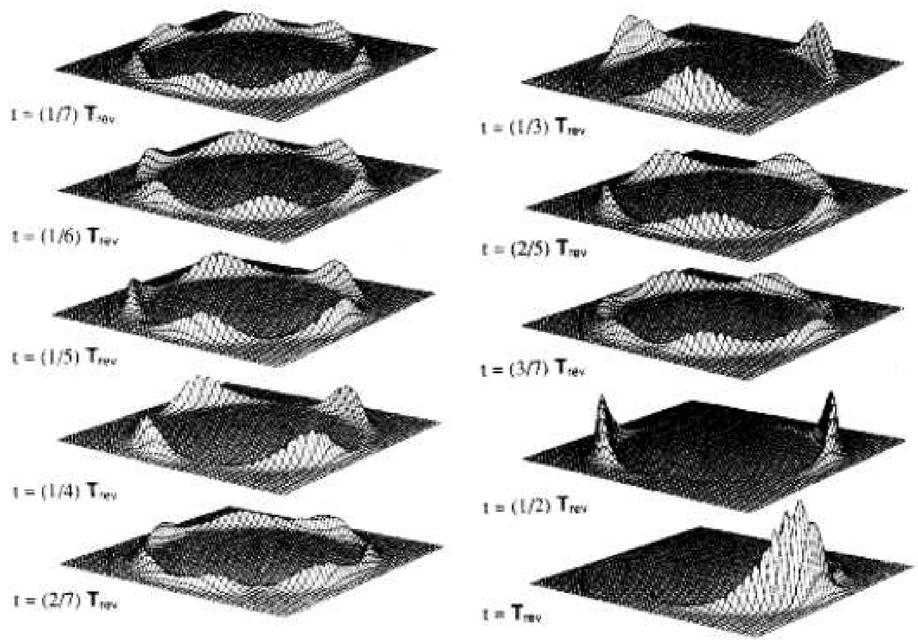

Fig. 9. Wave-packet revival and fractional revivals [82].

Suppose the initial wave function is

$$\psi(x;0)=\sum_{n=1}^{\infty}a_n\phi_n(x),$$

where $\phi_n(x)$ are the energy eigenfunctions. Time evolution of this wave function is described by

$$\psi(x;t)=\sum_{n=1}^{\infty}a_n e^{-\frac{i}{\hbar}E_nt}\phi_n(x).$$

Therefore, if there exists such a revival time $T_{\rm rev}$ that

$$\frac{E_n}{\hbar}T_{\rm rev}=2\pi N_n+\varphi \tag{16}$$

for all nonzero a_n, where N_n is an integer that can depend on n and φ does not depend on n, then $\psi(x;T_{\rm rev})$ will describe exactly the same state as $\psi(x;0)$.

But (16) is fulfilled if

$$T_{\rm rev}=\frac{4mL^2}{\pi\hbar}$$

and $\varphi = 0$. Therefore, any quantum state in an infinite square well will be exactly revived after a time $T_{\rm rev}$. Note that the classical period of bouncing back and forth between the walls is

$$T_{\rm cl} = \sqrt{\frac{2m}{E}}L$$

and for highly excited states ($n \gg 1$)

$$T_{\rm cl} = \frac{2mL^2}{n\pi\hbar} \ll T_{\rm rev}\,.$$

Now let us consider the quantum state $\psi(x;t)$ at half of revival time

$$\psi(x;T_{\rm rev}/2) = \sum_{n=1}^{\infty} a_n e^{-i\pi n^2}\phi_n(x) = \sum_{n=1}^{\infty}(-1)^n a_n\phi_n(x)\,,$$

where we have used $e^{-i\pi n^2} = (-1)^n$ identity. But

$$\phi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} = -(-1)^n\phi_n(L-x)\,.$$

Therefore,

$$\psi(x;T_{\rm rev}/2) = -\sum_{n=1}^{\infty} a_n\phi_n(L-x) = -\psi(L-x;0)\,,$$

and we have the perfect revival of the initial quantum state but at a location $L - x$ which mirrors the initial position about the center of the well.

