LATTICE COMPUTATION OF THE STRANGE QUARK MASS IN QCD*

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We present a determination of the strange quark mass using lattice QCD. Particular focus is put on the definition and renormalization of the mass. The latter is done non-perturbatively, using a recursive finite-size scaling technique. The hadronic regime of QCD, where the kaon mass is used as input of the calculation, is connected with the perturbative regime, where the strange quark mass can be translated into the $\overline{\text{MS}}$ scheme. A summary plot of the present lattice computations using dynamical (sea) quarks is included.

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1. Introduction

Quantum chromodynamics (QCD) as the theory of strong interactions between the $N_f = 3$ light quark flavours $q = u, d, s$ (we treat the c, b, t quarks as infinitely heavy) is described through the Lagrange density

$$\mathcal{L}_{\text{QCD}}(g_0, m_q) = -\frac{1}{2g_0^2}\text{Tr}\{F_{\mu\nu}F_{\mu\nu}\} + \sum_q \bar{q}\left(\gamma_\mu(\partial_\mu + A_\mu) + m_q\right)q \,. (1.1)$$

The bare gauge coupling g_0 and bare quark masses m_q are the free parameters of QCD. In order to make predictions, we need after a suitable regularization, *e.g.* on a Euclidean space-time lattice, to fix the free parameters through a set of $N_f + 1$ physical quantities from experiment. For example,

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we can take as experimental input hadronic quantities such as

$$\overbrace{\begin{bmatrix} F_\pi \\ m_\pi \\ m_K \end{bmatrix}}^{\text{Experiment}} \qquad \mathcal{L}_{\text{QCD}}(g_0, m_q) \atop \Longleftrightarrow \qquad \overbrace{\begin{bmatrix} \Lambda_{\text{QCD}} \\ \hat{M} = (M_u + M_d)/2 \\ M_s \end{bmatrix}}^{\text{QCD parameters (RGI)}} + \overbrace{\begin{bmatrix} m_{\text{Nucleon}} \\ B_K \\ \cdots \end{bmatrix}}^{\text{Predictions}}$$

and, using Eq. (1.1), extract from the high-energy regime of QCD the so-called renormalization group invariant (RGI) parameters associated with the running renormalized gauge coupling and masses. In addition a number of other predictions can be made for phenomenologically relevant quantities. A crucial point becomes immediately clear: we need a tool to relate the hadronic, low-energy regime of QCD with the high-energy (> 2–$10\,\text{GeV}$) regime, where perturbation theory in the gauge coupling applies. In the following we will demonstrate that such a tool is provided by the Euclidean space-time lattice.

The predictive power of lattice QCD can be seen in Fig. 1, which is taken from [1]. The data points represent various hadron masses m_h computed by the CP-PACS Collaboration [2]. The input used in the lattice simulations are m_ρ, m_π and m_ϕ, which fix the lattice spacing a, the mass of the degenerate up and down quarks, and the strange quark mass (a chiral extrapolation
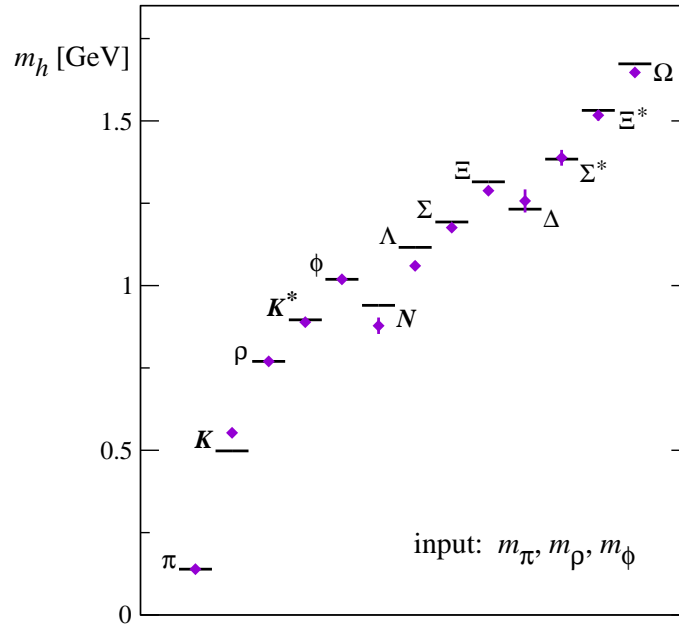

Fig. 1. The hadron spectrum from quenched lattice QCD.

is needed for the mass of the up and down quarks). The hadron spectrum is extrapolated to the continuum limit $a \rightarrow 0$. Given that quark polarization effects have been neglected (the so-called quenched approximation), the agreement between data points and experimental values marked by the horizontal lines is remarkable.

