STARUSZKIEWICZ'S THEORY OF THE ELECTRIC CHARGE IN A CUT FOCK SPACE — NUMERICAL INVESTIGATION

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In this paper we present our partial results of numerical investigation of the quantum Coulomb field $|u\rangle$. In particular we investigate the matrix of the $C_1 = -1/2 M_{\mu\nu} M^{\mu\nu}$ operator in Jacobi base, obtained from orthonormalization of $C_1^n|u\rangle$ vectors and photon distribution in the bound state of the C_1 operator, in search for some critical values of the coupling constant e^2 of the theory. So far our results are negative, that is, all characteristics we studied are smooth functions of $0 < e^2 < \pi$.

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1. Introduction

We use mechanical units such that $\hbar = 1 = c$ and the metric $(+,-,-,-)$. Staruszkiewicz's theory of the phase of the infrared part of electromagnetic field is formulated in the key papers [1, 2] and further developments are presented in [3–13]. The theory is defined by the action:

$$S_A = \frac{1}{8\pi e^2} \int\limits_{x\cdot x=-1} d^3\xi \sqrt{g} g^{ik} \partial_i S(x(\xi)) \partial_k S(x(\xi)) \, . \tag{1}$$

The phase $S(x)$ is a massless scalar field which lives on the (2+1)-dimensional hyperboloid $x \cdot x = -1$. Once the system is quantised, one can see that one of its degrees of freedom (namely quantum counterpart of a constant solution of equation of motion) is periodic. Once its scale is fixed the numerical value of the coupling e^2 becomes physically meaningful. In the original derivation of the theory [1], e^2 is the fine structure constant with the experimental value $e^2 = 1/137.03599911(46)$ [14]. The reader who does not like this connection can consider the action (1) as some field theory model in the curved (2+1)-dimensional space–time with some nontrivial dependence on the numerical value of e^2 being just some mathematical fact. ξ is any set of coordinates on the hyperboloid $x\cdot x=-1$. In the following we use $\xi^1=\psi$, $\xi^2=\theta$ and $\xi=\varphi$, where ψ is time coordinate: $x^0=\sinh\psi$ $(-\infty<\psi<\infty)$ and θ, φ are usual polar and azimuthal angles on a sphere. Solution of equation of motion and subsequent canonical quantisation leads to the following expansion and commutation rules ([1] p. 364):

$$S(x)=S_0-e\,Q\,\mathrm{th}\psi+\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left[f_{lm}^{(+)}(\psi,\theta,\varphi)c_{lm}+\mathrm{h.c.}\right], \tag{2}$$

$$[Q,\,S_0]=ie\,,\qquad [Q,\,c_{lm}]=0\,,\qquad [S_0,\,c_{lm}]=0\,, \tag{3}$$

$$[c_{lm},\,c^{\dagger}_{l'm'}]=4\pi e^2\,\delta_{ll'}\delta_{mm'}\,. \tag{4}$$

The explicit form of the functions $f_{lm}^{(+)}(\psi,\theta,\varphi)$, leading to the commutation rules (4), is given in [1,4]. The action (1) has the following symmetries: the Lorentz symmetry, which gives six constant of motion $M_{\mu\nu}=-M_{\nu\mu}$ and the "gauge" symmetry $S(x)\to S(x)+\mathrm{const}$, which gives additional constant of motion called the total charge. This constant is equal to the Q operator in the expansion (2):

$$Q=-\frac{1}{4\pi e}\int\limits_{\text{Cauchy surface}} d\Sigma^i\partial_i S(\xi)\,. \tag{5}$$

