Vol. 36 (2005) *ACTA PHYSICA POLONICA B* No 12

NON-PERTURBATIVE YANG–MILLS FROM SUPERSYMMETRY AND STRINGS, OR, IN THE JUNGLES OF STRONG COUPLING*

M. Shifman

William I. Fine Theoretical Physics Institute, University of Minnesota
Minneapolis, MN 55455, USA

(Received September 21, 2005)

I summarize some recent developments in the issue of planar equivalence between supersymmetric Yang–Mills theory and its orbifold/orientifold daughters. This talk is based on works carried out in collaboration with Adi Armoni, Sasha Gorsky and Gabriele Veneziano.

PACS numbers: 11.30.Pl, 11.15.Tk

1. Introduction

Unlike some theories whose relevance to nature is sill a big question mark, quantum chromodynamics and other similar strongly coupled gauge theories will stay with us forever. QCD is a very rich (and quite old) theory supposed to describe the widest range of strong interaction phenomena: from nuclear physics to Regge behavior at large E, from color confinement to quark–gluon matter at high temperatures/densities (neutron stars), the vastest horizons of hadronic physics: chiral dynamics, glueballs, exotics, light and heavy quarkonia and mixtures of thereof, exclusive and inclusive phenomena, interplay between strong forces and weak interactions, ... That's why I do not expect theoretical developments to culminate in full analytic solution of QCD. And yet, in spite of its age, advances in QCD continue. The most recent advances are due to proliferation of supersymmetry and string-inspired methods. I will summarize some recent results which, to my mind, are most promising, and pose some stimulating questions.

* Based on invited talks delivered at the Planck-05/Mohapatra-Fest, ICTP, Trieste, May 23–28, 2005, and PASCOS-05, Gyeongju, Korea, May 30–June 4, 2005 and the Cracow School of Theoretical Physics, Zakopane, Poland, June 3–12, 2005.

2. Planar equivalence

The main stumbling block in theoretical understanding of strongly coupled gauge theories is the absence of obvious expansion parameters. A hidden parameter which might serve the purpose, N (the number of colors) was suggested by 't Hooft long ago [1]. It governs expansion in topologies. The leading order at $N \to \infty$ corresponds to planar topology. Recently it was realized [2, 3] that the very same parameter can be used to parameterize deviations of certain non-supersymmetric theories, close relatives of QCD, from supersymmetric Yang–Mills (SYM) theory. These relatives — they go under the name orbifold and orientifold gauge field theories — are obtained from supersymmetric gluodynamics by means of orbifolding and orientifolding, procedures well known in string theory. For our purposes we do not need to delve in string-theoretic aspects since all results we need can be readily formulated in field-theoretic language. They are shown in Fig. 1.

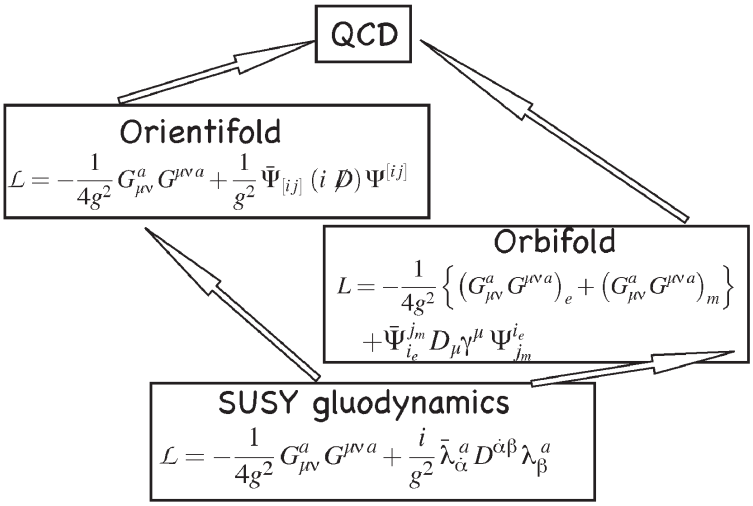

Fig. 1. Orbifold and orientifold (orienti-A) daughters are obtained from SYM theory by orientifolding and orbifolding, respectively.

SYM theory is assumed to be SU($2N$) or SU(N) gauge theory. The first case is pertinent to Z_2 orbifolding, the second to orientifolding. Then the Z_2 orbifold daughter[1] has the gauge group SU(N)$\times$ SU(N), and the fermion sector consisting of one bifundamental Dirac spinor. The gauge coupling of

[1] Z_n orbifold daughters with $n > 2$ will not be discussed since these theories are chiral and, hence, cannot be considered as close relatives of QCD. Two other lines of research that are marginally related to my main topic are left aside, namely *(i)* orbifold pairs with both theories, parent and daughter, supersymmetric; and *(ii)* orbifolding with one or more compact spatial dimensions. In both cases there are special circumstances whose consideration will lead me far astray.

the orbifold daughter is adjusted as follows

$$g_D^2 = 2\, g_P^2\,, \tag{1}$$

where the subscripts D and P mark the daughter and parent theories. For historic reasons the first SU(N) is often referred to as "electric" (and marked by e), while the second as "magnetic" (and marked by m).

The orientifold gauge theory is even closer to QCD. Its gauge group is SU(N), the same as in the parent SU(N) SYM theory. The gauge couplings of the parent SYM and its orientifold daughter are identical too. The fermion sector consists of one Dirac fermion either in two-index symmetric (orienti-S) or two-index antisymmetric (orienti-A) representation of $\mathrm{SU}(N)_{\mathrm{color}}$. In fact, at $N = 3$ orienti-A is identical to one-flavor QCD.

