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EXACT ANALYTICAL SOLUTION OF THE CONSTRAINED STATISTICAL MULTIFRAGMENTATION MODEL

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A novel powerful mathematical method is presented, which allows us to find an analytical solution of a simplified version of the statistical multifragmentation model with the restriction that the largest fragment size cannot exceed the finite volume of the system. A complete analysis of the isobaric partition singularities is done for finite system volumes. The finite size effects for large fragments and the role of metastable (unstable) states are discussed. These results allow us, for the first time, to exactly describe the finite volume analog of the bulk nuclear liquid phase and understand completely its contribution to the grand canonical partition.

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1. Introduction

Exactly solvable models with phase transitions play a special role in the statistical physics — they are the benchmarks of our understanding of critical phenomena that occur in more complicated substances. They are our theoretical laboratories, where we can study the most fundamental problems of critical phenomena which cannot be studied elsewhere. A great deal of progress was recently achieved in our understanding of the multifragmentation phenomenon [1–4] when an exact analytical solution of a simplified version of the statistical multifragmentation model (SMM) [5,6] was found in Refs. [7,8]. This exact solution allowed us to elucidate the role of the Fisher exponent τ on the properties of (tri)critical point and to show explicitly [9] that in the SMM the relations between τ and other critical indices differ from the corresponding relations of a well known Fisher droplet model [10]. Note that these questions *in principle* cannot be clarified either within the widely used mean-filed approach or numerically.

Despite this success, the application of the exact solution [7–9] to the description of experimental data is limited because this solution corresponds to an infinite system volume. Therefore, from a practical point of view it is necessary to extend the formalism for finite volumes. Such an extension is also necessary because, despite a general success in the understanding of the nuclear multifragmentation, there is a lack of a systematic and rigorous theoretical approach to study the phase transition phenomena in finite systems. For instance, even the best formulation of the statistical mechanics and thermodynamics of finite systems by Hill [11] is not rigorous while discussing the phase transitions. Exactly solvable models of phase transitions applied to finite systems may provide us with the first principle results unspoiled by the additional simplifying assumptions. Here we present a finite volume extension of the SMM.

To have a more realistic model for finite volumes, we would like to account for the finite size and geometrical shape of the largest fragments, when they are comparable with the system volume. For this we will abandon the arbitrary size of the largest fragment and consider the constrained SMM (CSMM) in which the largest fragment size is explicitly related to the volume V of the system. A similar model, but with the fixed size of the largest fragment, was recently analyzed in Ref. [12].

In this work we will: solve the CSMM analytically at finite volumes using a new powerful method; consider how the first order phase transition develops from the singularities of the SMM isobaric partition [13] in thermodynamic limit; study the finite volume analogs of phases; and discuss the finite size effects for large fragments.

2. Laplace–Fourier transformation

The system states in the SMM are specified by the multiplicity sets $\{n_k\}$ ($n_k = 0, 1, 2, \ldots$) of k-nucleon fragments. The partition function of a single fragment with k nucleons is [1] $V\phi_k(T) = V\,(mTk/2\pi)^{3/2}\; z_k$, where $k = 1, 2, \ldots, A$ (A is the total number of nucleons in the system), V and T are, respectively, the volume and the temperature of the system, m is the nucleon mass. The first two factors on the right-hand side (r.h.s.) of the single fragment partition originate from the non-relativistic thermal motion and the last factor, z_k, represents the intrinsic partition function of the k-nucleon fragment. Therefore, the function $\phi_k(T)$ is a phase space density of the k-nucleon fragment. For $k = 1$ (nucleon) we take $z_1 = 4$ (4 internal spin-isospin states) and for fragments with $k > 1$ we use the expression motivated by the liquid drop model (see details in Ref. [1]): $z_k = \exp(-f_k/T)$, with fragment free energy

$$f_k = -W(T)\; k + \sigma(T)\; k^{2/3} + \left(\tau + \frac{3}{2}\right) T \ln k \,, \tag{1}$$

with $W(T) = W_0 + T^2/\varepsilon_0$. Here $W_0 = 16$ MeV is the bulk binding energy per nucleon. T^2/ε_0 is the contribution of the excited states taken in the Fermi-gas approximation ($\varepsilon_0 = 16$ MeV). $\sigma(T)$ is the temperature dependent surface tension parametrized in the following relation: $\sigma(T) = \sigma_0[(T_{\rm c}^2 - T^2)/(T_{\rm c}^2 + T^2)]^{5/4}$, with $\sigma_0 = 18$ MeV and $T_{\rm c} = 18$ MeV ($\sigma = 0$ at $T \geq T_{\rm c}$). The last contribution in Eq. (1) involves the famous Fisher's term with dimensionless parameter τ.