In this article we will be concerned with the computation of the strange quark mass from lattice QCD. In the limit of vanishing quark masses (chiral limit) QCD possesses a large chiral symmetry. This symmetry is spontaneously broken and the spectrum contains eight pseudo-scalar Goldstone bosons, which correspond to the observed eight lightest hadrons (π's, K's, η). The latter are not exactly massless due to the non-zero quark masses. Quark masses are the explicit symmetry-breaking parameters [3, 4] and they are treated as perturbations of the chiral limit in the framework of chiral perturbation theory [5]. At lowest order the quark mass ratios $m_u/m_d = 0.56$ and $m_s/m_d = 20.1$ can be determined from the pion and kaon masses [6, 7]. Chiral perturbation theory, though, cannot determine the absolute scale of the quark masses; lattice computations are required for this. The strange quark mass can be determined almost directly through lattice computations; few assumptions are needed, which we will explain below. For the up and down quark masses, more difficult chiral extrapolations are needed.

The value of the strange quark mass is required as an input for phenomenological predictions of the Standard Model, such as the CP-violating ratio ϵ'/ϵ. Also for physics beyond the Standard Model the quark masses are very important parameters.

2. The strange quark mass from lattice QCD

In the following we will use Wilson's formulation of lattice QCD, including Symanzik's O(a) improvement. A review of principles of lattice computations can be found in Ref. [8]. The Wilson formulation explicitly breaks the chiral symmetry through an O(a) term in the Wilson–Dirac operator [9]. As a consequence the quark mass receives an additive renormalization. However, it is well understood how the chiral Ward identities can be implemented on the lattice up to cut-off effects, which are O(a^2) in the improved theory; for a short review see [10].

A renormalized quark mass on the lattice, translated into the continuum $\overline{\text{MS}}$ scheme, can be defined through

$$\overline{m}^{\overline{\text{MS}}}(\mu) \;=\; Z_m(g_0, a\mu)(m - m_{\text{crit}}) \, . \tag{2.1}$$

A definition that avoids the determination of the additive mass renormalization m_{crit} is through the partial conservation of the axial currents (PCAC) relation, *e.g.* in the form

$$\partial_\mu(\bar{u}\gamma_\mu\gamma_5 s)_{\text{lat}} = (m_u + m_s)_{\text{lat}}(\bar{u}\gamma_5 s)_{\text{lat}} + \text{O}(a^2)\,, \qquad (2.2)$$

where $A_\mu = (\bar{u}\gamma_\mu\gamma_5 s)_{\text{lat}}$ is an improved axial current [11], $P = (\bar{u}\gamma_5 s)_{\text{lat}}$ a pseudo-scalar density, and m_q^{lat} are the bare lattice current quark masses. A renormalized quark mass is defined from the PCAC relation Eq. (2.2) using the renormalized axial current $(A_{\text{R}})_\mu = Z_{\text{A}}A_\mu$ and pseudo-scalar density $P_{\text{R}} = Z_{\text{P}}P$ and comparing the bare and renormalized relations:

$$(\overline{m}_u + \overline{m}_s)^{\overline{\text{MS}}}(\mu) = (m_u + m_s)_{\text{lat}}\,\frac{Z_{\text{A}}(g_0)}{Z_{\text{P}}^{\overline{\text{MS}}}(g_0, a\mu)}\,. \qquad (2.3)$$

The renormalization factor $Z_{\text{A}}(g_0)$ has been calculated non-perturbatively, using the chiral Ward identities [12]. In order to compute $Z_{\text{P}}^{\overline{\text{MS}}}(g_0, a\mu)$, we could apply perturbation theory. At one loop,

$$Z_{\text{P}}^{\overline{\text{MS}}}(g_0, a\mu) = 1 + \frac{g_0^2}{4\pi}\left[\frac{2}{\pi}\ln(a\mu) + k\right] + \text{O}(g_0^4)\,, \qquad (2.4)$$

where k is a calculable constant that depends on the lattice regularization and the renormalization scheme employed. Bare perturbation theory is known to be unreliable at the coupling accessible to simulations $g_0 \simeq 1$ and the systematic error of the perturbative expansion is difficult to assess. A non-perturbative determination of Z_{P} avoids these problems.