At one quarter of the revival time

$$\psi(x;T_{\rm rev}/4) = \sum_{n=1}^{\infty} a_n e^{-i\pi n^2/2}\phi_n(x)\,.$$

But

$$e^{-i\pi n^2/2} = \cos\frac{n^2\pi}{2} - i\sin\frac{n^2\pi}{2} = \begin{cases} 1, & \text{if } n \text{ even}\,, \\ -i, & \text{if } n \text{ odd}\,. \end{cases}$$

Therefore,

$$\psi(x;T_{\rm rev}/4) = \sum_{n\ \rm even} a_n\phi_n(x) - i\sum_{n\ \rm odd} a_n\phi_n(x)\,.$$

Comparing this expression to

$$\psi(x;0) = \sum_{n\ \rm even} a_n\phi_n(x) + \sum_{n\ \rm odd} a_n\phi_n(x)$$

and

$$\psi(L-x;0) = -\sum_{n\ \mathrm{even}} a_n\phi_n(x) + \sum_{n\ \mathrm{odd}} a_n\phi_n(x)\,,$$

we deduce that [85]

$$\psi(x;T_{\mathrm{rev}}/4) = \frac{1-i}{2}\;\psi(x;0) - \frac{1+i}{2}\;\psi(L-x;0)\,.$$

Therefore, we have the perfect fractional revival at a time $T_{\mathrm{rev}}/4$ when two smaller copies of the initial wave packet appear at locations x and $L-x$.

Space–time structure of the probability density $|\psi(x;t)|^2$ is also very interesting. When plotted over long time periods (of the order of T_{rev}) it exhibits fine interference patterns known as quantum carpets [86,87]. Some examples, taken from [85], are shown in Fig. 10.

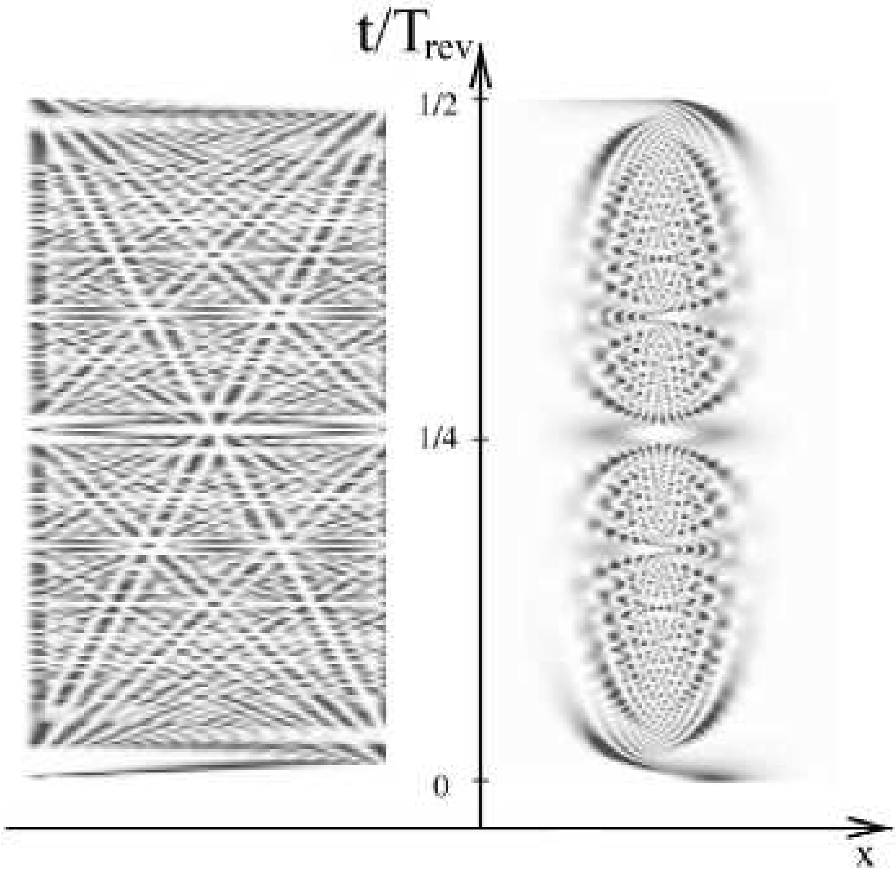

Fig. 10. The quantum carpets for the Pöschel–Teller and Rosen–Morse potentials.

Contemporary experimental technique allows to investigate quantum revivals and carpets and many theoretical results have been confirmed by experiments [83, 88]. Of course, nothing remotely similar to this weird phenomena happens during the Achilles and the tortoise race. And we come to one more paradox: why Zeno's paradoxes, formulated purely in classical terms, make complete sense for us? The answer may sound like this [81]: "The environment surrounding a quantum system can, in effect, monitor some of the systems observables. As a result, the eigenstates of these observables continuously de-cohere and can behave like classical states". Objects have no a priori classical properties. These properties are emergent phenomenon and come into being only through the very weak interaction with the ubiquitous degrees of freedom of the environment. Amazingly, the emergence of the classical world seems to be just another side of the quantum Zeno effect. As an example one can consider a large chiral molecule (like sugar) which can have both left-handed and right-handed classical spatial structures [89, 90]. For symmetry reasons, the ground state is equal mixture of both chiral states. Chiral molecules are never found in energy eigenstates and this is probably not surprising because such states are examples of Schrödinger cat states (like a superposition of a dead and an alive cat) which look truly absurd from classical viewpoint. But the real reason why non-classical states of the Chiral molecules are not observed is that it is chirality (not parity) that is recognised by the environment, for example by scattered air molecules. The chirality of the molecule is thus continuously "observed" by the environment and, therefore, cannot change because of quantum Zeno effect.