The generators of Lorentz transformation were explicitly given in [12], we rewrite here the boost generators for the reader's convenience:

$$\begin{aligned}
M_{0+} &= M_{01}+iM_{02}\\
&= -\frac{2\sqrt{z}}{\sqrt{3}}\frac{Q}{e}\left(a_{1(-1)}-a^{\dagger}_{11}\right)\\
&\quad +i\sum_{l=2}^{\infty}\sqrt{\frac{l^2-1}{4l^2-1}}\sum_{m=-l+2}^{l}\sqrt{(l+m-1)\,(l+m)}a^{\dagger}_{lm}a_{(l-1)(m-1)}\\
&\quad +i\sum_{l=2}^{\infty}\sqrt{\frac{l^2-1}{4l^2-1}}\sum_{m=-l}^{l-2}\sqrt{(l-m-1)\,(l-m)}a^{\dagger}_{(l-1)(m+1)}a_{lm}\,,
\end{aligned}$$

$$\begin{aligned}
M_{03} &= -\frac{\sqrt{2z}}{\sqrt{3}}\frac{Q}{e}\left(a_{10}+a^{\dagger}_{10}\right)\\
&\quad -i\sum_{l=2}^{\infty}\sqrt{\frac{l^2-1}{4l^2-1}}\sum_{m=-l+1}^{l-1}\sqrt{l^2-m^2}\left(a^{\dagger}_{lm}a_{(l-1)m}-a^{\dagger}_{(l-1)m}a_{lm}\right)\,,
\end{aligned}$$

where
$$a_{lm} = \frac{1}{\sqrt{4\pi e^2}}\, c_{lm} \qquad \text{and} \qquad a^\dagger_{lm} = \frac{1}{\sqrt{4\pi e^2}}\, c^\dagger_{lm}\,,$$
and $z = e^2/\pi$. In the following we use z instead of e^2/π. The vacuum state of the theory is defined by:
$$c_{lm}|0\rangle = 0 = \langle 0|c^\dagger_{lm} \tag{6}$$
and
$$Q|0\rangle = 0 = \langle 0|Q\,. \tag{7}$$
The first condition (6) follows directly from the commutation rules (4), while the second (7) is a consequence of the Lorentz invariance of the vacuum state:
$$M_{\mu\nu}|0\rangle = 0 = \langle 0|M_{\mu\nu}\,.$$
In [2, 11] the important theorem has been proved: The quantum Coulomb field[1] $|u\rangle$, (for definition see [2]) when decomposed into unitary, irreducible representations of the proper orthochronous Lorentz group contains

- only representations from the main series if $z > 1$,
- representations from the main series and a single representation from the supplementary series if $0 < z < 1$, corresponding to the special value of the Casimir operator:
$$0 < C_1 = -\frac{1}{2}M_{\mu\nu}M^{\mu\nu} = z(2-z) < 1\,.$$

This theorem shows that the value $z = 1$ is distinguished, as it separates two regimes with markedly different kinematical properties. This theorem is also interesting in a sense that it relates the numerical value of the fine structure constant with a purely kinematical quantity namely the parameter which labels unitary irreducible representations from the supplementary series. For the Coulomb field $|u\rangle$ the continuous part of the norm equals $1 - e^z(1-z)$,

[1] The quantum Coulomb field is defined as
$$|u\rangle = \exp\left[-iS(u)\right]|0\rangle\,,$$
where u is some 4-velocity, $u\cdot u = 1$, and $S(u)$ is the mean of the operator valued distribution $S(x)$ over a Cauchy surface $u\cdot x = 0$:
$$S(u) = \frac{1}{4\pi}\int\limits_{x\cdot x=-1} d^3\xi\sqrt{g}\,\delta(u\cdot x(\xi))\,S(\xi)\,.$$
$S_0 = S(u)$ in the rest frame of u.

while the discrete part of the norm equals $e^z(1-z)$. Thus for the small values of z almost entire norm of the Coulomb field is concentrated upon discrete part of the norm.