The running of the renormalized gauge coupling $\bar{g}$ and quark masses $\{\overline{m}_q\}$ depends on the renormalization scheme employed and is controlled by the renormalization group equations (RGEs)

$$\mu\frac{\text{d}\bar{g}}{\text{d}\mu} = \beta(\bar{g})\,, \quad \mu\frac{\text{d}\overline{m}_q}{\text{d}\mu} = \tau(\bar{g})\overline{m}_q\,, \qquad (2.5)$$

in terms of the β and τ functions. They are in general non-perturbatively defined. Their perturbative expansions are

$$\beta(\bar{g}) \underset{\bar{g}\to 0}{\sim} -\bar{g}^3\{b_0 + b_1\bar{g}^2 + b_2\bar{g}^4 + ...\}\,, \qquad (2.6)$$

$$\tau(\bar{g}) \underset{\bar{g}\to 0}{\sim} -\bar{g}^2\{d_0 + d_1\bar{g}^2 + ...\}\,. \qquad (2.7)$$

If the renormalization conditions are imposed at zero quark mass [13] the β and τ functions do not depend on the mass. One such scheme, the $\overline{\text{MS}}$ scheme, is the one most commonly used for perturbative QCD computations and there the β and τ functions are known up to the 4-loop coefficients b_3 [14, 15] and d_3 [16, 17]. The RGI quark masses are defined through

$$
M_q = \overline{m}_q(\mu) \left(2b_0 \bar{g}^2(\mu) \right)^{-d_0/(2b_0)} \exp \left\{ - \int\limits_0^{\bar{g}(\mu)} \mathrm{d}x \left[\frac{\tau(x)}{\beta(x)} - \frac{d_0}{b_0 x} \right] \right\} . \quad (2.8)
$$

Together with the Λ parameter, they are independent of the renormalization scale μ and, in fact, any physical quantity, whose total dependence on μ vanishes, can be considered as a function of Λ and $\{M_q\}$. The RGI masses are the same in all massless renormalization schemes.

A straightforward implementation of the definition Eq. (2.3) on the lattice poses a scale problem. In order to compute Z_P, the matrix element $\langle 0|P|K^+\rangle$ is needed and for this the spatial lattice size L has to be large enough with respect to a typical scale like the kaon decay constant F_K, so as to avoid finite-volume effects. The renormalization scale μ has to be large enough to make contact with perturbation theory and at the same time μ has to be small with respect to the cut-off $1/a$ in order to avoid large cut-off effects. We get a set of inequalities [18]

$$
L \quad \gg \quad \frac{1}{F_K} \sim \frac{1}{0.2\,\mathrm{GeV}} \quad \gg \quad \frac{1}{\mu} \sim \frac{1}{10\,\mathrm{GeV}} \quad \gg a \,,
$$

which imply $L/a \gg 50$. But lattices that big cannot be simulated. One elegant solution to this problem is to take a finite-size effect as the physical observable, that is to identify $L = 1/\mu$, with the only requirement $L/a \gg 1$. The renormalization scale μ is changed recursively in steps by factors of 2. Individually at each step the continuum limit can be taken. One therefore speaks of the recursive finite-size scaling (or step scaling) technique [19]. We take the strange quark mass as an example: a running mass $\overline{m}_s(\mu)$ is defined[1] that runs with the system's size $\mu = 1/L$. Starting at a reference maximal value $L_{\max}$, the system's size L is halved n times until contact with the perturbative regime is made; the RGI mass M_s can then be extracted:

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[1] For a specific renormalization scheme implementing this idea, see below.

$$L_{\max} = \text{const.}/F_K = \text{O}(0.5\,\text{fm}): \qquad \longrightarrow \qquad L_{\max}\,\overline{m}_s(\mu = 1/L_{\max})$$

$$\downarrow$$

$$L_{\max}\,\overline{m}_s(\mu = 2/L_{\max})$$

$$\downarrow$$

$$\text{always } a\mu = a/L \ll 1 \qquad\qquad\qquad \vdots$$

$$\downarrow$$

$$L_{\max}\,\overline{m}_s(\mu = 2^n/L_{\max})$$

$$\text{perturbation theory} \qquad\qquad\qquad \downarrow$$

$$L_{\max}\,M_s = \#$$

If the reference scale $L_{\max}$ is expressed in units of the kaon decay constant F_K, the result is a number for M_s/F_K, which is scheme-independent.

