

THREE-LOOP ELECTROWEAK CORRECTIONS TO THE ρ -PARAMETER, $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ AND THE W -BOSON MASS IN THE LARGE HIGGS MASS LIMIT*

RADJA BOUGHEZAL

Institut für Theoretische Physik und Astrophysik, Universität Würzburg
Am Hubland, D-97074 Würzburg, Germany

BAS TAUSK

Fakultät für Mathematik und Physik, Albert-Ludwigs-Universität Freiburg
D-79104 Freiburg, Germany

(Received November 11, 2005)

A review of the calculation of the leading three-loop electroweak corrections to the ρ -parameter, $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ and the W -boson mass is presented. The heavy Higgs mass expansion and the renormalization are discussed.

PACS numbers: 12.15.Lk, 14.80.Bn

1. Introduction

The Electroweak Standard Model (EWSM) is in good agreement with all experimentally known phenomena of electroweak origin, with the exception of the evidence of neutrino mixing. The only ingredient predicted by this model that has not been seen yet is the Higgs particle. The direct search at LEP provided us with a lower limit on the mass of the Standard Model Higgs-boson m_H excluding the region below 114.4 GeV [1]. The global fit of the experimental data to the Standard Model, based on confronting theoretical predictions for electroweak observables with their experimental values, favours a light Higgs-boson ($m_H \leq 186$ GeV one-sided 95% Confidence Level (CL) [2]).

* Presented at the XXIX International Conference of Theoretical Physics “Matter to the Deepest”, Ustroń, Poland, September 8–14, 2005.

However, there is a discrepancy of about 3.2σ between the two most precise measurements of the electroweak mixing angle $\sin^2 \theta_{\text{eff}}^{\text{lept}}$, which is used to set stringent bounds on the Higgs-boson mass. The measurement based on the leptonic asymmetry parameter A_l at SLD, together with the W -boson mass measurement from Tevatron and LEP, point to a light Higgs-boson with a mass slightly below the lower bound from the direct searches. On the other hand, the measurement based on the b -quark forward-backward asymmetry $A_{\text{FB}}^{0,b}$ at LEP favours a relatively heavy Higgs-boson with a mass around 500 GeV [2]. Since the center of mass energies of present colliders do not allow to probe the region of a heavy Higgs-boson, the sensitivity of radiative corrections to low energy electroweak observables to a heavy-Higgs boson mass becomes an important tool in setting limits on m_H [2].

As the Yukawa couplings are very small, the Higgs dependent effects are limited to corrections to the vector-boson propagators. They give rise to shifts, *e.g.*, in the ρ -parameter, the W -boson mass, and in the effective leptonic weak mixing angle $\sin^2 \theta_{\text{eff}}^{\text{lept}}$. These shifts are often parametrized by S , T and U or ϵ_1 , ϵ_2 , ϵ_3 [3]. For a light Higgs-boson, the Higgs mass dependence of theoretical predictions is mainly due to one-loop radiative corrections to the gauge-boson propagators which grow logarithmically with m_H [4]. However, because the Higgs self-interaction is proportional to m_H^2 , higher order radiative corrections which grow like powers of m_H could become important if the Higgs-boson mass is much larger than the Z -boson mass.

At the two-loop level, the leading corrections are proportional to m_H^2 , but the numerical coefficient of these terms turns out to be very small [5–7], and therefore they are not important for m_H less than a few TeV. However it has been suggested that the smallness of the two-loop corrections may be somewhat accidental [8], therefore important effects might first appear only at the three-loop level. The situation was only clarified recently by an explicit leading three-loop calculation for three precision variables, the ρ -parameter, the electroweak mixing angle $\sin^2 \theta_{\text{eff}}^{\text{lept}}$, and the W -boson mass M_W [9].

The electroweak ρ -parameter is a measure of the relative strengths of neutral and charged-current interactions in four-fermion processes at zero momentum transfer. In the Standard Model, at tree level, it is related to the W and Z boson masses by:

$$\rho = \frac{M_W^2}{c_W^2 M_Z^2} = 1, \quad (1)$$

where $c_W = \cos \theta_W$. Including higher order corrections modifies this relation into

$$\rho = \frac{1}{1 - \Delta\rho}. \quad (2)$$

Here $\Delta\rho$ parametrises all higher loop corrections which are sensitive to the existence of a heavy Higgs particle. The leading one- and two-loop corrections, which grow logarithmically and quadratically with m_H respectively, have been calculated [4, 6].