10. Zeno meets quantum gravity

Another twist to the story of localisation is added when one tries to incorporate gravity into a quantum theory. Any sharp localisation of the system creates a significant local energy density due to uncertainty relations and, therefore, changes the space–time metric according to the philosophy of general relativity. This can effect another localisation effort nearby in an unavoidable manner [91] and as a result it will matter whether x-position measurement is carried before or after y-position measurement. Moreover, localisation sharper than the Plankian scale creates a singularity in the space–time metric and, therefore, is problematic. We expect consequently that any coherent quantum gravity theory will not only bring the fundamental Plankian scale as the limit of space–time divisibility with it, but also a non-commutative space–time geometry.

Not only the concept of localisation of material objects but also operational meaning of the space–time itself is expected to be lost at Plankian scales. The principles of quantum mechanics and general relativity limit the accuracy of space–time distance measurements. The argument goes as follows [92].

Suppose we want to measure the initial distance between Achilles and the tortoise. We can attach a small mirror to the tortoise while a clock with light-emitter and receiver to Achilles. When the clock reads zero a light signal is sent to be reflected by the mirror. If the reflected signal arrives back at time t then the distance is $l = ct/2$. Note that this is a quite realistic scheme used, for example, in Lunar Laser Ranging experiments [93]. Quantum mechanics sets some limits on ultimate precision which can be reached in such distance measurements. Let, for example, the initial uncertainty in the clock's position be Δx. Then according to uncertainty relation its speed is also uncertain with the spread

$$\Delta v \geq \frac{1}{2}\frac{\hbar}{m\Delta x}\,,$$

where m is the mass of the clock. At time t the uncertainty of the clock's position will be larger

$$\Delta x(t) = \Delta(x(0)+vt) = \sqrt{(\Delta x)^2 + (t\Delta v)^2} = \sqrt{(\Delta x)^2 + \frac{1}{4}\frac{t^2\hbar^2}{m^2(\Delta x)^2}}\,.$$

The optimal uncertainty at initial time that minimises the uncertainty at time t is given by

$$(\Delta x)^2 = \frac{1}{2}\frac{t\hbar}{m}\,.$$

Therefore, the minimal uncertainty at time t is

$$\Delta x(t) = \sqrt{\frac{t\hbar}{m}} = \sqrt{\frac{2l\hbar}{mc}} = \lambda_C\sqrt{\frac{2l}{\lambda_{\mathrm{C}}}}\,,$$

where $\lambda_C = \hbar/mc$ is the Compton wavelength of the clock. The mirror contributes similarly to the overall uncertainty of the distance l to be measured. Therefore, ignoring small factors of the order of 2, we obtain an order of magnitude estimate [92, 94] (the argument actually goes back to Wigner [95, 96])

$$\Delta l \geq \lambda_{\mathrm{C}}\sqrt{\frac{l}{\lambda_{\mathrm{C}}}}\,. \tag{17}$$

Hence we need massive clock to reduce the uncertainty. But the mass of the clock cannot be increased indefinitely because the distance l should be greater than the clock's Schwarzschild radius. Otherwise the clock will collapse into a black hole as it is assumed that its size is smaller than l. Therefore,

$$l \geq \frac{Gm}{c^2}$$

and inserting this into (17) we get

$$\Delta l \geq l_{\mathrm{P}}\,, \tag{18}$$

where

$$l_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^3}}$$

is the Planck length.

Ng and Van Dam argue even for a more restrictive bound [92]. suppose the clock consists of two parallel mirrors a distance d apart. Then its one tick cannot be less than d/c and this implies $\Delta l \geq d$ if the clock is used for timing in distance measurements. But d should be greater than the clock's Schwarzschild radius. Therefore,

$$\Delta l \geq \frac{Gm}{c^2}. \tag{19}$$

Squaring (17) and multiplying the result by (19), we eliminate the clock's mass and obtain $(\Delta l)^3 \geq l l_{\mathrm{P}}^2$. Therefore,

$$\Delta l \geq l_{\mathrm{P}} \left(\frac{l}{l_{\mathrm{P}}}\right)^{1/3}. \tag{20}$$