Our aim was to find the wave function of the bound state and if possible to find some characteristics of the bound state which depend on the numerical value of z. So far results are negative: the wave function is still not known and all the characteristics we studied seem to be smooth functions of z. We used the method of the cut Fock space [13]. We confirmed all the results presented in [13] and investigated the mean number of infrared photons in the bound state. It seems that this number is finite for all $0 < z < 1$. The paper is organised as follows. In Sec. 2 we summarise the method of [13]. In Sec. 3 we give some general properties of the C_1 matrix in the base defined in Sec. 2. In Sec. 4 we briefly discuss our implementation of building the orthogonal base of Sec. 2 and successive approximation of the bound state. In Sec. 5 we discuss some properties of the bound state, in particular its mean number of photons.

2. Staruszkiewicz's theory in the cut Fock space

2.1. Recurrence relations for a matrix of a general Hermitian operator in Jacobi form

Let H be a Hermitian operator and $|e_0\rangle$ a normalised state. Acting with H^n on $|e_0\rangle$ we get a set of linearly independent states: $|e_0\rangle$, $H\,|e_0\rangle$, $H^2\,|e_0\rangle\,, \ldots$. Applying the Gram–Schmidt procedure on this set of states we obtain the orthononormal base: $|e_0\rangle$, $|e_1\rangle$, $|e_2\rangle$, $\ldots$. In this orthonormal base the H matrix is 3-diagonal. The base $|e_i\rangle$ can be obtained in the following way:

$$|e_n\rangle = E_n\,|e_0\rangle\ , \tag{8}$$

where

$$E_n = \frac{1}{\sqrt{Z_n}}\left(H^n - \sum_{k=0}^{n-1}\langle E_k H^n\rangle\, E_k\right) \tag{9}$$

and the mean values are taken in the state $|e_0\rangle$. From the construction $\langle E_n E_k\rangle = \delta_{kn}$. Then the H matrix elements can be calculated from the following recurrence relations:

$$\langle e_{n+1}|H|e_n\rangle \ = \ \frac{\sqrt{Z_{n+1}}}{\sqrt{Z_n}}\,, \tag{10}$$

$$\langle e_n|H|e_n\rangle \ = \ \frac{\left\langle E_n H^{n+1}\right\rangle}{\sqrt{Z_n}} - \frac{\langle E_{n-1}H^n\rangle}{\sqrt{Z_{n-1}}}\,, \tag{11}$$

where

$$Z_n = \left\langle H^{2n} \right\rangle - \sum_{k=0}^{n-1} \langle E_k H^n \rangle^2 = \langle E_n H^n \rangle^2 \,, \tag{12}$$

$$\langle E_m H^n \rangle = \frac{1}{\sqrt{Z_n}} \left(\left\langle H^{m+n} \right\rangle - \sum_{k=0}^{m-1} \langle E_k H^m \rangle \langle E_k H^n \rangle \right) \tag{13}$$

and all mean values are taken in the state $|e_0\rangle$. Thus all H matrix elements are given in terms of $\left\langle H^k \right\rangle$. In our case $H \equiv C_1$ and $|e_0\rangle \equiv e^{-iS_0}|0\rangle$.

2.2. Evaluation of C_1 moments

The moments $\left\langle C_1^k \right\rangle$ are evaluated from a generating function

$$\left\langle u \left| e^{i\tau C_1} \right| u \right\rangle = \frac{1}{\pi}\, e^{i\tau} \int_0^\infty d\nu\, \nu e^{i\tau\nu^2} \int_{-\infty}^{+\infty} d\lambda\, (\sin \nu\lambda)(\sinh \lambda) e^{-z(\lambda \coth \lambda - 1)} \,. \tag{14}$$

On the right-hand side we exchange the order of integration, integrate term by term and then compare coefficients at successive powers of τ at both sides. We get

$$\langle u \left| C_1^n \right| u \rangle = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} A_k \,,$$

where

$$A_k = (-1)^k \frac{d^{2k+1}}{d\lambda^{2k+1}} \left[(\sinh \lambda) e^{-z(\lambda \coth \lambda - 1)} \right] \Big|_{\lambda=0} \,.$$

It is useful to note that

$$(\lambda \cot \lambda - 1) = \sum_{n \geq 1} 4^n B_{2n} \frac{\lambda^2 n}{(2n)!} \,.$$

It should be noted that we checked that this procedure works *experimentally* rather then *theoretically*: in fact the autocorrelation function (14) has zero radius of convergence, but all moments calculated in a direct way agree exactly with the moments obtained from the above procedure.