The canonical partition function (CPF) of nuclear fragments in the SMM has the following form:

$$Z_A^{\rm id}(V,T) = \sum_{\{n_k\}} \left[\prod_{k=1}^{A} \frac{[V\ \phi_k(T)]^{n_k}}{n_k!} \right] \delta(A - \sum_k k n_k)\,. \qquad (2)$$

In Eq. (2) the nuclear fragments are treated as point-like objects. However, these fragments have non-zero proper volumes and they should not overlap in the coordinate space. In the excluded volume (Van der Waals) approximation this is achieved by substituting the total volume V in Eq. (2) by the free (available) volume $V_f \equiv V - b\sum_k k n_k$, where $b = 1/\rho_0$ ($\rho_0 = 0.16$ fm^{-3} is the normal nuclear density). Therefore, the corrected CPF becomes: $Z_A(V,T) = Z_A^{\rm id}(V - bA, T)$. The SMM defined by Eq. (2) was studied numerically in Refs. [5,6]. This is a simplified version of the SMM, *e.g.* the symmetry and Coulomb contributions are neglected. However, its investigation appears to be of principal importance for studies of the liquid–gas phase transition.

The calculation of $Z_A(V,T)$ is difficult due to the constraint $\sum_k k n_k = A$. This difficulty can be partly avoided by evaluating the grand canonical partition (GCP)

$$\mathcal{Z}(V,T,\mu) \;\equiv\; \sum_{A=0}^{\infty} \exp\left(\tfrac{\mu A}{T}\right) Z_A(V,T)\ \Theta(V - bA)\,, \qquad (3)$$

where μ denotes a chemical potential. The calculation of $\mathcal{Z}$ is still rather difficult. The summation over $\{n_k\}$ sets in Z_A cannot be performed analytically because of additional A-dependence in the free volume V_f and the restriction $V_f > 0$. This problem was resolved [7,8] by the Laplace transformation method to the so-called isobaric ensemble [13].

In this work we would like to consider a more strict constraint $\sum_k^{K(V)} k\ n_k = A$, where the size of the largest fragment $K(V) = \alpha V/b$ cannot exceed the total volume of the system (the parameter $\alpha \leq 1$ is introduced for convenience). A similar restriction should be also applied to the upper limit of the product in all partitions $Z_A^{\rm id}(V,T)$, $Z_A(V,T)$ and $\mathcal{Z}(V,T,\mu)$ introduced above (how to deal with the real values of $\alpha V/b$, see later). Then the model with this constraint, the CSMM, cannot be solved by the Laplace transform method, because the volume integrals cannot be evaluated due to a complicated functional V-dependence. However, the CSMM can be solved analytically with the help of the following identity

$$G(V) = \int\limits_{-\infty}^{+\infty} d\xi \int\limits_{-\infty}^{+\infty} \frac{d\eta}{2\pi}\, e^{i\eta(V-\xi)}\, G(\xi)\,, \tag{4}$$

which is based on the Fourier representation of the Dirac δ-function. The representation (4) allows us to decouple the additional volume dependence and reduce it to the exponential one, which can be dealt with by the usual Laplace transformation in the following sequence of steps