It seems there is a common consensus about the validity of (18). While the more stronger bound (20) is still under debate (see, for example, [94]). Nevertheless this latter bound is consistent with the holographic principle [97] which states that the information content of any region of space cannot exceed its surface area in Planck units. Indeed [98], suppose we have a cube of dimension $l \times l \times l$ and every cell $\Delta l \times \Delta l \times \Delta l$ of this cube can be used to store one bit of information. Δl cannot be made less than dictated by the bound (20), otherwise it will be possible to measure the size of the cube in Δl units and reach the precision superior to the limit (20). Therefore, the information content of the cube

$$N = \frac{l^3}{(\Delta l)^3} \leq \frac{l^2}{l_{\mathrm{P}}^2}\,.$$

In any case, we can conclude that basic principles of quantum mechanics and general relativity strongly suggest discreteness of space–time at some fundamental scale. Zeno anticipated such possibility and attacked it with another couple of paradoxes. The first one, the Arrow, states that [2]

- *If everything is either at rest or moving when it occupies a space equal to itself, while the object moved is always in the instant, a moving arrow is unmoved.*

Stated differently, if a particle exists only at a sequence of discrete instants of time, what is the instantaneous physical properties of a moving particle which distinguishes it from the not moving one? And if there is no such properties (well, the notion of instantaneous velocity requires the concept of limit and thus is inappropriate in discrete space–time) how the motion is possible?

Modern physics changed our perspective of particles and motion and the Arrow seems not so disturbing today. For example we can defy it by stating that the arrow cannot be at rest at definite position according to the uncertainty principle. Alternatively, we can evoke special relativity and say that there is a difference how the world looks for a moving arrow and for an arrow at rest [2]: they have different planes of simultaneity. Special relativity can cope also with the second Zeno paradox against space–time discreteness, the Stadium:

- *Consider two rows of bodies, each composed of an equal number of bodies of equal size. They pass each other as they travel with equal velocity in opposite directions. Thus, half a time is equal to the whole time.*

If motion takes place in discrete quantum jumps then there should be an absolute upper bound on velocity. The maximum velocity is achieved then all jumps are in the same direction. The Stadium is intended to show logical impossibility of the maximum relative velocity. Suppose the rows of bodies from the paradox move at maximum or nearly maximum speed. Then in the rest frame of the first row other bodies are approaching at twice or so the maximum possible speed. Now we know that the latter inference is not sound. But Zeno was right that radical change of classical concepts of space and time is necessary to assimilate an observer-independent maximum velocity. "Space by itself and time by itself are to sink fully into shadows and only a kind of union of the two should yet preserve autonomy" — to quote Minkowski from his famous Cologne lecture in 1908.

In fact space–time discreteness coupled with the relativity principle assumes two invariant scales: not only the maximum velocity but also the minimum length. In this respect the special relativity refutation of Zeno is not complete. Only recently a significant effort was invested in developing Doubly Special Relativity [99, 100] — relativity theories with two observer-independent scales. These developments, although interesting, still are lacking experimental confirmation.

Doubly Special Relativity, if correct, should be a limiting case of quantum gravity — an ultimate theory of quantum space–time and a major challenge of contemporary physics to combine general relativity with quantum mechanics. This synthesis is not achieved yet but curiously enough its outcome may turn out to vindicate Parmenidean view that time and change are not fundamental reality. In any case the problem of time seems to be central in quantum gravity because time plays conceptually different roles in quantum mechanics and general relativity not easy to reconcile [101–105].

In quantum mechanics time is an external parameter, not an observable in the usual sense — it is not represented by an operator. Indeed, suppose there is an operator $\hat{T}$, representing a perfect clock, such that in the Heisenberg picture $\hat{T}(t) = e^{i\hat{H}t}\hat{T}e^{-i\hat{H}t} = t$. Then (we are using again $\hbar = 1$ convention)

$$i[\hat{H}, \hat{T}(t)] = \frac{d}{dt}\hat{T}(t) = 1\,.$$

Therefore, $[\hat{T}(t), \hat{H}] = i$ and $[\hat{H}, e^{i\alpha\hat{T}}] = \alpha e^{i\alpha\hat{T}}$. If ψ is an energy eigenstate with energy E then

$$\hat{H}e^{i\alpha\hat{T}}\psi = \left\{\left[\hat{H}, e^{i\alpha\hat{T}}\right] + e^{i\alpha\hat{T}}\hat{H}\right\}\psi = (E+\alpha)e^{i\alpha\hat{T}}\psi\,.$$

Therefore, $e^{i\alpha\hat{T}}\psi$ is also the energy eigenstate with energy $E+\alpha$ and the Hamiltonian spectrum cannot be bounded from below as it usually is. This argument goes back to Pauli [106]. Alternatively we can resort to the Stone–von Neumann theorem [107] and argue that the canonically conjugate $\hat{T}$ and $\hat{H}$ are just the disguised versions of the position and momentum operators and, therefore, must have unbounded spectra.

