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PERTURBATIVE ODDERON IN THE COLOR GLASS CONDENSATE* **

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We derive small-x evolution equations for odderon exchange processes in the color glass condensate formalism. We consider the dipole–color glass scattering and the three-quark–color glass scattering, with particular emphasis on the gauge invariant coupling to the external probes. In the low energy regime where the classical gluon field is not so strong, our result is equivalent to the Bartels–Kwiecinski–Praszalowicz (BKP) equation.

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1. Introduction

The odderon is a C-odd compound state of reggeons which dominates the hadronic cross section difference between the direct and crossed channel processes at very high energies. Its existence is as natural in perturbative QCD as that of the C-even counterpart, the Pomeron. To lowest order in perturbative QCD, the odderon is a three-gluon exchange. Indeed, with the definition of the C-conjugation operation for the non-Abelian field

$$A_\mu \to -A_\mu^T\,,$$

it is easy to see that the following combination is C-odd:

$$d^{abc}\,\mathrm{Tr}\,A_\mu^a A_\nu^b A_\rho^c\,,$$

where d^{abc} is the totally symmetric SU(3) d-symbol. Shortly after the Balitsky–Fadin–Kuraev–Lipatov (BFKL) equation [1,2] which describes the

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hard Pomeron was established, the small-x evolution equation for the odderon exchange was written down by Bartels [3] and Kwiecinski and Praszalowicz [4], known as the BKP equation. Although the BKP equation is simply a pairwise iteration of the BFKL kernel, its solution remained unattainable for a long time. A theoretical breakthrough took place when Lipatov [5] and Faddeev and Korchemsky [6] discovered that the evolution equation of the odderon (or more generally, of the multireggeon exchange in the large $N_{\rm c}$ limit) can be mapped into the eigenvalue problem of exactly solvable one dimensional Heisenberg spin chains. At the moment two leading solutions of the odderon evolution equation are available [7,8] and the subject continues to be under intensive study [9, 10].

In these notes I report on the recent attempt [11] to formulate the odderon problem in QCD in the framework of the color glass condensate (CGC) formalism [12–14] which is an effective theory of gluon saturation [15]. So far, the CGC formalism has been restricted to the C-even (Pomeron) channel where one can study linear and nonlinear evolution equations for the gluon distribution. The odderon problem will be a good starting point towards understanding multireggeon exchange amplitudes in the CGC. The built-in gauge invariance of the CGC formalism allows one to obtain direct insights into the physical solution to the odderon evolution equations.

In Section 2, I introduce the master equation of the CGC formalism, the JIMWLK equation and its simpler version. In Section 3, I derive the odderon evolution equations for the dipole–color glass scattering and the proton–color glass scattering both in the linear, weak field approximation and in the nonlinear, strong field regime. In Section 4 I discuss the connection to the BKP equation.

2. JIMWLK equation with the dipole kernel

The idea of the color glass condensate is that the small-x gluons of a fast-moving hadron (nucleus) can be described by classical strong color fields $\alpha_a \equiv A_a^+$ created by the high-x partons. The fields α_a are distributed according to an unknown weight function $W_\tau[\alpha]$. The JIMWLK equation is an evolution equation of $W_\tau[\alpha]$ in rapidity $\tau = \ln \frac{1}{x}$.

$$\frac{\partial}{\partial\tau} W_\tau[\alpha] = \frac{1}{2}\int_{xy} \frac{\delta}{\delta\alpha_\tau^a(x_\perp)}\eta^{ab}(x_\perp, y_\perp)\frac{\delta}{\delta\alpha_\tau^b(y_\perp)} W_\tau[\alpha]\,, \qquad (1)$$

where the subscript xy denotes the integration over two dimensional transverse coordinates $x_\perp$ and $y_\perp$. [See, [14] for the details.] The kernel $\eta^{ab}(x_\perp, y_\perp)$

is a function of Wilson lines in the adjoint representation $\widetilde{V}^\dagger(x_\perp)$:

$$\begin{aligned}\eta^{ab}(x_\perp, y_\perp) &= \frac{1}{\pi}\int \frac{d^2 z_\perp}{(2\pi)^2}\,\frac{(x_\perp - z_\perp)\cdot(y_\perp - z_\perp)}{(x_\perp - z_\perp)^2 (z_\perp - y_\perp)^2} \\ &\quad\times\Big(1 + \widetilde{V}^\dagger(x_\perp)\widetilde{V}(y_\perp) - \widetilde{V}^\dagger(x_\perp)\widetilde{V}(z_\perp) - \widetilde{V}^\dagger(z_\perp)\widetilde{V}(y_\perp)\Big)^{ab}, \\ \widetilde{V}^\dagger(x_\perp) &= \mathrm{P}\exp\left(ig\int\limits_{-\infty}^{\infty} dx^- \alpha^a(x^-, x_\perp) T^a\right), \end{aligned} \tag{2}$$