A particularly suitable renormalization scheme to implement the finite-size scaling technique is the Schrödinger functional (SF). It has been defined for QCD in Refs. [20, 21] and is the "work horse" of the ALPHA Collaboration. QCD is formulated in a box where Dirichlet boundary conditions are imposed on the fields at Euclidean time $x_0 = 0$ and $x_0 = T$. They provide an infrared cut-off proportional to $1/T$ to the frequency spectrum of quarks and gluons, making it possible to perform simulations, and hence impose renormalization conditions, at zero quark mass. A renormalized gauge coupling $\bar{g}(L)$, running with the scale $\mu = 1/L$, is defined through a variation of the effective action with respect to a change of the boundary gluon fields [22, 23]. Keeping $\bar{g}(L)$ fixed, while changing the bare lattice parameters, is equivalent to keeping the physical size L (and hence the renormalization scale μ) fixed.

The recursive finite-size scaling technique in the SF has been applied to compute the running of the renormalized coupling with $N_f = 2$ flavours of massless quarks [23]. The non-perturbative β function is shown by the points in Fig. 2, together with a non-perturbative fit. It deviates from the 3-loop β function for $\alpha_{\text{SF}}(L) = \bar{g}^2(L)/(4\pi) > 0.25$.

The renormalization factor Z_{P} can be defined and computed in the SF scheme [24–26]. The running of the renormalized quark mass is extracted from it through

$$\frac{\overline{m}^{\text{SF}}(\mu)}{\overline{m}^{\text{SF}}(\mu/2)} \;=\; \lim_{a\to 0} \left.\frac{Z_{\text{P}}^{\text{SF}}(g_0, 2L/a)}{Z_{\text{P}}^{\text{SF}}(g_0, L/a)}\right|_{\mu=1/L}, \qquad (2.9)$$

which is independent of the quark flavour q. The results for $N_f = 2$ flavours of massless quarks have been presented in detail in Ref. [27] and are shown by the points in Fig. 3. The running starts at the low energy $\mu = 1/L_{\max} \sim 0.4\,\text{GeV}$ corresponding to the leftmost point and extends to high energies,
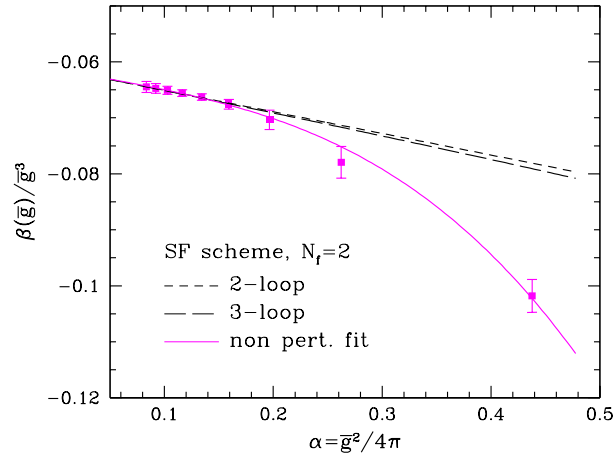
Fig. 2. The non-perturbative β function in the SF scheme for $N_f = 2$ flavours of massless quarks.

where perturbation theory can be safely applied to extract the RGI mass. The curves are obtained by using the perturbative expressions for the τ/β functions at the loop order indicated in the legend. In this case perturbation theory works surprisingly well down to small energies. For the running gauge coupling, instead, at $\alpha_{\rm SF}(L_{\rm max}) = 0.367$, clear deviations between non-perturbative and perturbative curves can be seen in Fig. 2.
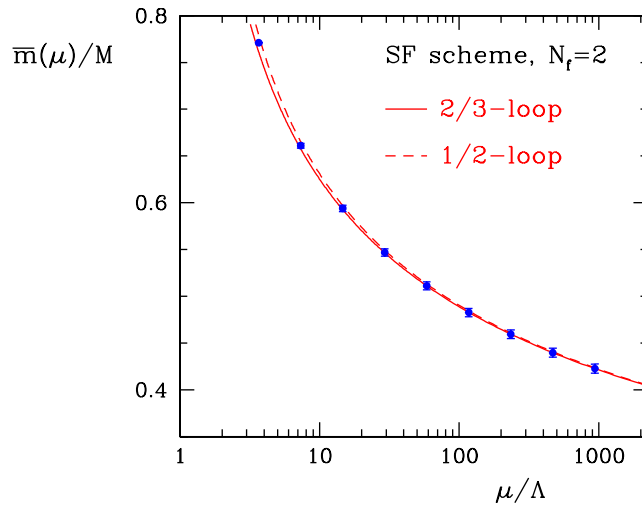

Fig. 3. The running quark mass in the SF scheme for $N_f = 2$ flavours of massless quarks. Plot taken from [27].