However, there are subtleties in both the Pauli's argument [106] as well as in the Stone–von Neumann theorem [108] and various time operators are suggested occasionally. Note that many useful concepts in physics are ambiguous or even incorrect from the mathematical point of view. In an extremal manner this viewpoint was expressed by Dieudonné [109]: "When one gets to the mathematical theories which are at the basis of quantum mechanics, one realizes that the attitude of certain physicists in the handling of these theories truly borders on the delirium". Of course, this is an exaggeration but sometimes mathematical refinement leads to new physical insights and it is not excluded that the last word about the time operator is not said yet. In any case, subtle is the time in quantum mechanics!

The idea of an event happening at a given time plays a crucial role in quantum theory [110] and at first sight introduces unsurmountable difference between space and time. For example [101], if $\Psi(\vec{x}, t)$ is a normalised wave function then $\int |\Psi(\vec{x}, t)|^2 d\vec{x} = 1$ for all times, because the particle must be somewhere in space at any given instant of time. While $\int |\Psi(\vec{x}, t)|^2 dt$ can fluctuate wildly for various points of space.

Nevertheless, the conceptual foundation of quantum theory is compatible with special relativity. Absolute Newtonian time is simply replaced by

Minkowski spacetime fixed background and unitary representations of the Poincaré group can be used to develop quantum theory. Situation changes dramatically in general relativity. "There is hardly any common ground between the general theory of relativity and quantum mechanics" [95]. The central problem is that spacetime itself becomes a dynamical object in general relativity. Not only matter is influenced by the structure of spacetime but the metric structure of spacetime depends on the state of ambient matter. As a result, the spatial coordinate $\vec{x}$ and the temporal coordinate t lose any physical meaning whatsoever in general relativity. If in non-general-relativistic physics (including special relativity) the coordinates correspond to readings on rods and clocks, in general relativity they correspond to nothing at all and are only auxiliary quantities which can be given arbitrary values for every event [95, 111].

Already at classical level, general relativity is a great deal Parmenidean and usually some tacit assumptions which fix the coordinate system is needed to talk meaningfully about time and time-evolution. To quote Wigner [95] "Evidently, the usual statements about future positions of particles, as specified by their coordinates, are not meaningful statements in general relativity. This is a point which cannot be emphasised strongly enough and is the basis of a much deeper dilemma than the more technical question of the Lorentz invariance of the quantum field equations. It pervades all the general theory, and to some degree we mislead both our students and ourselves when we calculate, for instance, the mercury perihelion motion without explaining how our coordinate system is fixed in space, what defines it in such a way that it cannot be rotated, by a few seconds a year, to follow the perihelion's apparent motion".

In quantum theory the situation only worsens. The Hamiltonian generates the time evolution of quantum system. But the equations of motion of general relativity are invariant under time re-parametrisation. Therefore, the time evolution is in fact unobservable — it is a gauge. The Hamiltonian vanishes and the Schrödinger equation in cosmology — the Wheeler–DeWitt equation does not contain time. "General relativity does not describe evolution in time: it describes the relative evolution of many variables with respect to each other" [111]. Zeno and Parmenides with their strange idea that time and change are some kind of illusion still have chance in quantum gravity!

11. Conclusion

The main conclusion of this paper is that physics is beautiful. Questions aroused two and half millennium ago and scrutinised many times are still not exhausted. Zeno's paradoxes deal with fundamental aspects of reality like localisation, motion, space and time. New and unexpected facets of these notions come into sight from time to time and every century finds it worthwhile to return to Zeno over and over. The process of approaching to the ultimate resolution of Zeno's paradoxes seems endless and our understanding of the surrounding world is still incomplete and fragmentary.

"Nevertheless, I believe that there is something great in astronomy, in physics, in all the natural sciences that allows the human being to look beyond its present place and to arrive at some understanding of what goes on beyond the insignificant meanness of spirit that so often pervades our existence. There is a Nature; there is a Cosmos; and we walk towards the understanding of it all. Is it not wonderful? There are many charms in the profession; as many charms as in love provided, of course, that they are not in the service of mercantile aims" [112].

REFERENCES

[1] A.N. Whitehead, *Essays in Science and Philosophy*, Philosophical Library: New York 1948, p. 87.

[2] K. Brown, *Reflections on Relativity*, p. 210, `http://www.mathpages.com/ rr/rrtoc.htm`.

[3] Diogenes Laertius, *The Lives and Opinions of Eminent Philosophers*, translated by C.D. Yonge, Henry G. Bohn, London 1853, `http://classicpersuasion.org/pw/diogenes/dlzeno-eleatic.htm`

[4] J.J. O'Connor, E.F. Robertson, *Zeno of Elea*, `http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Zeno_of_Elea.html`

[5] J. Burnet, *Early Greek Philosophy*, Black, London 1930.