The moments $\langle C_1^n \rangle$ are polynomials in z, of degree n, with all coefficients positive.

$$\langle C_1 \rangle = 2z \qquad \text{and} \qquad \langle C_1^n \rangle = O\left(z^2\right) \quad \text{for} \quad n \geq 2 \,. \tag{15}$$

3. General properties of the C_1 matrix

We have calculated the elements of the C_1 matrix in the base $|e_0\rangle$, $|e_1\rangle$, $|e_2\rangle$, ... , defined in the previous section, in function of z up to the size 21×21. The general structure of the non-zero elements is

$$[C_1]_{n,n} = c_{n,n} \frac{W_{(n-3)\times(n-2)+(n-2)\times(n-1)+1}}{W_{(n-3)\times(n-2)} W_{(n-2)\times(n-1)}} \,, \tag{16}$$

$$[C_1]_{n,n+1} = c_{n,n+1} \frac{\sqrt{W_{(n-3)\times(n-2)} W_{(n-1)\times n}}}{W_{(n-2)\times(n-1)}} \,, \tag{17}$$

where W_k is a polynomial of degree k in z, with all coefficients positive.

We note that all zeros of these polynomials have negative real part but there is a numerical evidence [13] that the zero with the biggest real part (which happens to be always real) approaches zero in the limit $n \to \infty$, reducing the convergence radius of successive matrix elements to zero. Similarly we checked the zeros with the smallest real part (which again happens to be real) are not bounded from below (they decrease linearly with n, where $n(n+1)$ is polynomial degree). This indicates that the C_1 matrix is not analytic at $z = 0$ nor at $z = \infty$. This is consistent with the following approximations for C_1 matrix at small and large z:

$$\begin{aligned}
[C_1]_{n,n} &= 2n(n-1) + \left(\frac{4}{3}n(n-1) + 2\right) z + O\left(z^2\right) \,, \\
[C_1]_{n,n+1} &= \frac{2(n+1)n(n-1)}{\sqrt{(2n-1)(2n+1)}} + \frac{4(n+1)n(n-1) + 8n}{3\sqrt{(2n-1)(2n+1)}} z + O\left(z^2\right) \,,
\end{aligned}$$

and

$$\begin{aligned}
[C_1]_{n,n} = &\left(\frac{8}{3}(n-1) + 2\right) z + \frac{4\,(n-1)\,(4\,(n-1)+1)}{5} \\
&- \frac{12\,(n-2)\,(n-1)\,(8n-3)}{175} \, \frac{1}{z} \\
&+ \frac{6\,(n-2)\,(n-1)\left(80n^2 - 152n + 51\right)}{875} \, \frac{1}{z^2} \\
&- \frac{108\,(n-2)\,(n-1)\left(2248n^3 - 8001n^2 + 8361n + 2451\right)}{336875} \, \frac{1}{z^3} \\
&+ O\left(\frac{1}{z^4}\right) \,,
\end{aligned}$$

$$
\begin{aligned}
[C_1]_{n,n+1} = &\ \frac{2\sqrt{2n\,(2n+1)}}{3}\, z \\
&+ \frac{4\,(n-1)\,\sqrt{2n\,(2n+1)}}{5} - \frac{3\,(n-1)\,(8n-9)\,\sqrt{2n\,(2n+1)}}{175}\,\frac{1}{z} \\
&+ \frac{6\,(n-1)\left(20n^2-43n+27\right)\sqrt{2n\,(2n+1)}}{875}\,\frac{1}{z^2} \\
&- \frac{27\,(n-1)\left(8992n^3-29756n^2+35055n-13859\right)\sqrt{2n\,(2n+1)}}{1347500}\,\frac{1}{z^3} \\
&+ O\left(\frac{1}{z^4}\right).
\end{aligned}
$$

