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THE PENROSE INEQUALITY IN PERTURBED SCHWARZSCHILD GEOMETRIES

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There exists a scenario of a proof of the general Penrose inequality that requires a convexity property and the no-twist condition of a foliation of the Cauchy hypersurface. This paper shows that the no-twist condition can be removed and that, in the Schwarzschild geometry with linear axial perturbations, there do exist foliations that are convex in the required sense.

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1. Introduction

The Penrose inequality can constitute a necessary condition for the validity of the cosmic censorship ([4,5]). It has been proved a few years ago in the Riemannian case by Huisken and Ilmanen [7], and Bray [8]. That corresponds, roughly speaking, to momentarily static initial data of Einstein equations. It is well known that the inequality holds true in spherically symmetric systems [9], for any foliation choice of the hypersurface ([10,11]), but under a dominant energy condition.

Malec, Mars and Simon presented a scheme for the proof of the general Penrose inequality [13], which requires the existence of such a three-dimensional Cauchy hypersurface that *(i)* can be foliated by a family of two-spheres that conform to the Geroch flow condition [6] and *(ii)* are convex in the sense explained in the next section. If the above is true in a chosen space-like hypersurface and the dominant energy condition is satisfied, then the inequality is valid.

The purpose of this paper is twofold. Firstly, we show that one of the conditions of [13] can be relaxed — the foliation in question can have a nonzero twist. Secondly, and that is the main result, we demonstrate that there exists a foliation satisfying all other conditions of [13] in axial perturbations

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of maximal slices of Schwarzschild spacetimes. Therefore, one can conclude that the Penrose inequality is satisfied in a perturbed Schwarzschild spacetime.

2. Definitions

Let a Cauchy surface Σ be endowed with a metric

$$ {}^{(3)}\mathrm{d}s^2 = (N^2 + N_A N^A)\,\mathrm{d}r^2 + 2N_A\,\mathrm{d}r\,\mathrm{d}x^A + {}^{(2)}g_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B \tag{1} $$

and an extrinsic curvature K_{ij}. Assume that the surfaces S_r, defined as the level sets $r = \text{const}$, have spherical topology. The radial variable can be always reparametrized, so that $r = \sqrt{A_r/4\pi}$, where A_r is the area of S_r [17]. The Latin small case indices $i, j, k \ldots$ range from 1 to 3 while Latin capital indices $A, B, C \ldots$ denote tangential components and they range from 2 to 3. The index r always means the radial component — normal to the two-dimensional foliation — and is never used in summation. S_r has the induced two-dimensional metric ${}^{(2)}g_{AB}$ with an inverse ${}^{(2)}g^{AB}$. The trace of K_{ij} will be denoted as $\mathrm{Tr}\,K = K_{ij}{}^{(3)}g^{ij}$. K^T_{AB} is a two-dimensional trace-less tensor defined by:

$$ K^T_{AB} = K_{AB} - \frac{1}{2}{}^{(2)}g_{AB}(K_{DE}{}^{(2)}g^{DE}) = K_{AB} - \frac{1}{2}{}^{(2)}g_{AB}q\,. \tag{2} $$

The embedding of S_r in the three-dimensional manifold Σ is described by the second fundamental form p_{AB} with the mean curvature $p = \mathrm{Tr}\,p = p_{AB}{}^{(2)}g^{AB}$. The trace-less part of p_{AB} is defined as

$$ p^T_{AB} = p_{AB} - \frac{1}{2}{}^{(2)}g_{AB}p\,, \tag{3} $$

N is the three-dimensional lapse while N_A denotes the covariant components of the two-dimensional shift vector. We will say that the metric (1) has a nonzero twist if N_A is nonzero. It is assumed in what follows that $prN = 2$ — the *inverse mean curvature flow condition* (IMCF) of Geroch [6].

The embedding of S_r in a four-dimensional space–time can be specified with the help of the so-called optical scalars — objects that are built from the two fundamental forms, K_{ij} and p_{AB}. One has

$$ \theta = p + q\,, \tag{4a} $$

$$ \theta' = p - q\,. \tag{4b} $$

The apparent horizon H in the original formulation [1] is defined as the outermost level set S_H of $\theta = 0$ in Σ such that $\theta'(S_H) \geq 0$. Penrose

conjectured that the mass m of a space–time and the area A_H of the apparent horizon do satisfy the inequality

$$m \geq \sqrt{\frac{A_H}{16\pi}}\,. \tag{5}$$

This definition of apparent horizons can be ambiguous [19]; a recent discussion on that can be found in [20].

The main result of [13] proves Eq. (5) assuming that *(i)* the twist vanishes, *(ii)* there exists an inverse mean curvature flow extending from the outermost apparent horizon S_H (being one of the foliation leaves for some $r = r_0$) to spatial infinity, *(iii)* outside S_H the optical scalar $\theta'(S_r) > 0$, *(iv)* the ratio q/p is constant on each foliation leaf and, finally *(v)* the dominant energy condition holds.