where P denotes the path-ordering along the light cone direction x^- [1]. The functional derivative $\frac{\delta}{\delta\alpha^a_\tau(x_\perp)}$ is meant to act on the Wilson line in the following way:

$$\frac{\delta \widetilde{V}^\dagger_x}{\delta \alpha^a_y} = ig\delta^2(x_\perp - y_\perp) T^a \widetilde{V}^\dagger_x\,, \qquad \frac{\delta \widetilde{V}_x}{\delta \alpha^a_y} = -ig\delta^2(x_\perp - y_\perp)\widetilde{V}_x T^a\,. \tag{3}$$

An evolution equation for a generic observable $\mathcal{O}$ is given by

$$\frac{\partial}{\partial\tau}\langle\mathcal{O}\rangle_\tau = \int\limits_{xy}\left\langle \frac{\delta}{\delta\alpha^a_\tau(x_\perp)}\eta^{ab}(x_\perp, y_\perp)\frac{\delta}{\delta\alpha^b_\tau(y_\perp)}\mathcal{O}\right\rangle_\tau, \tag{4}$$

where the average of the observable is defined by

$$\langle\mathcal{O}\rangle_\tau \equiv \int \mathcal{D}\alpha\, \mathcal{O}[\alpha]\, W_\tau[\alpha]\,. \tag{5}$$

The z-integral in the kernel η^{ab} is a potential source of infrared ($z\to\infty$) logarithmic divergence. This divergence cancels in the final evolution equation if the projectile is a color singlet. However, in practical calculations it is very tedious to show the actual cancellation for complicated observables like those considered in the next section. In [11] it has been suggested that for an observable $\mathcal{O}$ describing the scattering between a color singlet projectile and a CGC, the evolution equation can equivalently be rewritten in the following way:

$$\begin{aligned}\frac{\partial}{\partial\tau}\langle\mathcal{O}\rangle_\tau &= \langle H_{\mathrm{dp}}\mathcal{O}\rangle \equiv -\frac{1}{16\pi^3}\int\limits_{xyz}\frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2 (z_\perp - y_\perp)^2} \\ &\quad\times\left\langle \Big(1 + \widetilde{V}^\dagger_x\widetilde{V}_y - \widetilde{V}^\dagger_x\widetilde{V}_z - \widetilde{V}^\dagger_z\widetilde{V}_y\Big)^{ab}\frac{\delta}{\delta\alpha^a_x}\frac{\delta}{\delta\alpha^b_y}\mathcal{O}\right\rangle_\tau. \end{aligned} \tag{6}$$

[1] We assume that the color glass moves in the positive z direction.

Note that the $z_\perp$-integral in the new kernel ("dipole kernel") is now infrared-finite. Moreover, one of the functional derivatives has been moved to the right of the kernel. These two modifications considerably simplify the derivation of evolution equations.

As an example, consider the dipole-color glass scattering. In the eikonal approximation, the S-matrix is given by

$$\begin{aligned}
\mathcal{O} &= S_\tau(x_\perp, y_\perp) \equiv \frac{1}{N_c} \langle \mathrm{Tr}\,(V_x^\dagger V_y) \rangle_\tau , \\
V^\dagger(x_\perp) &= \mathrm{P}\exp\left(ig \int_{-\infty}^{\infty} dx^- \alpha^a(x^-, x_\perp) t^a \right) ,
\end{aligned} \tag{7}$$

where t^a are the generators in the fundamental representation. $V_x^\dagger$ (V_y) represents the propagation of a (anti-)quark at the transverse coordinate $x_\perp(y_\perp)$ in the background gauge field α. Since a dipole is color singlet, one can use Eq. (6). After some simple algebra (much simpler than using the original form, Eq. (1)) one obtains the Balitsky–JIMWLK equation [19]:

$$\begin{aligned}
\frac{\partial}{\partial \tau} \langle \mathrm{Tr}\,(V_x^\dagger V_y) \rangle_\tau &= \frac{\bar{\alpha}_s}{2\pi} \int d^2 z_\perp \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2 (y_\perp - z_\perp)^2} \\
&\quad \times \left\langle \frac{1}{N_c} \mathrm{Tr}\,(V_x^\dagger V_z) \mathrm{Tr}\,(V_z^\dagger V_y) - \mathrm{Tr}\,(V_x^\dagger V_y) \right\rangle_\tau ,
\end{aligned} \tag{8}$$

where $\bar{\alpha}_s = \frac{N_c \alpha_s}{\pi}$.