$$\begin{aligned}\hat{\mathcal{Z}}(\lambda,T,\mu) &\equiv \int\limits_0^{\infty} dV\, e^{-\lambda V}\, \mathcal{Z}(V,T,\mu) = \int\limits_0^{\infty} dV' \int\limits_{-\infty}^{+\infty} d\xi \int\limits_{-\infty}^{+\infty} \frac{d\eta}{2\pi}\, e^{i\eta(V'-\xi)-\lambda V'} \\ &\times \sum_{\{n_k\}} \left[\prod_{k=1}^{K(\xi)} \frac{1}{n_k!} \left\{ V'\, \phi_k(T)\, e^{\frac{(\mu-(\lambda-i\eta)bT)k}{T}} \right\}^{n_k} \right] \Theta(V') \\ &= \int\limits_0^{\infty} dV' \int\limits_{-\infty}^{+\infty} d\xi \int\limits_{-\infty}^{+\infty} \frac{d\eta}{2\pi}\, e^{i\eta(V'-\xi)-\lambda V' + V'\mathcal{F}(\xi,\lambda-i\eta)}\,. \end{aligned} \tag{5}$$

After changing the integration variable $V \to V' = V - b\sum_k^{K(\xi)} k\ n_k$, the constraint of Θ-function has disappeared. Then all n_k were summed independently leading to the exponential function. Now the integration over V' in Eq. (5) can be straightforwardly done resulting in

$$\hat{\mathcal{Z}}(\lambda,T,\mu) = \int\limits_{-\infty}^{+\infty} d\xi \int\limits_{-\infty}^{+\infty} \frac{d\eta}{2\pi}\, \frac{e^{-i\eta\xi}}{\lambda - i\eta\ -\ \mathcal{F}(\xi,\lambda-i\eta)}\,, \tag{6}$$

where the function $\mathcal{F}(\xi,\tilde{\lambda})$ is defined as follows

$$\begin{aligned}\mathcal{F}(\xi,\tilde{\lambda}) &= \sum_{k=1}^{K(\xi)} \phi_k(T)\ e^{\frac{(\mu-\tilde{\lambda}bT)k}{T}} \\ &= \left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} \left[z_1\ e^{\frac{\mu-\tilde{\lambda}bT}{T}} + \sum_{k=2}^{K(\xi)} k^{-\tau} e^{\frac{(\mu+W-\tilde{\lambda}bT)k-\sigma k^{2/3}}{T}} \right]. \end{aligned} \tag{7}$$

As usual, in order to find the GCP by the inverse Laplace transformation, it is necessary to study the structure of singularities of the isobaric partition (7).

3. Isobaric partition singularities

The isobaric partition (7) of the CSMM is, of course, more complicated than its SMM analog [7, 8] because for finite volumes the structure of singularities in the CSMM is much richer than in the SMM, and they match in the limit $V \rightarrow \infty$ only. To see this let us first make the inverse Laplace transform:

$$\begin{aligned}
\mathcal{Z}(V,T,\mu) \;&=\; \int\limits_{\chi-i\infty}^{\chi+i\infty} \frac{d\lambda}{2\pi i}\ \hat{\mathcal{Z}}(\lambda,T,\mu)\ e^{\lambda\,V} \\
&=\; \int\limits_{-\infty}^{+\infty} d\xi \int\limits_{-\infty}^{+\infty} \frac{d\eta}{2\pi} \int\limits_{\chi-i\infty}^{\chi+i\infty} \frac{d\lambda}{2\pi i}\ \frac{e^{\lambda\,V - i\eta\xi}}{\lambda - i\eta \;-\; \mathcal{F}(\xi,\lambda-i\eta)} \\
&=\; \int\limits_{-\infty}^{+\infty} d\xi \int\limits_{-\infty}^{+\infty} \frac{d\eta}{2\pi}\ e^{i\eta(V-\xi)} \sum_{\{\lambda_n\}} e^{\lambda_n\,V} \left[1 - \frac{\partial\mathcal{F}(\xi,\lambda_n)}{\partial\lambda_n}\right]^{-1}, \qquad (8)
\end{aligned}$$

where the contour λ-integral is reduced to the sum over the residues of all singular points $\lambda = \lambda_n + i\eta$ with $n = 1, 2, \ldots$, since this contour in the complex λ-plane obeys the inequality $\chi > \max(\mathrm{Re}\{\lambda_n\})$. Now both remaining integrations in (8) can be done, and the GCP becomes