[6] S. Makin, *Zeno of Elea*, Routledge Encyclopedia of Philosophy, Routledge, London 1998, Vol. 9, p. 843.

[7] A. Fairbanks, *The First Philosophers of Greece*, K. Paul, Trench, Trubner, London 1898, p. 112, `http://history.hanover.edu/texts/presoc/zeno.htm`

[8] F. Cajori, *History of Zeno's Arguments on Motion, Am. Math. Monthly* **22**, 1–6; 39–47; 77–82; 109–115; 143–149; 179–186; 215–220; 253–258; 292–297 (1915).

[9] A. Grunbaum, *Modern Science and Zeno's Paradoxes*, George Allen & Unwin Ltd., London 1968.

[10] W.C. Salmon, *Zeno's Paradoxes*, Bobbs Merrill, New York 1970.

[11] N. Huggett, *Zeno's Paradoxes*, The Stanford Encyclopedia of Philosophy (Summer 2004 Edition), Ed. E.N. Zalta, `http://plato.stanford.edu/archives/sum2004/entries/paradox-zeno/`

[12] Alexander Pushkin, *Speaking In Tongues Guided by Voices*, translated by Andrey Kneller, `http://spintongues.vladivostok.com/pooshkin3.htm`

[13] Aristotle, *Physics*, Book 8, translated by R.P. Hardie and R.K. Gaye, `http://www.abu.nb.ca/Courses/GrPhil/Physics.htm`

[14] W.I. McLaughlin, S.L. Miller, *Synthese* **92**, 371 (1992).

[15] W.I. McLaughlin, *Sci. Am.* **271**, 84 (1994).

[16] J.L. Bell, *The Art of the Intelligible: An Elementary Survey of Mathematics in Its Conceptual Development*, Kluwer Academic Pub. Dordrecht, The Netherland 1999, p. 175.

[17] L.M. Blumenthal, *Am. Math. Monthly* **47**, 346 (1940).

[18] V. Runde, *The Banach–Tarski Paradox or What Mathematics and Miracles Have in Common*, π in the Sky 2000, Vol. 2, p. 13, `http://ru.arxiv.org/abs/math.GM/0202309`

[19] S. Wagon, *The Banach–Tarski Paradox*, Cambridge University Press, New York 1993.

[20] K. Stromberg, *Am. Math. Monthly* **86**, 151 (1979).

[21] I. Pitowsky, *Phys. Rev. Lett.* **48**, 1299, 1768 (1982); *Phys. Rev. Lett.* **49**, 1216 (1982).

[22] I. Pitowsky, *Phys. Rev.* **D27**, 2316 (1983).

[23] K. Svozil, N. Neufeld, *Chaos Solitons Fractals* **7(5)**, 785 (1996).

[24] B.W. Augenstein, *Chaos Solitons Fractals* **7(11)**, 1761 (1996).

[25] R. Dougherty, M. Foreman, *J. Am. Math. Soc.* **7(1)**, 75 (1994).

[26] J.J. O'Connor, E.F. Robertson, *Georg Ferdinand Ludwig Philip Cantor*, `http://www-groups.dcs.st-andrews.ac.uk/ history/Mathematicians/Cantor.html`

[27] A. Peres, *Am. J. Phys.* **48**, 931 (1980),

[28] D. Wallace, *Phys. Rev.* **A63**, 022109 (2001).

[29] B. Misra, E.C.G. Sudarshan, *J. Math. Phys.* **18**, 756 (1977).

[30] D. Home, M.A.B. Whitaker, *Ann. Phys.* **258**, 237 (1997).

[31] A. Bokulich, *International Studies Philosophy of Science* **17**, 127 (2003).

[32] P. Facchi, S. Pascazio, *Phys. Lett.* **A241**, 139 (1998).

[33] M.C. Fischer, B. Gutiérrez-Medina, M.G. Raizen, *Phys. Rev. Lett.* **87**, 040402 (2001).

[34] D. Home, M.A.B. Whitaker, *J. Phys.* **A19**, 1847 (1986).

[35] B. Kaulakys, V. Gontis, *Phys. Rev.* **A56**, 1138 (1997).

[36] A.G. Kofman, G. Kurizki, *Phys. Rev.* **A54**, R3750 (1996).

[37] A.P. Balachandran, S.M. Roy, *Phys. Rev. Lett.* **84**, 4019 (2000).

[38] J.D. Franson, B.C. Jacobs, T.B. Pittman, *Phys. Rev.* **A70**, 062302 (2004).