In the weak field approximation ($\alpha \sim 1$) one can expand both the Hamiltonian H_{dp} and the operator in powers of α's. Eq. (6) becomes

$$\begin{aligned}
\frac{\partial}{\partial \tau} \langle \mathcal{O} \rangle_\tau &= \frac{g^2}{16\pi^3} \int_{xyz} \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2 (z_\perp - y_\perp)^2} \\
&\quad \times \left\langle (\alpha_x - \alpha_z)^d (\alpha_y - \alpha_z)^e f^{dfg} f^{egh} \frac{\delta}{\delta \alpha_x^f} \frac{\delta}{\delta \alpha_h^b} \mathcal{O} \right\rangle_\tau ,
\end{aligned} \tag{9}$$

where

$$\alpha_x^a \equiv \int dx^- \alpha^a(x^-, x_\perp) \,. \tag{10}$$

(The nonlocality in x^- due to the path ordering is irrelevant in the linear term.) Note that in the weak field approximation, the evolution equation conserves the number of α fields. In the perturbative QCD language, this corresponds to processes in which the number of reggeons in the t-channel is fixed. For the imaginary part of the scattering amplitude (the real part of the S-matrix),

$$\mathcal{O} = \langle N(x_\perp, y_\perp)\rangle_\tau \equiv \frac{1}{N_\mathrm{c}}\mathrm{Re}\left\langle \mathrm{Tr}\left(1 - V_x^\dagger V_y\right)\right\rangle_\tau \approx \frac{g^2}{4\pi}\left\langle \left(\alpha_x^a - \alpha_y^a\right)^2\right\rangle_\tau, \quad (11)$$

one obtains the dipole version of the BFKL equation.

$$\begin{aligned}\frac{\partial}{\partial\tau}\langle N(x_\perp, y_\perp)\rangle_\tau \;=\; & \frac{\bar{\alpha}_\mathrm{s}}{2\pi}\int d^2z_\perp \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2(y_\perp - z_\perp)^2} \\ & \times\Big(\langle N(x_\perp, z_\perp)\rangle_\tau + \langle N(y_\perp, z_\perp)\rangle_\tau - \langle N(x_\perp, y_\perp)\rangle_\tau\Big). \quad (12)\end{aligned}$$

3. Odderon exchanges in high energy scatterings

The analysis of the odderon exchange proceeds in the same way as the Pomeron exchange. First one identifies the relevant observable which describes the odderon amplitude. Then one writes down the evolution equation both in the linear, weak field approximation and in the nonlinear saturated regime with the use of the dipole JIMWLK equation obtained in the previous section.

3.1. Odderon contribution to the dipole–color glass scattering

Consider first the odderon exchange in the dipole–color glass scattering. It is well known that the odderon dominates the real part of the scattering amplitude, or equivalently, the imaginary part of the S-matrix. Therefore, we define the odderon amplitude

$$\langle O(x_\perp, y_\perp)\rangle_\tau \equiv \mathrm{Im}S_\tau(x_\perp, y_\perp) = \frac{1}{2iN_\mathrm{c}}\left\langle \mathrm{Tr}\left(V_x^\dagger V_y - V_y^\dagger V_x\right)\right\rangle_\tau. \quad (13)$$

Note that the amplitude is odd in the exchange of $x_\perp$ and $y_\perp$.

In the weak field limit, this reduces to

$$\langle O(x_\perp, y_\perp)\rangle_\tau \approx \frac{-g^3}{24N_\mathrm{c}} d^{abc}\left\langle 3\alpha_x^a\alpha_y^b\alpha_y^c - 3\alpha_x^a\alpha_x^b\alpha_y^c + \alpha_x^a\alpha_x^b\alpha_x^c - \alpha_y^a\alpha_y^b\alpha_y^c\right\rangle_\tau. \quad (14)$$

A notable point is that, thanks to the totally symmetric d-symbol, this part of the amplitude is free from the complexity due the path ordering in x^-.