$$\mathcal{Z}(V,T,\mu) \;=\; \sum_{\{\lambda_n\}} e^{\lambda_n\,V} \left[1 - \frac{\partial\mathcal{F}(V,\lambda_n)}{\partial\lambda_n}\right]^{-1}, \qquad (9)$$

i.e. the double integral in (8) simply reduces to the substitution $\xi \rightarrow V$ in the sum over singularities. This is a remarkable result which can be formulated as the following theorem: *if the Laplace–Fourier image of the excluded volume GCP exists, then for any additional V-dependence of $\mathcal{F}(V,\lambda_n)$ or $\phi_k(T)$ the GCP can be identically represented by Eq. (9).*

The simple poles in (8) are defined by the equation

$$\lambda_n \;=\; \mathcal{F}(V,\lambda_n)\,. \qquad (10)$$

In contrast to the usual SMM [7, 8] the singularities λ_n are *(i)* functions of volume V, and *(ii)* they can have a non-zero imaginary part, but in this case there exist pairs of complex conjugate roots of (10) because the GCP is real.

Introducing the real R_n and imaginary I_n parts of $\lambda_n = R_n + iI_n$, we can rewrite Eq. (10) as a system of coupled transcendental equations

$$R_n = \sum_{k=1}^{K(V)} \tilde{\phi}_k(T)\; e^{\frac{\mathrm{Re}(\nu)\,k}{T}} \cos(I_n b k)\,, \tag{11}$$

$$I_n = -\sum_{k=1}^{K(V)} \tilde{\phi}_k(T) e^{\frac{\mathrm{Re}(\nu)\,k}{T}} \sin(I_n b k)\,, \tag{12}$$

where we have introduced the effective chemical potential $\nu = \mu + W(T) - \lambda_n b T$, and the reduced distributions $\tilde{\phi}_1(T) = z_1 \exp(-W(T)/T)$ and $\tilde{\phi}_{k>1}(T) = k^{-\tau} \exp(-\sigma(T)\; k^{2/3}/T)$ for convenience.

Consider the real root $(R_1 > 0, I_1 = 0)$, first. For $I_n = I_1 = 0$ the real root R_1 exists for any T and μ. Comparing R_1 with the expression for vapor pressure of the analytical SMM solution [7,8] shows that TR_1 is a constrained grand canonical pressure of the gas. Eq. (12) shows that for $I_{n>1} \neq 0$ the inequality $\cos(I_n b k) \leq 1$ never become the equality for all k-values simultaneously. Then from Eq. (11) one obtains $(n > 1)$

$$R_n < \sum_{k=1}^{K(V)} \tilde{\phi}_k(T)\; e^{\frac{\mathrm{Re}(\nu)\,k}{T}} \quad \Rightarrow \quad R_n < R_1\,, \tag{13}$$

where the second inequality (13) immediately follows from the first one. In other words, the gas singularity is always the rightmost one. This fact plays a decisive role in the thermodynamic limit $V \to \infty$.

It is instructive to use the effective chemical potential $\nu(\lambda)$ as an independent variable instead of μ. Like in the usual SMM [7,8], for infinite volume the effective chemical potential must be real and non-positive, $\nu \leq 0$, because otherwise the function $\mathcal{F}(V, \lambda)$ (7) diverges and the formal manipulations in (5) to establish (6) cannot be used. The limiting value $\nu = 0$ defines the liquid phase singularity of the isobaric partition which gives the liquid pressure $p_l(T, \mu) = R_1 T = (\mu + W(T))/b$ [7,8]. But for finite volumes and finite $K(V)$ the effective chemical potential can be complex (with either sign for its real part) and its value defines the number and position of the imaginary roots $\{\lambda_{n>1}\}$ in the complex plane. Positive and negative values of the effective chemical potential for finite systems were considered [14] within the Fisher droplet model, but, to our knowledge, its complex values have never been discussed. From the definition of the effective chemical potential $\nu(\lambda)$ it is evident that its complex values for finite systems exist only because of the excluded volume interaction, which is not taken into account in the Fisher droplet model [10].