[39] A.U. Schmidt, *Mathematics of the Quantum Zeno Effect*; `http://arxiv.org/abs/math-ph/0307044`

[40] J. Thomson, *Analysis* **15**, 1 (1954).

[41] P. Benacerraf, *Journal of Philosophy* **59**, 765 (1962).

[42] J. Earman, J.D. Norton, *Infinite Pains: the Trouble with Supertasks*, Eds. A. Morton & S.P. Stich, *Benacerraf and his Critics*; Blackwell, Oxford 1996, p. 231.

[43] J. Pérez Laraudogoitia, *Mind* **105**, 81 (1996).

[44] J.D. Hamkins, *Supertask Computation*, `http://arxiv.org/abs/math.LO/0212049`

[45] J. Pérez Laaraudogoitia, M. Bridger, J.S. Alper, *Synthese* **131**, 173 (2002).

[46] J.S. Alper, M. Bridger, J. Earman, J.D. Norton, *Synthese* **124**, 281 (2000).

[47] J. Earman, J.D. Norton, *British Journal for the Philosophy of Science* **49**, 123 (1998).

[48] J.D. Norton, *Found. Phys.* **29**, 1265 (1999).

[49] J. Earman, *Determinism: What We Have Learned and What We Still Don't Know*, Eds. Joseph Keim Campbell, Michael O'Rourke and David Shier, *Freedom and Determinism*; MIT Press, Cambridge 2004, p. 21.

[50] Z. Xia, *Ann. Math.* **135**, 411 (1992).

[51] D.G. Saari, Z. Xia, *AMS Notices* **42**, 538 (1995).

[52] D.G. Korycansky, G. Laughlin, F.C. Adams, *Astrophys. Space Sci.* **275**, 349 (2001).

[53] H. Weyl, *Philosophy of Mathematics and Natural Science*; Princeton University Press, Princeton 1949, p. 42.

[54] J. Earman, J.D. Norton, *Philosophy of Science* **60**, 22 (1993).

[55] I. Pitowsky, *The Physical Church's Thesis and Physical Computational Complexity*, Iyun **39**, 81 (1990).

[56] M. Hogarth, *Found. Phys. Lett.* **5**, 173 (1992).

[57] G. Etesi, I. Nemeti, *Int. J. Theor. Phys.* **41**, 341 (2002).

[58] F.C. Adams, G. Laughlin, *Rev. Mod. Phys.* **69**, 337 (1997).

[59] L.M. Krauss, G.D. Starkman, *Astrophys. J.* **531**, 22 (2000).

[60] K. Freese, W.H. Kinney, *Phys. Lett.* **B558**, 1 (2003).

[61] S. Lloyd, *Nature* **406**, 1047 (2000).

[62] M.B. Plenio, V. Vitelli, *Contemp. Phys.* **42**, 25 (2001).

[63] H.S. Leff, A.F. Rex, (Eds.) *Maxwell's Demon 2: Entropy, Classical and Quantum Information, Computing*; IOP Publishing, Bristol 2003.

[64] C.H. Bennett, *Studies in History and Philosophy of Modern Physics* **34**, 501 (2003).

[65] F.J. Dyson, *Rev. Mod. Phys.* **51**, 447 (1979).

[66] T.D. Newton, E.P. Wigner, *Rev. Mod. Phys.* **21**, 400 (1949).

[67] A.S. Wightman, *Rev. Mod. Phys.* **34**, 845 (1962).

[68] H. Halvorson, *Philosophy of Science* **68**, 111 (2001).

[69] G.N. Fleming, *Philosophy of Science* **67**, S495 (2000).

[70] H. Halvorson, *Locality, Localization, and the Particle Concept: Topics in the Foundations of Quantum Field Theory*; http://philsci-archive.pitt.edu/archive/00000346/

[71] Z.K. Silagadze, *The Newton–Wigner position operator and the domain of validity of one particle relativistic theory*, preprint SLAC-PUB-5754-REV, 1993.

[72] L.L. Foldy, S.A. Wouthuysen, *Phys. Rev.* **78**, 29 (1950).

[73] H. Feshbach, F. Villars, *Rev. Mod. Phys.* **30**, 24 (1958).

[74] E. de Vries, *Fortschr. der Phys.* **18**, 149 (1970).

[75] J.A. Lock, *Am. J. Phys.* **47**, 797 (1979).

[76] A.O. Barut, M.G. Cruz, *Eur. Phys. J.* **15**, 119 (1994).

[77] M. Abramowitz, I.A. Stegun, (Eds.) *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*, U.S. Government Printing Office, Washington 1972, p. 374.