In the following we summarize the computation of the strange quark mass done in Ref. [27]. The steps involved are visible in the equation

$$\overline{m}_s^{\overline{\text{MS}}}(\mu_2) = \left(\frac{\overline{m}(\mu_2)}{M}\right)^{\overline{\text{MS}}}_{\text{PT}} \times \left(\frac{M}{\overline{m}(\mu_1)}\right)^{\text{SF}}_{\text{NP}} \times (m_s)_{\text{lat}} \frac{Z_\text{A}(g_0)}{Z_\text{P}(g_0, a\mu_1)} . \quad (2.10)$$

The scale in the $\overline{\text{MS}}$ scheme is conventionally $\mu_2 = 2\,\text{GeV}$. The first factor on the right-hand side of Eq. (2.10) is the perturbative (PT) evolution in the $\overline{\text{MS}}$ scheme. The second factor is the non-perturbative (NP) running in the SF scheme that starts at the hadronic scale $\mu_1 = 1/L_\text{max}$. The third factor is the renormalized (at the scale μ_1) strange quark mass in the SF scheme and it has to be computed at several lattice spacings to take the continuum limit. This part involves in principle

1. simulations of u, d and s quark flavours at their physical masses, and

2. setting the overall scale of the simulations, $i.e.$ determining the lattice spacing a in MeV^{-1}.

Because of computer cost and technical difficulties, simulations of two light and non-degenerate u and d quarks with an additional s quark have not been possible so far. There are very promising developments reported in [28], which show that these difficulties can be overcome in the near future. For the time being, the strategy of [29] is adopted. Simulations are performed with two degenerate flavours at a reference mass m_ref. The latter is tuned so that the pseudo-scalar (PS), made of two identical flavours, has a mass

$$m_\text{PS}(m_\text{ref}, m_\text{ref}) \ = \ m_K = \frac{1}{2}\left(m_{K^+}^2 + m_{K^0}^2\right)_\text{QCD} = 495\,\text{MeV} . \quad (2.11)$$

The subscript "QCD" means that an estimate of the electromagnetic effects has been subtracted from the experimental numbers [29] in order to obtain a pure QCD kaon mass, as we have on the lattice.

The lattice spacing could be set by computing on the lattice the kaon decay constant, a number aF_K, and dividing it by the experimental value of F_K, which can be obtained from the decay rate of $K^+ \longrightarrow \mu^+\nu_\mu$. Instead, in the present computation, the lattice spacing is set through the scale r_0 extracted from the static quark potential [30]. Data for r_0/a as well as pseudo-scalar masses needed to tune m_ref in Eq. (2.11) are taken from [31], where they are available at three lattice spacings in the approximate range 0.092–0.071 fm. The data for r_0/a are extrapolated to zero quark mass [23]. The phenomenological value is $r_0 = 0.5\,\text{fm}$, obtained from potential models.

The results for the reference RGI quark mass show a strong dependence on the lattice spacing [27]. No systematic continuum extrapolation is therefore performed, instead a continuum estimate is given:

$$M_{\text{ref}} \; = \; 72(3)(13) \, \text{MeV} \,, \tag{2.12}$$

where the value and the first error are the result at the smallest lattice spacing, the second error is a systematic one and is the difference with the value at the largest lattice spacing. The latter dominates. The result Eq. (2.12) is consistent with the one obtained in the quenched ($N_f = 0$) theory[2] [29]. It is assumed to hold in the $N_f = 3$ theory as well. In order to translate Eq. (2.12) into a value for the strange quark mass, the Gell-Mann–Oakes–Renner formula is used [3, 5, 6]:

$$m_K^2 \; = \; \left(\hat{M} + M_s \right) B_{\text{RGI}} \; = \; 2 M_{\text{ref}} \, B_{\text{RGI}} \,, \tag{2.13}$$

where $\hat{M} = \frac{1}{2}(M_u + M_d)$ and B_{RGI} is a constant of the chiral Lagrangian. Together with $M_s/\hat{M} = 24.4(1.5)$ [32], Eq. (2.13) can be solved for M_s, and using 4-loop running in the $\overline{\text{MS}}$ scheme yields

$$\overline{m}_s^{\overline{\text{MS}}}(\mu = 2 \, \text{GeV}) \; = \; 97(22) \, \text{MeV} \,. \tag{2.14}$$

The $\overline{m}_s^{\overline{\text{MS}}}$ values for the three lattice spacings are displayed in Fig. 4 (filled squares).