Consider the natural values of $K(V)$, first. As it is seen from Fig. 1, the r.h.s. of (12) is the amplitude and frequency modulated sine-like function of dimensionless parameter $I_n\,b$. Therefore, depending on T and $\mathrm{Re}(\nu)$ values, there may exist either zero, or finite, or infinite number of complex roots $\{\lambda_{n>1}\}$. In Fig. 1 we show a special case which corresponds to exactly three roots of Eq. (10) for each value of $K(V)$: the real root ($I_1 = 0$) and two complex conjugate roots ($\pm I_2$). Since the r.h.s. of (12) is monotonously increasing function of $\mathrm{Re}(\nu)$, when the former is positive, then it is possible to map the $T - \mathrm{Re}(\nu)$ plane into regions of a fixed number of roots of Eq. (10).

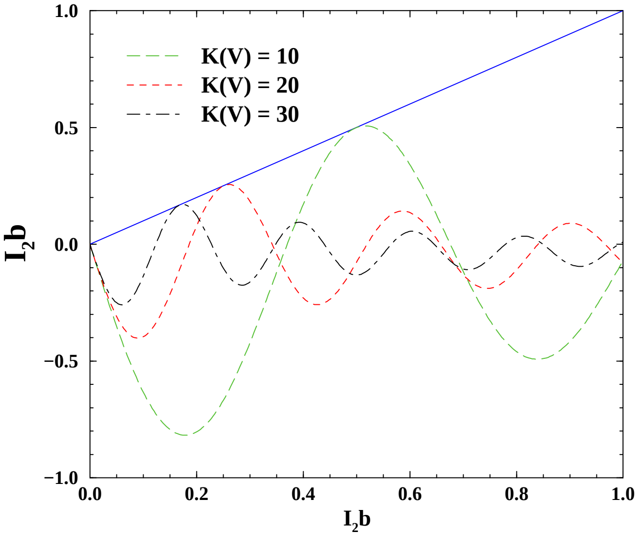

Fig. 1. A graphical solution of Eq. (12) for $T = 10$ MeV and $\tau = 1.825$. The l.h.s. (straight line) and r.h.s. of Eq. (12) (all dashed curves) are shown as the function of dimensionless parameter $I_2\,b$ for the three values of the largest fragment size $K(V)$. The intersection point at (0; 0) corresponds to a real root of Eq. (10). Each tangent point with the straight line generates two complex roots of (10).

Each curve in Fig. 2 divides the $T - \mathrm{Re}(\nu)$ plane into three parts: for $\mathrm{Re}(\nu)$-values below the curve there is only a real root (gaseous phase), for points on the curve there exist three roots, and above the curve there are five or more roots of Eq. (10) which represent the finite volume analog of the mixed phase (for more details see next section).

A similar situation occurs for the real values of $K(V)$. In this case all sums in Eqs. (10)–(13) should be expressed via the Euler–MacLaurin formula

$$\sum_{k=1}^{K(V)} f_k = f(1) + \int\limits_2^{K(V)} dk\, f(k) + \tfrac{f(K(V))+f(2)}{2} + \mathit{\Delta}_f(K(V)) - \mathit{\Delta}_f(2)\,, \quad (14)$$

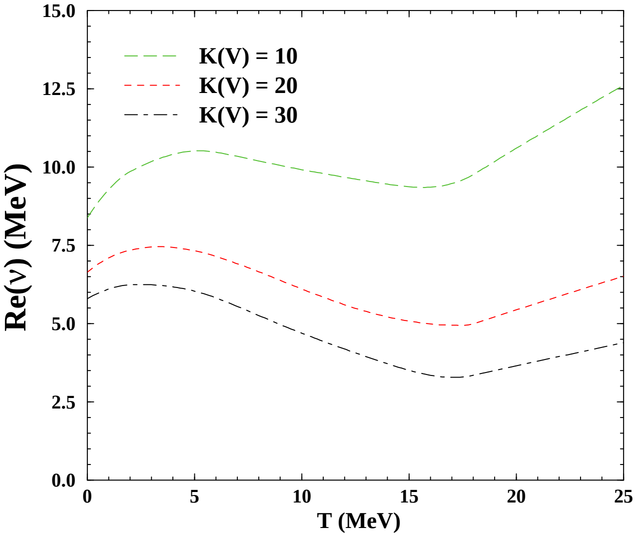

Fig. 2. Each curve separates the $T-\mathrm{Re}(\nu)$ region of one real root of Eq. (10) (below the curve), three complex roots (at the curve) and five and more roots (above the curve) for three values of $\alpha V/b$ and the same parameters as in Fig. 1.