The expansion for large z can be easily continued, but its form suggests that the expansion is asymptotic: the ratio of the two successive terms in the expansion behaves like n/z and becomes large for $n \to \infty$. This suggests that C_1 matrix is not analytic at $z = \infty$. We have not managed to guess the exact form of the next terms of the expansion for small z, but again the ratio of successive terms in the expansion behaves as nz and becomes large for $n \to \infty$. Otherwise our result to the order $O\left(z^2\right)$ can be easily checked due to (15).

4. Implementation in Mathematica

The C_1 matrix and coefficients of the bound state in the orthonormal base (8) can be obtained as described in the previous section. However, to get the explicit form of $|e_i\rangle$ vectors as well as the bound state one has to evaluate (8) explicitly from first principles. To do such calculation programs like Mathematica or Maple can be used. We use the method of [13] with a slight modification in representing the states. Any state can be represented as superposition of the vectors from the Fock basis:

$$
\prod_{l=1}^{\infty}\prod_{m=-l}^{l}\left(a^{\dagger}_{lm}\right)^{n_{lm}}\ e^{-in_Q S_0}|0\rangle\,.
$$

Thus in Mathematica we represent a state from $n_Q = Q/e = 1$ sector as a flexible list of the form

$$
|\mathrm{state}\rangle = \{N, \{\underbrace{\alpha_1,\ldots,\alpha_N}_{N\text{elements}}\}, \underbrace{\{\{p^1_1, n_{p^1_1}\},\ldots,\{p^1_{k_1}, n_{p^1_{k_1}}\}\},\ldots,\{\{p^N_1, n_{p^N_1}\},\ldots,\{p^N_{k_N}, n_{p^N_{k_N}}\}\}}_{N\text{elements}}\}.
$$

The first element of this list (N) gives the number of Fock basis states in the superposition. The second is the list of (N) amplitudes multiplying basis states in the superposition. Next elements are lists, each representing one of the (N) Fock basis states entering superposition. Each of these lists is a list of pairs, the first element (position p) encodes l, m: $p = l^2 + l + m$, the second is the number of quanta for a given p. Only non-zero occupation numbers are explicitly encoded. We give here a few examples:

$$\begin{aligned}
|0\rangle &\leftrightarrow \{0, \{\}\}\,, \\
|e_0\rangle \equiv e^{-iS_0}\,|0\rangle &\leftrightarrow \{1, \{1\}, \{\}\}\,, \\
|e_1\rangle \equiv \frac{1}{\sqrt{6}} &\left(-2\, a^\dagger_{1(-1)}\, a^\dagger_{11} + \left(a^\dagger_{10}\right)^2\right) e^{-iS_0}\;|0\rangle \\
&\leftrightarrow \left\{2, \left\{-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right\}, \{\{1,1\},\{3,1\}\},\ \{\{2,2\}\}\right\}\,.
\end{aligned}$$

We implement all basic operations on states, like linear superposition, action of creation (annihilation) operators, as operations on the lists. This approach allows to do calculation from first principles (commutation rules), analytically on a computer.

5. The properties of the bound state of C_1 operator

For any eigenstate $|x\rangle$ of the C_1 operator for the eigenvalue x one can define a series

$$F_x(z) = \sum_{n=0}^{\infty} \left|a_n^{(x)}\right|^2\,, \tag{18}$$

where

$$|x\rangle = \sum_{n=0}^{\infty} |e_n\rangle\,\langle e_n|\,x\rangle = \sum_{n=0}^{\infty} a_n^{(x)}\,|e_n\rangle\,.$$

Due to the fact that in the Jacobi base $|e_0\rangle$, $|e_1\rangle$, ... the C_1 matrix is three-diagonal, the eigenequation $C_1\,|x\rangle = x\,|x\rangle$ becomes a recurrence equation for the coefficients $a_n^{(x)}$. The Staruszkiewicz's theorem says that the series (18) is finite only if $0 < z < 1$ and $x = z(2-z)$. If this theorem was broken for some value of z it would exclude this value from the theory. So far no numerical evidence that this happens has been found [13]. It seems that the series (18) is finite for all $0 < z < 1$.