Let us observe that one of the assumptions can be replaced by another that allows a more transparent geometric interpretation. Namely, the condition *(iv)* is equivalent to *(iv')* $\theta = c\theta'$ with a positive c being constant on all foliation leaves outside S_H. Indeed, it was already shown in [13] that if $q/p =$const, then $|q/p| < 1$ and $\theta = \gamma\theta'$, where $\gamma = (1 + q/p)/(1 - q/p) > 0$ is constant on all leaves in question. Conversely, if we start from $\theta = c\theta'$ and $c > 0$, then $q/p = (1 - c)/(1 + c)$ is constant on all foliation leaves outside S_H. Obviously, $|q/p| < 1$. The assumption $\theta = c\theta'$ appears in the Hayward's approach [21].

The optical scalars can be related to the expansions $\theta'(S_r)$ and $\theta(S_r)$ of bundles of light rays that start orthogonally from S_r inward or outward, respectively. The positivity of θ and θ' on a particular two-surface S_r implies that the in-going null rays are convergent everywhere and the outgoing null rays are divergent everywhere; that probably ensures that S_r is locally geodesically convex (*i.e.*, for any two close enough points lying on S_r there exists a geodesic segment joining them that does not leave the ball enclosed by S_r). In the light of that and the preceding Lemma, the condition *(iv)* can be interpreted as demanding a kind of convexity.

3. Nonzero twist

The initial data of Einstein evolution equations have to satisfy the nonlinear constraints

$$R^{(3)} = 16\pi\rho + K_{ij}K^{ij} - (\operatorname{Tr} K)^2\,, \tag{6a}$$

$$D_i\left(K^i_l - \delta^i_l \operatorname{Tr} K\right) = -8\pi j_l\,, \tag{6b}$$

where D is the covariant derivative and $R^{(3)}$ is the Ricci scalar on Σ, while ρ and j_l are the energy and momentum densities, respectively.

The rest of this section contains only a sketch of the derivations, which are straightforward but laborious. Calculational details will be described elsewhere [16]. Let $\mathcal{K}$ and n be the Gauss curvature and a normal of S_r, correspondingly. The Hamiltonian constraint can be written as

$$n^l D_l p = -8\pi\rho - \frac{1}{2}K_{ij}K^{ij} + \frac{1}{2}(\operatorname{Tr} K)^2 + \mathcal{K} - \frac{3}{4}p^2 - \frac{1}{2}p^{TA}_{\ \ B}p^{TB}_{\ \ A} + D_l a^l\,, \quad (7)$$

where $a^l = n^i D_i n^l$ is the "acceleration". Let us remark that Eq. (7) coincides, assuming $K_{ij} = 0$ and changing normalization of p, with the Eq. (2) of [17]. We will need the Hawking mass associated with a two-dimensional surface S_r, it is defined in terms of the optical scalars as follows

$$M_{\mathrm{H}}(S) = \frac{\sqrt{\frac{A_S}{16\pi}}}{4\pi}\int\limits_S \mathrm{d}^2 S\left(\mathcal{K} - \frac{\theta\theta'}{4}\right). \quad (8)$$

Assume that $V(S)$ is a volume of an annulus that is contained between two leaves S_1 and S_2, having radii r_1 and r_2 ($r_1 < r_2$), respectively. Let us multiply both sides of Eq. (7) by $pr/2$ and the momentum constraint in Eq. (6) by $n^l qr/2$, respectively, and integrate over the volume $V(S)$. One obtains, after subtracting the two expressions, integration by parts and numerous suitable rearrangements

$$\begin{aligned} M_{\mathrm{H}}(S_2) &= M_{\mathrm{H}}(S_1) + M(V) + J(V) \quad (9)\\ &+ \frac{1}{16\pi}\int\limits_{V(S)} \mathrm{d}V\Bigg[r\,{}^{(2)}g^{AB}\nabla^{(2)}_A(NK^r_B)(\operatorname{Tr} K - K^j_l n^l n_j)\\ &+ \frac{pr}{2}\,{}^{(2)}g^{AD}\,{}^{(2)}g^{EB}\\ &\times\left(p^T_{AB}p^T_{DE} - 2p^T_{AB}K^T_{DE}\frac{\operatorname{Tr} K - K^j_l n^l n_j}{p} + K^T_{AB}K^T_{DE}\right)\\ &+ p r^{(2)}g^{AB}\\ &\times\left(\frac{\nabla^{(2)}_A N}{N}\frac{\nabla^{(2)}_B N}{N} + 2NK^r_B\frac{\nabla^{(2)}_A N}{N}\,\frac{\operatorname{Tr} K - K^j_l n_j n^l}{p} + NK^r_A NK^r_B\right)\Bigg]. \end{aligned}$$

Here $M(V) = \frac{1}{2}\int_{V(S)} rp\rho\,\mathrm{d}V$ and $J(V) = \frac{1}{2}\int_{V(S)} r(\operatorname{Tr} K - K^b_a n^a n_b)j_i n^i\,\mathrm{d}V$ are the total mass and "radial" momentum, correspondingly. Notice that the right-hand side of Eq. (9) coincides with the volume form of the Eq. (11) in [13].