The sine of the effective leptonic weak mixing angle $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ is defined in terms of the couplings of the Z -boson to leptons. The complete electroweak fermionic corrections to $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ at the two-loop level are known [10]. Recently, the Higgs-dependent electroweak two-loop bosonic contributions to this observable have been completed [11]. For the W -boson mass, both the fermionic and the bosonic corrections have been calculated at the two-loop level [12].

In this contribution we review the leading three-loop bosonic corrections to the ρ -parameter, $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ and M_W [9], which grow like m_H^4 in the large Higgs mass limit.

The calculation is organized in such a way that the leading contributions come only from self energy corrections to the gauge boson propagators, often referred to as *oblique* corrections in the literature, and not from vertex or box diagrams. This is achieved by our choice of renormalization scheme. We should mention at this point that the renormalization is performed up to the two-loop level only, removing sub-divergences from the gauge boson self-energies, but not yet the overall divergences. This is due to the fact that the three-loop counter terms cancel in $\Delta^{(3)}\rho$, $\Delta^{(3)}\sin^2 \theta_{\text{eff}}^{\text{lept}}$ and $\Delta^{(3)}M_W$. The renormalized self-energies are then related to physical observables through the formalism of S , T and U parameters, which was developed by Peskin and Takeuchi to describe the effect of heavy particles on electroweak precision observables [13]. They are defined in terms of the transverse gauge boson self-energies at zero momentum transfer and their first derivative w.r.t their momentum. These self-energies are not observable individually and may still contain ultraviolet divergences. However, the three combinations:

$$S \equiv \frac{4s_W^2 c_W^2}{\alpha} \left(\Sigma_T^{\prime ZZ} - \frac{c_W^2 - s_W^2}{c_W s_W} \Sigma_T^{\prime AZ} - \Sigma_T^{\prime AA} \right), \quad (3)$$

$$T \equiv \frac{1}{\alpha M_W^2} (c_W^2 \Sigma_T^{ZZ} - \Sigma_T^{WW}), \quad (4)$$

$$U \equiv \frac{4s_W^2}{\alpha} (\Sigma_T^{\prime WW} - c_W^2 \Sigma_T^{\prime ZZ} - 2c_W s_W \Sigma_T^{\prime AZ} - s_W^2 \Sigma_T^{\prime AA}), \quad (5)$$

where $s_W = \sin\theta_W$ and α is the fine structure constant, are finite and observable. Using this formalism, the shifts to ρ , $\sin^2\theta_{\text{eff}}^{\text{lept}}$ and M_W are parametrised as:

$$\begin{aligned}\Delta\rho &= \alpha T, \\ \Delta M_W &= \frac{\alpha M_W}{2(c_W^2 - s_W^2)} \left(-\frac{1}{2}S + c_W^2 T + \frac{c_W^2 - s_W^2}{4s_W^2} U \right), \\ \Delta \sin^2\theta_{\text{eff}}^{\text{lept}} &= \frac{\alpha}{c_W^2 - s_W^2} \left(\frac{1}{4}S - s_W^2 c_W^2 T \right).\end{aligned}\tag{6}$$

2. Calculation and renormalization

The bare gauge boson self-energies are decomposed into transversal and longitudinal components according to

$$\Sigma_{\mu\nu}^X(p) = \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \Sigma_{\text{T}}^X(p^2) + \frac{p_\mu p_\nu}{p^2} \Sigma_{\text{L}}^X(p^2),\tag{7}$$

where $X = AA, AZ, ZZ, WW$. The scalar functions $\Sigma_{\text{T}}^X(p^2)$ are then expanded in a Taylor series in their momentum p up to order p^2 . Higher order terms in p^2 are suppressed by powers of the heavy Higgs boson mass. For the T parameter, only the constant term of this expansion (*i.e.* $p^2 = 0$) is required. After this step, we are left with vacuum integrals which, in general, depend on three different scales: m_H , M_W and M_Z . In order to separate the dependence of these integrals on the large scale m_H from the small scales M_W and M_Z and extract the leading m_H terms, we perform an asymptotic large mass expansion following the method of the expansion by regions [14]. The expansion is constructed by considering different regions in loop momentum space, distinguished by the set of propagator momenta which are large or small in those regions. In each region, a Taylor expansion of all propagators in the small masses and in the small momenta of that region is performed. Typically, the expansion generates extra scalar products of loop momenta in the numerator and higher powers of denominators, as compared to the original diagrams. The resulting expression is then integrated over the whole loop momentum space. For the three-loop vacuum topology shown in Fig. 1, there are 15 regions in loop momentum space to consider (see [9] for more details). The distribution of large and small masses in each diagram decides the number of regions that contribute to the corresponding integral up to the leading terms we are interested in. For example, only four regions give a non vanishing contribution to the diagram shown in Fig. 2, up to m_H^4 order.