We find an explicit form of the first 8 Jacobi base vectors $|e_0\rangle$, ..., $|e_7\rangle$. The number of Fock basis states entering $|e_i\rangle$ grows rapidly and reaches 27085 components for $|e_7\rangle$. In general in $|e_n\rangle$ there are at most $2n$ photons, with $l \leq n$. Then we get successive approximations of the bound state $C_1\,|\mathrm{bound}\rangle = z(2-z)\,|\mathrm{bound}\rangle$:

$$|\mathrm{bound}_N\rangle = \frac{1}{\sqrt{\sum\limits_{n=0}^{N} |a_n|^2}} \sum_{n=0}^{N} a_n\,|e_n\rangle\,.$$

The first 8 coefficients for $z = 1/2$ and $z = 1$ is listed in Table I. Formally, we do not know if the bound state exists for $z = 1$, however, it is useful to compare characteristics of the bound state for $0 < z < 1$, with the limiting value $z = 1$. In the following the bound state for $z = 1$ means, the eigenstate for the eigenvalue 1. We searched for some nontrivial dependence of the mean number of photons and photon distribution on the numerical value of z. So far results are negative that is all observables we looked at are smooth functions of z. For example the probabilities of finding n photons in successive approximations of the bound state for $z = 1/2$ and $z = 1$ are given in Table II and presented in Fig. 1. For other values of $0 < z < 1$ the photon distributions look very similar. We plot the probability of finding n photons in $|\mathrm{bound}_7\rangle$ for $z = 1$, $1/2$, $1/100$, $1/500$ in Fig. 2.

TABLE I

Coefficients of the first 8 Jacobi base states $|e_0\rangle, \ldots, |e_7\rangle$ in the linear superposition giving the bound state of the C_1 operator for $z = 1/2$ and $z = 1$.

	n	0	1	2	3	4	5	6	7
$z=1/2$	a_n	1	−0.306	0.188	−0.137	0.108	−0.090	0.077	−0.067
$z=1$	a_n	1	−0.765	0.702	−0.670	0.651	−0.638	0.628	−0.621

In Fig. 3 we plot mean number of photons in successive approximations of the bond state for $z = 1/2$ and $z = 1$. The $z = 1/2$ case is well fitted with the dependence, which indicates that it converges to ~ 0.47 (we get this value for different types of convergent fits). The $z = 1$ case is well fitted with logarithmic dependence, which may indicate that the mean number of photons diverges for $z = 1$. If this is the case it indicates once again, that the value $z = 1$ is the distinguished point of the theory.

Figs. 1, 2 suggests that for $0 < z < 1$ the probability of finding n photons in the bound state is given by

$$P(n) = \begin{cases} c(z)e^{-\gamma(z)\,n} & \text{for} \quad n = 0, 2, 3, 4, \ldots \\ 0 & \text{for} \quad n = 1, \end{cases} \tag{19}$$

where the exponent γ and the normalisation constant are functions of z. The consistency of this assumption is supported in the following way: once the exponent γ is determined the value of the normalisation constant c is set by the normalisation condition $\sum_{n=1}^{\infty} P(n) = 1$; then the mean number of photons can be calculated. For the distribution (19), c is also the probability of finding 0 photons in the bound state and can be independently estimated as a limiting value of the $n = 0$ row of Table II for the corresponding

TABLE II

The probabilities of finding n photons in successive approximations of the bound state for $z=1/2$ and $z=1$.