[78] W. Zawadzki, *Zitterbewegung and Its Effects on Electrons in Semiconductors*; http://arxiv.org/abs/cond-mat/0411488

[79] W. Zawadzki, S. Klahn, U. Merkt, *Phys. Rev. Lett.* **55**, 983 (1985).

[80] J. Schliemann, D. Loss, R.W. Westervelt, *Zitterbewegung of Electronic Wave Packets in Semiconductor Nanostructures*, http://arxiv.org/abs/cond-mat/0410321

[81] W.H. Zurek, *Phys. Today* **44**, 36 (1991).

[82] Z.D. Gaeta, C.R. Stroud (Jr.), *Phys. Rev.* **A42**, 6308 (1990).

[83] M.V Berry, I. Marzoli, W. Schleich, *Physics World* **14N6**, 39 (2001).

[84] D.F. Styer, *Am. J. Phys.* **69**, 56 (2001).

[85] W. Loinaz, T.J. Newman, *J. Phys.* **A32**, 8889 (1999).

[86] W. Kinzel, *Physikalische Blätter* **51**, 1190 (1995).

[87] M.V. Berry, *J. Phys.* **A29**, 6617 (1996).

[88] R.W. Robinett, *Quantum Wave Packet Revivals*, http://xxx.lanl.gov/abs/quant-ph/0401031

[89] C. Kiefer, E. Joos, *Decoherence: Concepts and Examples*, http://arxiv.org/abs/quant-ph/9803052

[90] H.D. Zeh, *Lect. Notes Phys.* **538**, 19 (2000).

[91] D.V. Ahluwalia, *Phys. Lett.* **B339**, 301 (1994).

[92] Y.J. Ng, H. Van Dam, *Mod. Phys. Lett.* **A9**, 335 (1994).

[93] J.G. Williams, D.H. Boggs, S.G. Turyshev, J.T. Ratcliff, *Lunar Laser Ranging Science*, http://arxiv.org/abs/gr-qc/0411095

[94] J.C. Baez, S.J Olson, *Class. Quant. Grav.* **19**, L121 (2002).

[95] E.P. Wigner, *Rev. Mod. Phys.* **29**, 255 (1957).

[96] H. Salecker, E.P. Wigner, *Phys. Rev.* **109**, 571 (1958).

[97] R. Bousso, *Rev. Mod. Phys.* **74**, 825 (2002).

[98] Y.J. Ng, H. Van Dam, *Class. Quant. Grav.* **20**, 393 (2003).

[99] G. Amelino-Camelia, *Nature* **418**, 34 (2002).

[100] G. Amelino-Camelia, *Int. J. Mod. Phys.* **D11**, 1643 (2002).

[101] G. Belot, J. Earman, *PreSocratic Quantum Gravity*, Eds. C. Callender & N. Huggett, *Physics Meets Philosophy at the Planck Scale*, Cambridge University Press, Cambridge 2001, p. 213.

[102] C.J. Isham, *Canonical Quantum Gravity and the Problem of Time*, `http://arxiv.org/abs/gr-qc/9210011`

[103] C. Rovelli, *Quantum Gravity*, Cambridge University Press, Cambridge 2004.

[104] W.G. Unruh, *Time, Gravity, and Quantum Mechanics*, `http://arxiv.org/abs/gr-qc/9312027`

[105] K.V. Kuchar, *Time and Interpretations of Quantum Gravity*, Eds. G. Kunstatter, D.E. Vincent, J.G. Williams, proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics, World Scientific, Singapore 1992, p. 211.

[106] Z.Y. Wang, B. Chen, *J. Phys. A* **36**, 5135 (2003).

[107] J. Rosenberg, *A Selective History of the Stone-von Neumann Theorem*, `http://www.math.umd.edu/jmr/StoneVNart.pdf`

[108] S.J. Summers, *On the Stone-von Neumann Uniqueness Theorem and its Ramifications*, Eds. M. Rédei, M. Stöltzner, *John von Neumann and the Foundations of Quantum Physics*, Kluwer, Dordrecht 2001, p. 135.

[109] F. Gieres, *Rep. Prog. Phys.* **63**, 1893 (2000).

[110] J. Butterfield, C.J. Isham, *Spacetime and the Philosophical Challenge of Quantum Gravity*, Eds. C. Callender, N. Huggett, *Physics Meets Philosophy at the Planck Scale*, Cambridge University Press, Cambridge 2001, p. 33.

[111] C. Rovelli, *Quantum Space–Time: What Do We Know?*, Eds. C. Callender, N. Huggett, *Physics Meets Philosophy at the Planck Scale*, Cambridge University Press, Cambridge 2001, p. 101.

[112] M. Lopez-Corredoira, *What do Astrophysics and the World's Oldest Profession Have in Common?* `http://arxiv.org/abs/astro-ph/0310368`