As expected, in lowest order the amplitude reduces to the three-point function of α which corresponds to the three-gluon exchange. It is straightforward to write down the evolution equation of $O(x_\perp, y_\perp)$ using Eq. (9). The resulting equation is exactly the same as the BFKL equation.

$$\begin{aligned}\frac{\partial}{\partial\tau}\langle O(x_\perp, y_\perp)\rangle_\tau &= \frac{\bar{\alpha}_s}{2\pi}\int d^2z \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2(y_\perp - z_\perp)^2}\Big(\langle O(x_\perp, z_\perp)\rangle_\tau \\ &\quad + \langle O(y_\perp, z_\perp)\rangle_\tau - \langle O(x_\perp, y_\perp)\rangle_\tau\Big), \end{aligned} \tag{15}$$

in agreement with the observation by Kovchegov, Szymanowski and Wallon [17]. In order to construct the solution, one has to select the eigenfunctions of the BFKL equation which is antisymmetric in the exchange of $x_\perp$ and $y_\perp$. We refer to [8, 17] for the details. Let us just remark that the computation of the initial condition of the evolution is particularly simple in the CGC formalism. For example, suppose that the target is a single quark. The corresponding gauge field is given by

$$\alpha_x^a = -\frac{1}{2\pi}\ln(|x_\perp - z_\perp|\mu) g t^a\,, \tag{16}$$

where $z_\perp$ is the transverse coordinate of the quark and μ is an infrared cutoff which will disappear from the final result. Substituting Eq. (16) in Eq. (14), we get[2]

$$\langle O(x_\perp, y_\perp)\rangle_{\tau=0} = \frac{\alpha_s^3}{12N_c^2}(N_c^2 - 4)(N_c^2 - 1)\ln^3\frac{|x_\perp - z_\perp|}{|y_\perp - z_\perp|}\,. \tag{17}$$

Since our odderon amplitude is just an imaginary part of the S-matrix, the nonlinear equation for $O(x, y)$ can be obtained by simply taking the imaginary part of the Balitsky equation Eq. (8). We get

$$\begin{aligned}\frac{\partial}{\partial\tau}\langle O(x_\perp, y_\perp)\rangle_\tau &= \frac{\bar{\alpha}_s}{2\pi}\int d^2z_\perp \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2(z_\perp - y_\perp)^2}\Big\langle O(x_\perp, z_\perp) + O(z_\perp, y_\perp) \\ &- O(x_\perp, y_\perp) - O(x_\perp, z_\perp)N(z_\perp, y_\perp) - N(x_\perp, z_\perp)O(z_\perp, y_\perp)\Big\rangle_\tau\,, \end{aligned} \tag{18}$$

The large-N_c factorized form of Eq. (18) was previously obtained in [17] within the framework of Mueller's dipole model [18].

[2] Note that, in the Gaussian approximation for the weight function W_τ as in the original McLerran–Venugopalan model, the imaginary part of $\langle \mathrm{Tr}\,(V_x^\dagger V_y)\rangle_\tau$ vanishes. Therefore, one has to go beyond the Gaussian approximation to keep track of the imaginary part [16].

3.2. Odderon contribution to the three-quark–CGC scattering

Next, we consider the odderon exchange between a three-quark system ("proton") and a color glass. We employ the following S-matrix [20]

$$S_\tau(x_\perp, y_\perp, z_\perp) = \frac{1}{6}\epsilon^{abc}\epsilon^{ijk}\left\langle V^\dagger_{ai}(x_\perp)V^\dagger_{bj}(y_\perp)V^\dagger_{ck}(z_\perp)\right\rangle_\tau \,, \tag{19}$$

where $x_\perp, y_\perp$ and $z_\perp$ are three coordinates of the valence quarks inside a proton. The odderon contribution is again given by the imaginary part of the S-matrix.

$$\langle B(x_\perp, y_\perp, z_\perp)\rangle_\tau = \frac{1}{12i}\epsilon^{abc}\epsilon^{ijk}\left\langle V^\dagger_{ai}(x_\perp)V^\dagger_{bj}(y_\perp)V^\dagger_{ck}(z_\perp) - \text{c.c.}\right\rangle_\tau . \tag{20}$$

In the weak field approximation, we expand B up to cubic order in α.