3. Conclusions

In Fig. 4 we show a summary of the lattice determinations of the strange quark mass with dynamical (sea) quarks at finite values of the lattice spacing a. This is an update of the plot of Refs. [27, 33]. There are points obtained with $N_f = 2$ dynamical quarks [27, 31, 34–36] and $N_f = 2 + 1$ [37, 38]. In the legend we indicate the gauge action/fermion action employed and whether non-perturbative or perturbative renormalization has been used. The dictionary for the gauge actions is: W: Wilson action; I: Iwasaki action; LW: 1-loop tadpole improved Lüscher–Weisz action. The dictionary for the fermion actions is: W: Wilson action; CW: Wilson-clover action; KS: Asqtad staggered action. Except the Wilson fermion action used in [36], all the other lattice formulations have $\text{O}(a^2)$ discretization errors. The non-perturbative renormalization is done in the SF or the RI-MOM [39] scheme. The physical kaon mass is always used as input to fix the strange quark mass.

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[2] Also for the Λ parameter there is no significant difference between $N_f = 0$ and $N_f = 2$ [23].
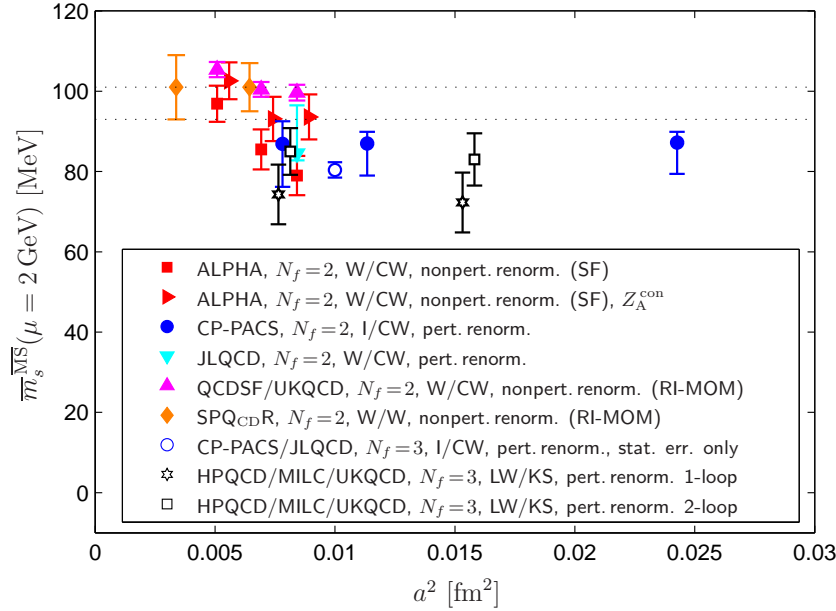
Fig. 4. Summary plot of the strange quark mass from dynamical simulations with $N_f = 2$ and $N_f = 3$.

There is a systematic effect due to the use of perturbative renormalization, which leads to values of the quark mass smaller than those for non-perturbative renormalization. Figure 4 also shows the impact in the quark mass values of [38] when the perturbative renormalization factor Z_P in Eq. (2.4) is evaluated at two loops [40]: the mass values increase by 15%. There is good agreement between data from different non-perturbative renormalization procedures and, in fact, between them and the $N_f = 0$ result [29] marked by the dotted lines. We also mention the most recent compilation of the strange quark mass from QCD sum rules [41], which quotes $\overline{m}_s(2\,\text{GeV}) = 99(28)\,\text{MeV}$, in agreement with the range of the lattice results; see also [42–44].

In conclusion, Fig. 4 demonstrates that non-perturbative renormalization is essential for a reliable lattice determination of the strange quark mass and that data at smaller lattice spacing(s) are needed for a systematic continuum extrapolation.

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