In the case when both optical scalars are nonnegative along the foliation, from the centre up to the surface S and

$$\frac{\operatorname{Tr} K - K^i_j n^j n_i}{p} \;=\; F(r)\,,$$

$$\textit{i.e.,} \qquad \nabla^{(2)}_A \left.\frac{\operatorname{Tr} K - K^i_j n_i n^j}{p}\right|_{r=\text{const}} \;=\; 0\,,$$

then the following can be observed.

(i) The first integral of Eq. (9) vanishes.

(ii) The positivity of both optical scalars implies $p \geq |\operatorname{Tr} K - K^j_i n_j n^i| = |q|$. Therefore, the second and third group of expressions appearing in the integrand

$$^{(2)}g^{AD}\,{}^{(2)}g^{EB}\left(p^T_{AB}p^T_{DE} - 2p^T_{AB}K^T_{DE}\frac{\operatorname{Tr} K - K^j_l n^l n_j}{p} + K^T_{AB}K^T_{DE}\right) \tag{10}$$

and

$$^{(2)}g^{AB}\left(\frac{\nabla^{(2)}_A N}{N}\frac{\nabla^{(2)}_B N}{N} + 2NK^r_B\frac{\nabla^{(2)}_A N}{N}\frac{\operatorname{Tr} K - K^j_l n_j n^l}{p} + NK^r_A NK^r_B\right) \tag{11}$$

are nonnegative.

(iii) Moreover, if the dominant energy condition is assumed and $p \geq |\operatorname{Tr} K - K^j_i n^i n_j|$, then $M(V) \geq |J(V)|$.

The Eq. (9) clearly demonstrates, that the Hawking mass $M_{\mathrm{H}}(S_2)$ is non-decreasing and nonnegative if the Hawking mass is nonnegative at S_1. If S_1 is the apparent horizon, then $M_{\mathrm{H}}(S_1) = \sqrt{A_{\mathrm{H}}/16\pi}$; assuming that S_2 is taken at spatial infinity, one obtains that the asymptotic mass $m \equiv m(S_2) \geq \sqrt{A_{\mathrm{H}}/16\pi}$. The existence of the foliation satisfying conditions *(i)–(v)* implies the validity of the Penrose inequality.

4. Axial perturbations of the Schwarzschild spacetime

In this section we will check that the Schwarzschild geometry with linear axial perturbations does indeed fulfill the needed assumptions.

Regge and Wheeler [14] found the general form of axial perturbations,

$$g_{tt} \;=\; -\left(1 - \frac{2m}{r}\right)\,, \tag{12a}$$

$$g_{tr} \;=\; 0, \tag{12b}$$

$$g_{t\theta} = -\varepsilon h_0(t,r)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}Y_{LM}\,, \tag{12c}$$

$$g_{t\phi} = \varepsilon h_0(t,r)\sin\theta\frac{\partial}{\partial\theta}Y_{LM}\,, \tag{12d}$$

$$g_{rr} = \frac{1}{1-2m/r}\,, \tag{12e}$$

$$g_{r\theta} = -\varepsilon h_1(t,r)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}Y_{LM}\,, \tag{12f}$$

$$g_{r\phi} = \varepsilon h_1(t,r)\sin\theta\frac{\partial}{\partial\theta}Y_{LM}\,, \tag{12g}$$

$$g_{\theta\theta} = r^2 + \varepsilon h_2(t,r)\left(\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi} - \frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi}\right)Y_{LM}\,, \tag{12h}$$

$$g_{\theta\phi} = \frac{\varepsilon}{2}h_2(t,r)\left(\frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2} + \cos\theta\frac{\partial}{\partial\theta} - \sin\theta\frac{\partial^2}{\partial\theta^2}\right)Y_{LM}\,, \tag{12i}$$

$$g_{\phi\phi} = r^2\sin^2\theta - \varepsilon h_2(t,r)\left(\sin\theta\frac{\partial^2}{\partial\theta\partial\phi} - \cos\theta\frac{\partial}{\partial\phi}\right)Y_{LM}\,, \tag{12j}$$

where ε is a "smallness" parameter (*i.e.*, we neglect terms of the order of ε^2 and higher). There is a gauge freedom, which allows one to set $h_2 \equiv 0$ (this is the so-called "Regge–Wheeler gauge"). The background geometry (*i.e.*, with $\varepsilon = 0$) is given by the standard Schwarzschild slicing,