$$\begin{aligned}
&\langle B(x_\perp, y_\perp, z_\perp)\rangle_\tau \approx \\
&\quad -\frac{g^3}{144}d^{abc}\langle 12\alpha^a_x\alpha^b_y\alpha^c_z - 3\alpha^a_x\alpha^b_x(\alpha^c_y+\alpha^c_z) - 3\alpha^a_y\alpha^b_y(\alpha^c_z+\alpha^c_x) \\
&\quad -3\alpha^a_z\alpha^b_z(\alpha^c_x+\alpha^c_y) + 2\alpha^a_x\alpha^b_x\alpha^c_x + 2\alpha^a_y\alpha^b_y\alpha^c_y + 2\alpha^a_z\alpha^b_z\alpha^c_z\rangle_\tau \,.
\end{aligned} \tag{21}$$

Note that when two of the coordinates are equal, the baryonic operator reduces to the dipole operator.

$$B(x_\perp, z_\perp, z_\perp) = O(x_\perp, z_\perp) = -B(x_\perp, x_\perp, z_\perp) \quad (N_c = 3)\,. \tag{22}$$

After some algebra, we obtain the evolution equation in the linear regime.

$$\begin{aligned}
&\frac{\partial}{\partial\tau}\langle B(x_\perp, y_\perp, z_\perp)\rangle_\tau = \frac{3\alpha_s}{4\pi^2}\int d^2w_\perp \frac{(x_\perp - y_\perp)^2}{(x_\perp - w_\perp)^2(y_\perp - w_\perp)^2}\Big(\langle B(x_\perp, w_\perp, z_\perp)\rangle_\tau \\
&+\langle B(w_\perp, y_\perp, z_\perp)\rangle_\tau - \langle B(x_\perp, y_\perp, z_\perp)\rangle_\tau - \langle B(w_\perp, w_\perp, z_\perp)\rangle_\tau \\
&-\langle B(x_\perp, x_\perp, w_\perp)\rangle_\tau - \langle B(y_\perp, y_\perp, w_\perp)\rangle_\tau - \langle B(x_\perp, y_\perp, w_\perp)\rangle_\tau\Big) \\
&+(2\text{ cyclic permutations})\,.
\end{aligned} \tag{23}$$

The ultraviolet poles at $x_\perp = w_\perp$ and $y_\perp = w_\perp$ are innocuous; the numerator automatically vanishes at $x_\perp = w_\perp$ and $y_\perp = w_\perp$. If one sets $y_\perp = z_\perp$, Eq. (23) reduces to Eq. (15) due to Eq. (22).

4. Relation to the BKP equation

4.1. Decomposition of the scattering amplitude

The BKP equation for the three gluon–target scattering amplitude $f_\tau(k_1, k_2, k_3)$ reads

$$\frac{\partial}{\partial\tau} f_\tau(k_1,k_2,k_3) = \frac{1}{2}\sum_{i=1}^{3}\int d^2k'_1 d^2k'_2 d^2k'_3 \delta^2(k'_1+k'_2+k'_3-q) \times H_{\mathrm{BFKL}}(k_{i-1},k_{i+1};k'_{i-1},k'_{i+1})\delta^2(k_i-k'_i) f_\tau(k'_1,k'_2,k'_3)\,, \tag{24}$$

where k_i's are transverse momenta of gluons, $k_4 \equiv k_1$, *etc.*, and q is the momentum transfer. H_{BFKL} is the usual non-forward BFKL kernel including the virtual terms. The factor $\frac{1}{2}$ accounts for the fact that two of the gluons are in the color octet state. Although expected, it is still nontrivial to see whether our equations in the weak field approximation are indeed equivalent to the BKP equation. This is because the baryonic amplitude $\langle B(x_\perp, y_\perp, z_\perp)\rangle_\tau$ is not the Fourier transform of $f_\tau(k_1, k_2, k_3)$

$$f_\tau(x_\perp,y_\perp,z_\perp) \equiv \int d^2k_1 d^2k_2 d^2k_3 f_\tau(k_1,k_2,k_3) e^{ik_1x_\perp + ik_2y_\perp + ik_3z_\perp}\,. \tag{25}$$

In fact, each term in Eq. (21) represents a different coupling between the three quarks and three gluons, and only the sum of all possible couplings is gauge invariant. Therefore, we are led to the following identification: (Compare with Eq. (14) and Eq. (21).)