	$\vert\mathrm{bound}_3\rangle$	$\vert\mathrm{bound}_4\rangle$	$\vert\mathrm{bound}_5\rangle$	$\vert\mathrm{bound}_6\rangle$	$\vert\mathrm{bound}_7\rangle$
n			$z=1/2$		
0	0.8710	0.8622	0.8562	0.8519	0.8486
2	0.1001	0.1018	0.1027	0.1032	0.1035
3	0.0212	0.0247	0.0269	0.0284	0.0295
4	0.0067	0.0091	0.0108	0.0120	0.0130
5	0.0009	0.0019	0.0028	0.0034	0.0040
6	6.27E-5	2.87E-4	5.64E-4	8.35E-4	1.08E-3
7		2.34E-5	7.90E-5	1.53E-4	2.34E-4
8		8.13E-7	7.11E-6	2.05E-5	3.97E-5
9			3.54E-7	1.88E-6	5.04E-6
10			7.37E-9	1.09E-7	4.58E-7
11				3.57E-9	2.84E-8
12				5.0E-11	1.13E-9
13					2.6E-11
14					2.6E-13
n			$z=1$		
0	0.5608	0.5208	0.4917	0.4693	0.4513
2	0.2556	0.2440	0.2343	0.2260	0.2190
3	0.1030	0.1135	0.1187	0.1206	0.1214
4	0.0624	0.0799	0.0908	0.0981	0.1031
5	0.0160	0.0313	0.0431	0.0522	0.0594
6	0.0022	0.0090	0.0165	0.0234	0.0295
7		0.0014	0.0044	0.0079	0.0116
8		9.84E-5	7.56E-4	2.00E-3	3.63E-3
9			7.39E-5	3.52E-4	8.65E-4
10			3.08E-6	4.00E-5	1.51E-4
11				2.59E-6	1.83E-5
12				7.19E-8	1.44E-6
13					6.57E-8
14					1.30E-9

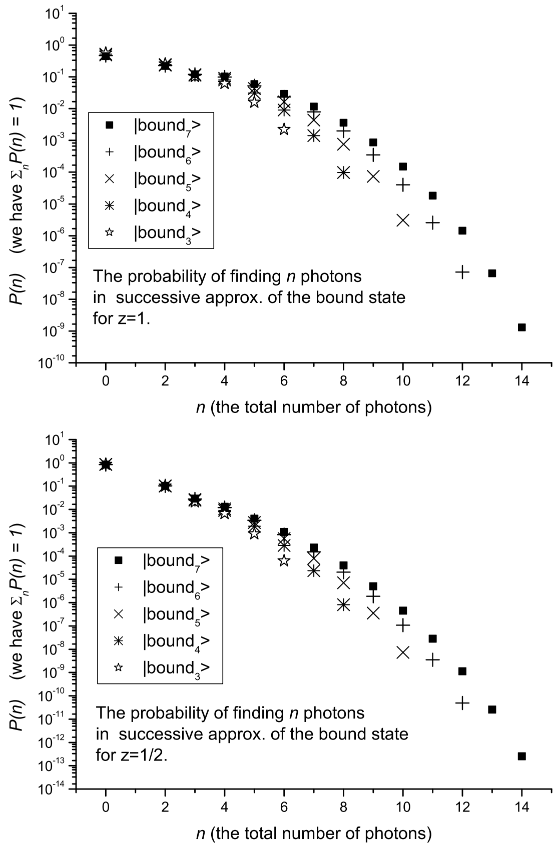

Fig. 1. The probabilities of finding n photons in successive approximations of the bound state for $z = 1/2$ and $z = 1$.