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-2m/r} + r^2\,\mathrm{d}\Omega^2\,. \tag{13}$$

All the following calculations were performed using a computer algebra package GRTensorII [24]. Firstly, we change the background slicing to the maximal one ($\mathrm{Tr}\,K = 0$), by performing a coordinate change:

$$\overline{t} = t - \int_r^\infty \frac{c\ \mathrm{d}r'}{(1-2m/r')\sqrt{r'^4 - 2mr'^3 + c^2}}\,, \tag{14}$$

leaving r, θ, ϕ unaltered. We then choose the slicing by Cauchy hypersurfaces $\Sigma_{\overline{t}}$ defined by $\overline{t} =$ const. On the new slices the extrinsic curvature vanishes up to terms linear in ε. The Ricci scalar of such hypersurfaces $\Sigma_{\overline{t}}$ reads

$$R^{(3)} = \frac{6c^2}{r^6} + \mathcal{O}(\varepsilon^2)\,. \tag{15}$$

In the next step we choose a foliation in a fixed hypersurface $\Sigma_{\overline{t}}$ by two-dimensional surfaces S_r, defined by $r =$ const, and calculate the second

fundamental form p_{AB} and its trace p. The coordinate radius r coincides with the areal radius. The apparent horizon resides at the two-surface having $r = 2m$. It appears that the IMCF condition is satisfied in the chosen slicing (S_r) of $\Sigma_{\overline{t}}$:

$$Npr = 2\,. \tag{16}$$

It might be regarded as surprising that the above equality is exact — there are no terms with higher orders of ε. Finally we find the value of q/p,

$$\frac{\operatorname{Tr} K - K^j_i n_j n^i}{p} = -\frac{c}{\sqrt{r^4 - 2mr^3 + c^2}} + \mathcal{O}(\varepsilon^2); \tag{17}$$

it is constant on S_r up to terms linear in ε. That concludes the proof that linear axial perturbations of the Schwarzschild geometry satisfying the Regge–Wheeler gauge condition fall into the class of geometries considered in [13]. For that reason the Penrose inequality holds true. An open question is whether the linear perturbations can be integrated to a full solution of the constraint equations. One can expect that in the nonlinear case the gauge should be left free; the function h_2 can be specified due to condition *(iv)* of Sec. 2.

An interesting sub-case of the above presented slicing is given by the condition $c = 0$, which defines the moment of time-symmetry slice of the background geometry. We have then $R^{(3)} = \mathcal{O}(\varepsilon^2)$; if the linear perturbations happen to be a proper approximation to the full nonlinear solution, then the scalar curvature $R^{(3)}$ can be nonzero — either positive or negative. Thus, it is possible, that in the Schwarzschild metric with full nonlinear perturbations, one could find a region with $R^{(3)} < 0$. That means that results of Huisken and Ilmanen [7] and Bray [8] cannot be directly applied to a perturbed Schwarzschild geometry. This indicates also that the approach outlined in [13] is not sensitive on the sign of the scalar curvature of the Cauchy hypersurface.

5. Remarks on polar perturbations

Thus far we have shown that the linear axial perturbations of maximal slices of the Schwarzschild geometry satisfying the Regge–Wheeler gauge condition still fulfill demands of the procedure outlined in the former section, up to terms linear in ε. There was no need to adjust the gauge function $h_2(t, r)$.

The situation is different for the polar perturbations. We explicitly checked that the Regge–Wheeler gauge is too rigid to allow linear perturbations to satisfy the condition of [13]. In the case of polar perturbations there is, however, a gauge freedom with three arbitrary functions. As we require

only two conditions (*i.e.*, the IMCF and convexity conditions) we expect that it is possible to choose those three gauge functions in such a way that the required conditions will be satisfied. In order to find such a gauge we have to perform all the required calculations, using the general form of perturbations. In this situation the algebra becomes very complex and further work is required in order to arrive at conclusive results.

6. Conclusions

We checked the validity of a particular scheme of proving the Penrose inequality in the case of Einstein initial data defined on a slice foliated by topological spheres and satisfying the IMCF and the convexity conditions. A class of spacetimes admitting those conditions — the Schwarzschild geometry with axial perturbations — was given. It can also be expected, that the proof shown here extends on even perturbations of the Schwarzschild metric. Another approach to the Penrose inequality is presented by Jezierski [18] in the case of the perturbed Reissner–Nordström metric.

The only assumptions needed in the presented proof are the assumptions *(i)–(v)* presented in Sec. 2. No other assumptions are required, in particular the three-dimensional slice may have negative scalar curvature (as long as the assumptions of the Sec. 2 are satisfied). The above results support the belief that the scheme presented in the paper [13] can be useful in proving the Penrose inequality in general case.

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