$$\begin{aligned}\langle O(x_\perp,y_\perp)\rangle_\tau &= \int d^2k_1 d^2k_2 d^2k_3 f_\tau(k_1,k_2,k_3)\Big(3e^{ik_1x+ik_2y+ik_3y} \\ &\quad -3e^{ik_1x+ik_2x+ik_3y} + e^{i(k_1+k_2+k_3)x} - e^{i(k_1+k_2+k_3)y}\Big) \\ &= \int d^2k_1 d^2k_2 d^2k_3 f_\tau(k_1,k_2,k_3)(e^{ik_1x_\perp}-e^{ik_1y_\perp}) \\ &\quad \times(e^{ik_2x_\perp}-e^{ik_2y_\perp})(e^{ik_3x_\perp}-e^{ik_3y_\perp})\,,\end{aligned} \tag{26}$$

for the odderon exchange in the dipole–color glass scattering and

$$\langle B(x_\perp, y_\perp, z_\perp)\rangle_\tau = \int d^2k_1 d^2k_2 d^2k_3 f_\tau(k_1,k_2,k_3)\left(2e^{ik_1x_\perp} - e^{ik_1y_\perp} - e^{ik_1z_\perp}\right) \times \left(2e^{ik_2y_\perp} - e^{ik_2z_\perp} - e^{ik_2x_\perp}\right)\left(2e^{ik_3z_\perp} - e^{ik_3x_\perp} - e^{ik_3y_\perp}\right), \quad (27)$$

for the odderon exchange in the proton–color glass scattering.

We define the impact factor which describes gauge invariant coupling of an odderon and a photon

$$\Phi_{\rm dp}(k_1,k_2,k_3) = \int dz d^2r_\perp |\Psi(z,r_\perp)|^2 \left(e^{ik_1\frac{r_\perp}{2}} - e^{-ik_1\frac{r_\perp}{2}}\right) \times \left(e^{ik_2\frac{r_\perp}{2}} - e^{-ik_2\frac{r_\perp}{2}}\right)\left(e^{ik_3\frac{r_\perp}{2}} - e^{-ik_3\frac{r_\perp}{2}}\right), \quad (28)$$

where $\Psi(z,r_\perp)$ is the light-cone wavefunction of a photon [21] (z is the fraction of momentum carried by the quark and $r_\perp = x_\perp - y_\perp$ is the relative coordinate.) Clearly, $\Phi_{\rm dp}(k_1 = 0, k_2, k_3) = 0$, *etc.*, as required by gauge invariance. The real part of the dipole–color glass scattering amplitude at nonzero momentum transfer can be decomposed in two ways.

$$\begin{aligned}\frac{\mathrm{Re}\,\mathcal{M}(q)}{s} &= \int d^2k_1 d^2k_2 d^2k_3 \delta(k_1+k_2+k_3-q)\,\Phi_{\rm dp}(k_1,k_2,k_3) f_\tau(k_1,k_2,k_3)\\ &= \int dz d^2r_\perp |\Psi(z,r_\perp)|^2 \int d^2b_\perp \langle O(x_\perp,y_\perp)\rangle_\tau e^{-iqb_\perp}, \qquad (29)\end{aligned}$$

where $b_\perp = \frac{x_\perp + y_\perp}{2}$ is the impact parameter.

Similarly, we introduce the impact factor for the "proton"–odderon coupling

$$\Phi_{\rm p}(k_1,k_2,k_3) = \int d^2x'_\perp d^2y'_\perp d^2z'_\perp |\Psi(x'_\perp,y'_\perp,z'_\perp)|^2 \left(2e^{ik_1x'_\perp} - e^{ik_1y'_\perp} - e^{ik_1z'_\perp}\right) \times \left(2e^{ik_2y'_\perp} - e^{ik_2z'_\perp} - e^{ik_2x'_\perp}\right)\left(2e^{ik_3z'_\perp} - e^{ik_3x'_\perp} - e^{ik_3y'_\perp}\right), \quad (30)$$

where $|\Psi(x'_\perp,y'_\perp,z'_\perp)|^2$ is the proton light-cone wavefunction. $x'_\perp$ are coordinates relative to the barycenter $b_\perp = (x_\perp + y_\perp + z_\perp)/3$. The condition $x'_\perp + y'_\perp + z'_\perp = 0$ is implicit in $|\Psi(x'_\perp,y'_\perp,z'_\perp)|^2$. The real part of the scattering amplitude becomes

$$\begin{aligned}\frac{\mathrm{Re}\,\mathcal{M}(q)}{s} &= \int d^2k_1 d^2k_2 d^2k_3 \delta(k_1+k_2+k_3-q) f_\tau(k_1,k_2,k_3)\,\Phi_{\mathrm{p}}(k_1,k_2,k_3), \\ &= \int d^2x'_\perp d^2y'_\perp d^2z'_\perp |\,\Psi(x'_\perp,y'_\perp,z'_\perp)|^2 \int e^{-iqb_\perp}\langle B(x_\perp,y_\perp,z_\perp)\rangle_\tau d^2b_\perp .\end{aligned} \tag{31}$$