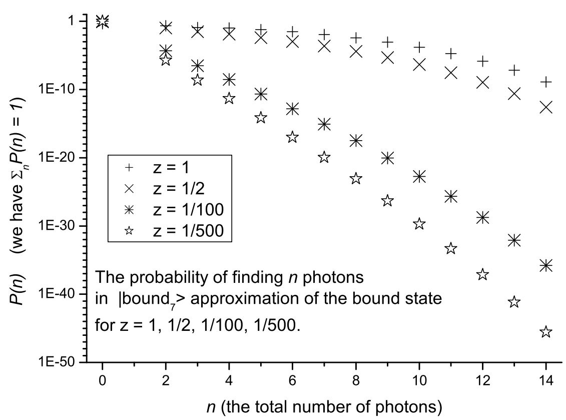

Fig. 2. The probability of finding n photons in $|\text{bound}_7\rangle$ for $z = 1,\ 1/2,\ 1/100,\ 1/500$.

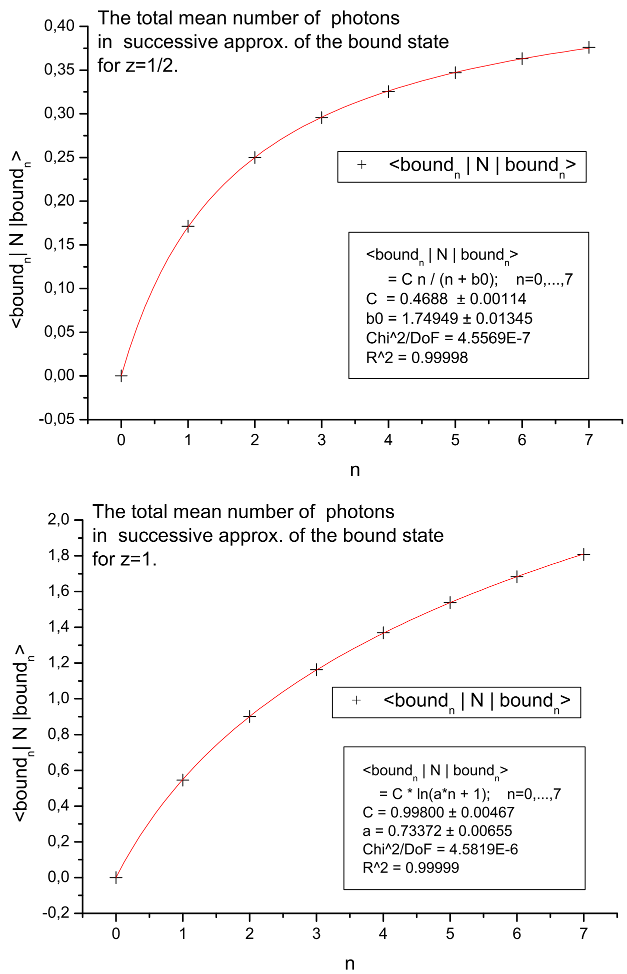

Fig. 3. The mean number of photons in successive approximations of the bound state for $z = 1/2$ and $z = 1$, together with schematic fits.

value of z. The mean number of photons from the distribution (19) can be compared with the value obtained form the fit like in Fig. 3. For example for $z = 1/2$ we fit the exponent γ from the first 7 points ($n = 0, \ldots, 6$) for $|\mathrm{bound}_7\rangle$ (see Table II) and we get $\gamma = 1$. This gives $c = 0.824$ and the mean number of photons $= 0.455$. This is consistent with the limiting value 0.826 of the first row of Table II (we get this value fitting $1 - ak/(k+b)$ dependence, with $k = 3, \ldots, 7$ for successive numbers in the row) and the mean number obtained from the fit in Fig. 3. The same consistency is obtained for other values of $0 < z < 1$.

6. Summary

It is known from the birth of the Staruszkiewicz's theory [1, 2] that the Hilbert space structure of the theory depends on the numerical value of the coupling constant, with the value $e^2/\pi = 1$ being distinguished. Below this value the bound state of the (first) Casimir operator C_1 appears. We searched for the other special values of e^2, looking at mean number of photons and photon distribution in the bound state. So far our results are negative, that is all observables we looked at seem to be smooth functions of $0 < e^2 < \pi$. However, we still believe that with deeper understanding of the Staruszkiewicz's theory the other critical values will be recognised.

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