where

$$\Delta_f(K) = \sum_{n=1} \frac{B_{2n}}{(2n)!}\, \frac{d^{2n} f(x)}{d\,x^{2n}}\bigg|_{x=K} \,. \tag{15}$$

Here B_{2n} are the Bernoulli numbers. The representation (15) allows one to study the effect of finite volume (FV) on the GCP (9).

4. Finite volume thermodynamics

In the CSMM there are two different ways of how the finite volume affects thermodynamical functions: it can be shown that for finite V and μ there is always a finite number of simple poles with $R_n > 0$ in (9) and all of them contribute to thermodynamic quantities; also the parameter $\alpha < 1$ describes a difference between the geometrical shape of the volume under consideration and that one of the largest fragment (assumed to be spherical). To see this, let us study the mechanical pressure which corresponds to the GCP (9):

$$\begin{aligned} p(T,\mu,V) &= T\, \frac{\partial \ln \mathcal{Z}(V,T,\mu)}{\partial V} \\ &= \tfrac{T}{\mathcal{Z}(V,T,\mu)} \sum_{\{\lambda_n\}} \left[\frac{\lambda_n\, e^{\lambda_n V}}{1 - \frac{\partial \mathcal{F}(V,\lambda_n)}{\partial \lambda_n}} + \frac{e^{\lambda_n V}}{\left[1 - \frac{\partial \mathcal{F}(V,\lambda_n)}{\partial \lambda_n}\right]^2} \right. \end{aligned} \tag{16}$$

$$
\begin{aligned}
&\times \Bigg\{ b^2 \, \frac{\partial \lambda_n}{\partial V} \sum_{k=1}^{K(V)} \tilde{\phi}_k(T)\, k^2 \; e^{\frac{\nu\, k}{T}} + \tilde{\phi}_{K(V)}(T) e^{\frac{\nu\, K(V)}{T}} K(V) \\
&\times \left[(1-\alpha) + \frac{\nu}{T} \left(\tfrac{1}{2} - \alpha \right) \right] + o(K(V)) \Bigg\} \Bigg] \,,
\end{aligned}
$$

where we give the main term for each λ_n and leading FV corrections explicitly for $\mathrm{Re}(\nu)/T < 1$, whereas $o(K(V))$ accumulates the higher order corrections due to the Euler–MacLaurin Eq. (15). In evaluation of (16) we used an explicit representation of the derivative $\partial \lambda_n / \partial V$ which can be found from Eqs. (10) and (15). The first term in the r.h.s. of (16) describes the partial pressure generated by the simple pole λ_n weighted with the "probability" $e^{\lambda_n V} / \mathcal{Z}(V,T,\mu)$, whereas the second and third terms appear due to the volume dependence of $K(V)$. Note that, instead of the FV corrections, the usage of natural values for $K(V)$ would generate the artificial delta-function terms in (16) for the volume derivatives.

As one can see from (16), for finite volumes the corrections can give a non-negligible contribution to the pressure because in this case $\mathrm{Re}(\nu) > 0$ can be positive. The real parts of the partial pressures $T\lambda_n$ may have either sign. Therefore, according to (13) the positive pressures $TR_{n>1} > 0$ are metastable and the negative ones $TR_{n>1} < 0$ are mechanically unstable compared to the gas pressure TR_1. The pair of complex conjugate roots with the same value of $TR_{n>1}$ corresponds to a formation and decay of those states in thermodynamical system at finite volumes. Taking all this into account, we conclude that the partial pressures $T\lambda_{n>1}$ contributions to the total pressure (16) correspond to the finite volume analog of the mixed phase which is metastable or unstable for finite values of μ.