4.2. The BKP equation with the dipole kernel

The above analysis shows that there is a nontrivial convolution with the impact factor in the relation between $f_\tau(k_1,\cdots)$ and $\langle B(x_\perp,\cdots)\rangle_\tau$. Moreover, the simple Fourier transform Eq. (25) contains infrared divergence which disappears in the gauge invariant amplitude $\langle B(x_\perp,\cdots)\rangle_\tau$. In order to relate the equations for $f_\tau(k_1,\cdots)$ and $\langle B(x_\perp,\cdots)\rangle_\tau$, we first consider the BFKL case. (See, Eq. (11).)

$$\langle N(x_\perp,y_\perp)\rangle_\tau = f_\tau(x_\perp,x_\perp) + f_\tau(y_\perp,y_\perp) - 2f_\tau(x_\perp,y_\perp)\,, \tag{32}$$

where $f_\tau(x_\perp,y_\perp)$ is the Fourier transform of the momentum space BFKL amplitude $f_\tau(k_1,k_2)$. Define

$$\tilde{f}_\tau(x_\perp,y_\perp) \equiv f_\tau(x_\perp,y_\perp) - \tfrac{1}{2}f_\tau(x_\perp,x_\perp) - \tfrac{1}{2}f_\tau(y_\perp,y_\perp)\,, \tag{33}$$

such that

$$\begin{aligned}\tilde{f}_\tau(x_\perp,x_\perp) &= 0\,, \\ \langle N(x_\perp,y_\perp)\rangle_\tau &= -2\tilde{f}_\tau(x_\perp,y_\perp)\,.\end{aligned} \tag{34}$$

In general, one has the freedom to add to $f_\tau(x_\perp,y_\perp)$ arbitrary functions which depend on only one coordinate, $x_\perp$ or $y_\perp$ [2, 22]. With the choice Eq. (33), the equation for $\tilde{f}_\tau(x_\perp,y_\perp)$ becomes the same as the dipole BFKL equation for $\langle N(x_\perp,y_\perp)\rangle_\tau$, Eq. (12). Note that Eq. (12) is well-defined only for functions which vanishes when two coordinates are equal, $f(x_\perp,x_\perp)=0$, due to the ultraviolet poles at $x_\perp=z_\perp$ and $y_\perp=z_\perp$. This is automatically the case for $\langle N(x_\perp,y_\perp)\rangle_\tau$. However, the function, $f_\tau(x_\perp,y_\perp)$ which is related via Fourier transform to the unintegrated gluon distribution in momentum space [23], does not have this property unless one subtracts the diagonal part as in Eq. (33). The dipole BFKL equation for $\tilde{f}$ implies the following equation for f.

$$\frac{\partial}{\partial\tau}f_\tau(x_\perp,y_\perp) = \frac{\bar{\alpha}_s}{2\pi}\int d^2z_\perp \frac{(x_\perp-y_\perp)^2}{(x_\perp-z_\perp)^2(y_\perp-z_\perp)^2}\Big(f_\tau(x_\perp,z_\perp)+f_\tau(y_\perp,z_\perp) -f_\tau(x_\perp,y_\perp)-f_\tau(z_\perp,z_\perp)\Big)\,. \tag{35}$$

The last term ensures that the residues of the poles at $x_\perp = z_\perp$ and $y_\perp = z_\perp$ automatically vanish so that $f_\tau(x_\perp, x_\perp)$ need not be zero. One can trace the origin of this term by directly Fourier transforming the momentum space BFKL kernel. In the complex coordinate notation ($z_1 = x^1_\perp + ix^2_\perp$, $z_2 = y^1_\perp + iy^2_\perp$), the real (as opposed to virtual) part of the BFKL kernel reads [5]

$$-\frac{\bar{\alpha}_s}{2|\partial_{z_1}|^2|\partial_{z_2}|^2}\Big(\partial_{\bar{z}_1}\partial_{z_2}\ln|z_1-z_2|^2\partial_{z_1}\partial_{\bar{z}_2}+\text{c.c.}\Big). \tag{36}$$

When commuting $\partial_{z_1}\partial_{\bar{z}_2}$ and $\ln|z_1-z_2|^2$, one obtains the delta function contribution $\delta^{(2)}(x_\perp - y_\perp)$ which corresponds to the last term in Eq. (35).