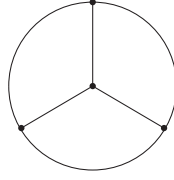


Fig. 1. The vacuum three-loop six-propagators topology.

They correspond to:

- the all internal momenta large region. In this case one expands the propagators in M_ϕ only, the result is a one-scale three-loop integral.
- the region where k_1 is small. Here we expand in M_ϕ and k_1 , which leads to the product of a one-loop integral depending on M_ϕ times a two-loop integral that depends on m_H .
- the region where $k_1 + k_2$ is small. After expanding in M_ϕ and $k_1 + k_2$ we get, similarly to the previous case, a product of one- and two-loop integrals.
- the region where $k_1 + k_2 + k_3$ is small. Here again we expand in M_ϕ and $k_1 + k_2 + k_3$ and get a product of one- and two-loop integrals.

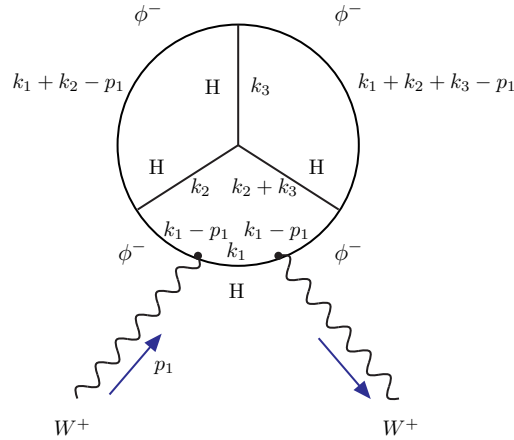


Fig. 2. An example of a three-loop W self-energy diagram on which the expansion by regions was applied. H and ϕ^- refer to the Higgs and Goldstone fields, respectively.

All the other regions produce either scaleless integrals, which are zero in dimensional regularization, or terms which do not have enough powers of m_H to contribute to the leading order.

From the expansion in the 15 regions we get two kinds of integrals:

- factorizable diagrams, which are products of one-(two)-loop vacuum integrals depending on m_H and two-(one)-loop vacuum integrals depending on M_W and M_Z .
- non-factorizable three-loop vacuum integrals, which are either single-scaled depending on m_H , or double-scaled depending on M_W and M_Z .

The Integration-By-Parts (IBP) method [15] is then used to reduce all the vacuum integrals to the master ones. We classify the single-scaled three-loop non-factorizable integrals into ten different kinds, depending on the distribution of masses in the propagators. Their reduction to a small set of master integrals [17, 18] was done in two ways. On the one hand, reduction formulae based on the Integration By Part identities have been constructed. On the other hand the Automatic Integral Reduction package AIR [16] was used as a cross check. The latter was also used to reduce the double-scaled three-loop non-factorizable integrals. Explicit formulae for their master integrals are not needed, as they all canceled once we have summed over all diagrams.

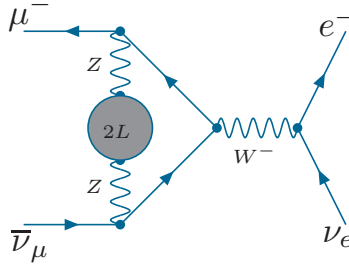


Fig. 3. An example of a three-loop vertex diagram which would contribute to the leading terms if $\Sigma_{ZZ}^{(2L)} \sim m_H^4$.

The longitudinal parts of the gauge boson self-energies are related to the self-energies of the Goldstones and mixings between gauge bosons and Goldstones by a set of Ward identities. We have verified that these Ward identities are satisfied by the full (*i.e.* including tadpoles) unrenormalised self-energies up to order p^2 .