Using Eqs. (11) and (12), one can show that the finite volume analog of the liquid phase corresponds to the densest possible state, which is defined by the inequality $\mathrm{Re}(\nu) \gg T$ or, equivalently, by $\mu \gg T$. Indeed, for $\mathrm{Re}(\nu) \gg T$ the leading contribution to the r.h.s. of (12) corresponds to the harmonic with $k = K(V)$, and, consequently, an exponentially large amplitude of this term can be only compensated by a vanishing value of $\sin\left(I_{n>1}\, bK(V)\right)$, *i.e.* $I_{n>1}\, bK(V) \approx \pi(n-1)$. For this result Eq. (11) gives

$$
R_n \;\approx\; \frac{p_l(T,\mu)}{T} - \frac{1}{K(V) b} \ln \left| \frac{R_n}{\tilde{\phi}_K(T)} \right| \to \frac{p_l(T,\mu)}{T} \tag{17}
$$

for $\mu \gg T$ and $K(V) \to \infty$. Since the partial pressure TR_n of (17) corresponds to a single fragment of largest size, we should identify it as a liquid phase for finite volumes.

When V increases, the number of simple poles with $R_n > 0$ in (8) also increases and imaginary part of the closest to the real λ-axis poles becomes very small. This can be seen for $\mathrm{Re}(\nu) \gg T$ from the above derivation of Eq. (17), for which we showed that $I_{n>1} \approx \pi(n-1)/bK(V) \to 0$. For infinite volume the infinite number of simple poles moves toward the real λ-axis to the vicinity of liquid phase singularity $p_l(T,\mu)/T$ and, thus, generates an essential singularity of function $\mathcal{F}(V, p_l/T)$ in (7). In this case the contribution of any remote poles from the real λ-axis to the GCP vanishes. Then it can be shown that the FV corrections in (16) become negligible because of the inequality $\mathrm{Re}(\nu) \leq 0$, and, consequently, the reduced distribution of largest fragment $\tilde{\phi}_{K(V)}(T) = K(V)^{-\tau} \exp(-\sigma(T)\ K(V)^{2/3}/T)$ and the derivative $\partial\lambda_n/\partial V$ vanish for all T-values, and we obtain the usual SMM solution [7, 8]. Its thermodynamics is governed by the farthest right singularity in the complex λ-plane.

5. Conclusions

In this work we presented an exact analytical solution of the grand canonical formulation of the CSMM for finite volumes. This was achieved with the help of a powerful mathematical method, the Laplace–Fourier transform, developed here. Using this method we were able to solve exactly three statistical models of surface partition [15]. The Laplace–Fourier transform allows one to identically rewrite the grand canonical partition in terms of the simple poles λ_n of the isobaric partition and study their behavior when the system volume increases to infinity.

As we showed, for finite volumes the gas singularity λ_1 is the only real singularity and the rightmost one, *i.e.* $\lambda_1 > \mathrm{Re}(\lambda_{n>1})$. The latter inequality means that the gaseous state has the largest pressure $T\lambda_1$ for given T and μ, and, consequently, the gaseous phase is mechanically stable compared to other $\lambda_{n>1}$ states of the same GCP. Therefore, the $\lambda_{n>1}$ singularities describe mechanically metastable states for $\mathrm{Re}(\lambda_{n>1}) > 0$ and unstable states for $\mathrm{Re}(\lambda_{n>1}) \leq 0$.

We proved that the complex $\lambda_{n>1}$ states consist of pairs of complex conjugate numbers, which automatically generate only the real values of the GCP. The complex $\lambda_{n>1}$ values lead to complex values of the effective chemical potential ν for finite volumes. To the best of our knowledge such a possibility has never been discussed in the literature on nuclear multifragmentation.

A detailed analysis of the isobaric partition singularities in the $T-\mathrm{Re}(\nu)$ plane allowed us to define the finite volume analogs of phases and study the behavior of these singularities in the limit $V \to \infty$. Such an analysis opens a possibility to study rigorously the nuclear liquid–gas phase transition directly from the finite volume partition. This may help to extract the phase diagram of the nuclear liquid–gas phase transition from the experiments on finite systems (nuclei) with more confidence.

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