Note that Eq. (35) is exactly the same as the equation satisfied by $\langle\alpha^a_x\alpha^a_y\rangle_\tau$, *provided* one uses the dipole JIMWLK Hamiltonian H_{dp}, Eq. (9). Strictly speaking, since $\alpha^a_x\alpha^a_y$ does not describe a color single projectile, one is not allowed to apply H_{dp} to $\alpha^a_x\alpha^a_y$. Nevertheless, under the premise that the CGC amplitudes $\alpha^a_x\alpha^a_y$ are eventually used as building blocks of the gauge-invariant amplitude $\langle N(x_\perp,y_\perp)\rangle_\tau$, one can identify $\langle\alpha^a_x\alpha^a_y\rangle_\tau$ with $f_\tau(x_\perp,y_\perp)$ and Eq. (35) for $\langle\alpha^a_x\alpha^a_y\rangle_\tau$ may be regarded as a regularized intermediate equation.

The above argument is somewhat formal and in fact not necessary for the two-body BFKL problem since one can always restrict oneself to functions which satisfy $\tilde{f}(x_\perp,x_\perp) = 0$. However, for the odderon problem, or more generally, for n-reggeon problem ($n \geq 3$), it is clear that one cannot always attain $\tilde{f}_\tau(x_\perp,x_\perp,z_\perp,\cdots) = 0$. The generalization of Eq. (35) for the odderon problem would be

$$\begin{aligned}\frac{\partial}{\partial\tau}f_\tau(x_\perp,y_\perp,z_\perp) = \frac{\bar{\alpha}_s}{4\pi}\int d^2w_\perp \frac{(x_\perp-y_\perp)^2}{(x_\perp-w_\perp)^2(y_\perp-w_\perp)^2}\Big(f_\tau(x_\perp,w_\perp,z_\perp)\\ +f_\tau(w_\perp,y_\perp,z_\perp)-f_\tau(x_\perp,y_\perp,z_\perp)-f_\tau(w_\perp,w_\perp,z_\perp)\Big)\\ +(2\text{ cyclic permutations})\,.\end{aligned} \tag{37}$$

This is the Fourier transform of the BKP equation (up to irrelevant terms containing a delta function) without assuming anything about the property of $f_\tau(x_\perp,y_\perp,z_\perp)$. Again this is the same as the equation for $d^{abc}\langle\alpha^a_x\alpha^b_y\alpha^c_z\rangle_\tau$, provided one uses the dipole JIMWLK Hamiltonian. Since Eq. (23) is a particular linear combination of the equation for $d^{abc}\langle\alpha^a_x\alpha^b_y\alpha^c_z\rangle_\tau$, this establishes the equivalence of the BKP equation and our Eq. (23). The problem of the last term $f_\tau(w_\perp, w_\perp, z_\perp)$ is that it breaks the holomorphic separability of the BKP (or BFKL) Hamiltonian which is the starting point of the construction of [5, 6]. One may restrict oneself to solutions which have the property $f_\tau(w_\perp, w_\perp, z_\perp) = 0$ to regain holomorphic separability. However, such solutions are less general. *A priori*, there is no reason to expect that $\langle B(x_\perp, z_\perp, z_\perp)\rangle_\tau = 0$. Instead, we have the general relation Eq. (22).

$$\langle B(x_\perp, z_\perp, z_\perp)\rangle_\tau = \langle O(x_\perp, z_\perp)\rangle_\tau \,. \tag{38}$$

5. Summary

We have shown that the CGC formalism contains the BKP dynamics which describes the odderon exchange in QCD. The master equation, the JIMWLK equation systematically produces both linear and nonlinear equations for different processes. For the dipole–color glass scattering our results agree with the previous results by Kovchegov *et al.* obtained in the color dipole approach. The advantage of the CGC formalism is that it is straightforward to treat the three-quark–color glass scattering within the same framework. Our formalism deals with the evolution equation of gauge invariant amplitudes directly in the coordinate space. This highlights the subtlety involved in Fourier transforming the momentum space BKP equation to the coordinate space. From the viewpoint of Eq. (23), the solutions which vanish at equal points $\langle B(x_\perp, z_\perp, z_\perp)\rangle_\tau = 0$ are very special ones. The equation naturally admits solutions which do not vanish at equal points $\langle B(x_\perp, z_\perp, z_\perp)\rangle_\tau \neq 0$. With this observation, it has been concluded in [11] that the Bartels–Lipatov–Vacca solution [8], which was originally discussed in the context of the dipole scattering, is the physically relevant solution also for the three-quark scattering Eq. (23).

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