As we have mentioned earlier, leaving out vertex and box contributions requires a proper way of renormalizing. Our renormalization conditions are fixed in such a way that the renormalization removes all the terms of

order m_H^2 and m_H^4 from the one- and two-loop gauge boson self-energies, the charged and neutral Goldstone self-energies and the mixings between gauge bosons and Goldstones. This ensures that no two- or three-loop vertex or box graphs containing such self-energies as subgraphs can give corrections that grow like m_H^2 or m_H^4 in the large Higgs mass limit (see Fig. 3).

As a check on the renormalization, we have verified that the renormalized longitudinal photon self-energy and photon- Z mixing are zero.

3. Results and conclusion

The shifts to the electroweak precision observables ρ , $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ and M_W relative to their tree level values, expressed in terms of α , G_F and M_Z , are given by

$$\rho = \frac{1}{1 - \Delta\rho}, \quad (8)$$

$$\sin^2 \theta_{\text{eff}}^{\text{lept}} = \Delta \sin^2 \theta_{\text{eff}}^{\text{lept}} + \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{\pi\alpha}{\sqrt{2}G_F M_Z^2}}, \quad (9)$$

$$M_W = \Delta M_W + M_Z \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\pi\alpha}{\sqrt{2}G_F M_Z^2}}}, \quad (10)$$

with $\Delta\rho$, $\Delta \sin^2 \theta_{\text{eff}}^{\text{lept}}$ and ΔM_W defined in (6) in terms of the parameters S , T and U . While the U -parameter vanishes in the approximation where only quartic terms or higher powers in m_H are kept at the three-loop level, the S - and the T -parameters give the following contributions

$$S^{(3)} = \frac{1}{4\pi} \left(\frac{g^2}{16\pi^2} \right)^2 \frac{m_H^4}{M_W^4} (1.1105), \quad (11)$$

$$T^{(3)} = \frac{1}{4\pi c_W^2} \left(\frac{g^2}{16\pi^2} \right)^2 \frac{m_H^4}{M_W^4} (-1.7282). \quad (12)$$

Using $g^2 = e^2/s_W^2 = 4\pi\alpha/s_W^2$ for the weak coupling constant, with $\alpha = 1/137$ and $s_W^2 = 0.23$, the shifts to ρ , $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ and M_W are

$$\Delta^{(3)}\rho = -8.3 \times 10^{-9} \times m_H^4/M_W^4, \quad (13)$$

$$\Delta^{(3)}\sin^2 \theta_{\text{eff}}^{\text{lept}} = 4.6 \times 10^{-9} \times m_H^4/M_W^4, \quad (14)$$

$$\Delta^{(3)}M_W = -6.3 \times 10^{-4} \text{MeV} \times m_H^4/M_W^4. \quad (15)$$

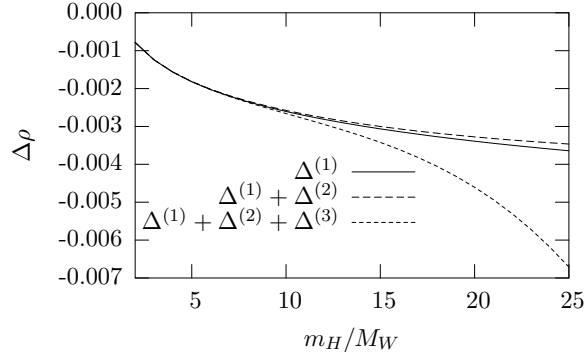


Fig. 4. One-, two- and three-loop shifts in ρ as a function of m_H/M_W .

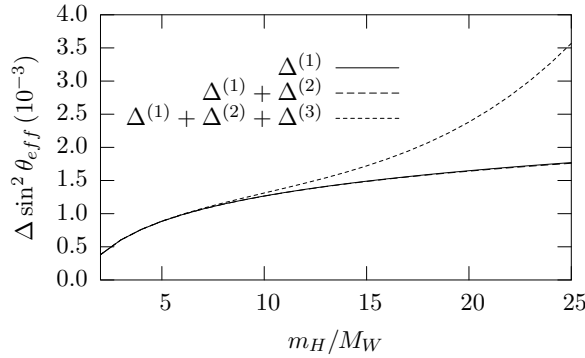


Fig. 5. Shifts in $\sin^2 \theta_{\text{eff}}^{\text{lept}}$ as a function of m_H/M_W .

The Higgs mass dependence of the ρ -parameter and $\Delta \sin^2 \theta_{\text{eff}}^{\text{lept}}$ is shown in Figs. 4 and 5. It turns out that the sign of the leading three-loop corrections to $\Delta \rho$, $\Delta \sin^2 \theta_{\text{eff}}^{\text{lept}}$ and ΔM_W is the same as the sign of the one-loop contributions.

The original question that motivated these calculations was, whether inclusion of the three-loop corrections with strong interactions could lead to an effect mimicking the one-loop effects of a light Higgs boson. The result of the investigations shows that this is highly unlikely. As the signs of the three-loop corrections are the same as the ones of the one-loop corrections, with increasing Higgs mass, the three-loop terms only make the effects grow faster, instead of partially cancelling the one-loop corrections. Therefore the presence of a strongly interacting heavy Higgs-sector appears to be extremely unlikely, and the electroweak precision data can indeed be taken as a strong indication for a light Higgs boson sector.

R. B. would like to thank the organizers of the conference for the pleasant atmosphere. This work was supported by the Sofja Kovalevskaja Award of the Alexander von Humboldt Foundation sponsored by the German Federal Ministry of Education and Research and the DFG within the project “(Nicht)-perturbative Quantenfeldtheorie”.

REFERENCES

- [1] R. Barate *et al.* [ALEPH Collaboration], *Phys. Lett.* **B565**, 61 (2003) [[hep-ex/0306033](#)].
- [2] The LEP Collaborations ALEPH, DELPHI, L3 and OPAL, the LEP Electroweak Working Group, the SLD Electroweak Group and the SLD Heavy Flavor Group, [hep-ex/0509008](#).
- [3] S. Eidelman *et al.* *Particle Data Group*, *Phys. Lett.* **B592**, 1 (2004).
- [4] M.J.G. Veltman, *Acta Phys. Pol.* **B8**, 475 (1977).
- [5] J.J. van der Bij, *Nucl. Phys.* **B248**, 141 (1984); J.J. van der Bij, *Nucl. Phys.* **B267**, 557 (1986).
- [6] J. van der Bij, M.J.G. Veltman, *Nucl. Phys.* **B231**, 205 (1984).
- [7] R. Barbieri, P. Ciafaloni, A. Strumia, *Phys. Lett.* **B317**, 381 (1993).
- [8] R. Akhoury, J. J. van der Bij, H. Wang, *Eur. Phys. J.* **C20**, 497 (2001) [[hep-ph/0010187](#)].
- [9] R. Boughezal, J.B. Tausk, J.J. van der Bij, *Nucl. Phys.* **B713**, 278 (2005) [[hep-ph/0410216](#)]; R. Boughezal, J.B. Tausk, J.J. van der Bij, *Nucl. Phys.* **B725**, 3 (2005) [[hep-ph/0504092](#)].
- [10] M. Awramik, M. Czakon, A. Freitas, G. Weiglein, *Phys. Rev. Lett.* **93**, 201805 (2004) [[hep-ph/0407317](#)]; W. Hollik, U. Meier, S. Uccirati, [hep-ph/0507158](#).
- [11] W. Hollik, U. Meier, S. Uccirati, [hep-ph/0509302](#).
- [12] M. Awramik, M. Czakon, A. Freitas, G. Weiglein, *Phys. Rev.* **D69**, 053006 (2004) [[hep-ph/0311148](#)].
- [13] M.E. Peskin, T. Takeuchi, *Phys. Rev. Lett.* **65**, 964 (1990); M.E. Peskin, T. Takeuchi, *Phys. Rev.* **D46**, 381 (1992).
- [14] V. A. Smirnov, *Applied asymptotic expansions in momenta and masses*, Springer-Verlag, Berlin 2002.
- [15] F. V. Tkachov, *Phys. Lett.* **B100**, 65 (1981); K.G. Chetyrkin, F.V. Tkachov, *Nucl. Phys.* **B192**, 159 (1981).
- [16] C. Anastasiou, A. Lazopoulos, *J. High Energy Phys.* **0407**, 046 (2004) [[hep-ph/0404258](#)].
- [17] D. J. Broadhurst, *Eur. Phys. J.* **C8**, 311 (1999) [[hep-th/9803091](#)].
- [18] J. Fleischer, M. Y. Kalmykov, *Phys. Lett.* **B470**, 168 (1999) [hep-ph/9910223](#).