Both daughter theories, in the limit $N \to \infty$, were shown [3] to be perturbatively equivalent to their parent, supersymmetric gluodynamics. In other words, all planar Feynman graphs of the daughter theories that can be mapped onto the parent theory are in one-to-one correspondence with the parent planar graphs.

This remarkable observation motivated [2] a non-perturbative orbifold (NPO) conjecture, according to which the above planar equivalence holds beyond perturbation theory, non-perturbatively, in a common sector, *i.e.* the sector of both theories, orbifold and SYM, which admits mapping in both directions.

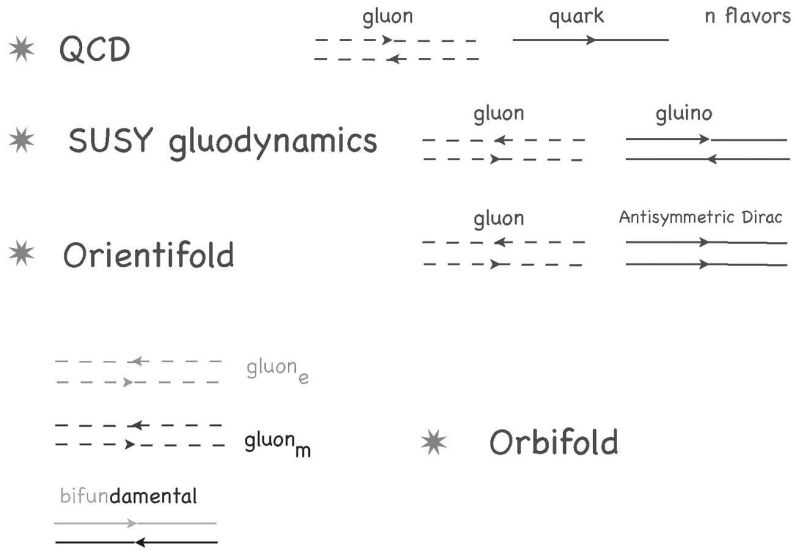

Fig. 2. Feynman rules in QCD, SYM theory and orbifold and orientifold (orienti-A) daughters in the 't Hooft notation.

As we will see shortly, radical distinctions in the vacuum structure of orbifold and SYM theories make NPO planar equivalence unlikely. At the same time, planar equivalence between orientifold daughter and its supersymmetric parent was solidly demonstrated, see [4,5] and the review paper [6], with quite a few far-reaching consequences that ensued almost immediately. Corresponding results were reported a year ago at various conferences, and I will not discuss them now (except for a few marginal remarks), referring the interested reader to [6]. Instead, I will dwell on the Z_2 orbifold daughter, a theory whose dynamics is rich and interesting irrespective of its (highly probable) non-perturbative non-equivalence to SYM theory.

Concluding this section I would like to display the 't Hooft large-N diagrammar for supersymmetric gluodynamics and its daughters (Fig. 2) which makes the perturbative proof of planar equivalence almost self-evident.

3. Non-perturbative planar equivalence: what does it mean?

As we will see shortly, SYM theory and its orbifold daughter have distinct vacuum structures even at $N = \infty$. The number of underlying (short-distance) degrees of freedom is also different. Under the circumstances one should carefully define what is expected to be equivalent. To calibrate the answer to this question it is instructive to consider an example where the answer is known, namely, let us compare SYM theory with *itself* in two cases:

$$N_c \;=\; 2N\,, \qquad g_P^2 \equiv g^2\,, \tag{2}$$

$$N_c \;=\; N\,, \qquad g_D^2 \equiv 2g^2\,. \tag{3}$$

The 't Hooft coupling in both cases is one and the same, $\lambda = 2Ng^2$, which entails, in turn, the equality of the dynamical scales, $\Lambda_P = \Lambda_D$. We will refer to the theories (2), (3) as parent/daughter. Having one and the same 't Hooft coupling, these theories are planar equivalent. This is as good as it gets, indeed.

Note that for the purpose of infrared regularization we will introduce a small gluino mass term $-(m/g^2)\mathrm{Tr}\lambda^2+$ h.c. The mass parameter m is assumed to be real and positive (a phase can be introduced through the θ term). The value of m must be the same in both theories since this parameter is physically observable.

Now, both theories are confining and have coincident spectra of composite bosons, up to $O(1/N^2)$ corrections. More exactly, we compare here excitation spectra over vacua which can be mapped one onto another, for instance, those characterized by real (and negative) gluino condensate $\langle \mathrm{Tr}\,\lambda^2\rangle$.

One must be careful since the parent theory has $2N$ vacua while the daughter one N vacua. In fact, the boson spectra in adjacent vacua differ only by terms $O(m/N^2)$ (one should remember that at $m \neq 0$ the true vacuum is the one in which $\mathrm{Tr}\lambda^2$ is real and negative; others are quasistable, with an exponentially suppressed decay rate [7], $\Gamma \sim \exp(-N^4)$). Only for distant vacua, *e.g.* those with negative and positive $\mathrm{Re}\,\langle \mathrm{Tr}\,\lambda^2\rangle$, the spectra are shifted by $O(N^0)$. This fact is related to another similar statement. Switching on $\theta \neq 0$ changes bosonic spectra since at $m \neq 0$ particle masses are θ-dependent. However, changing θ from zero to $\delta\theta \sim 2\pi$ produces an impact on the spectrum suppressed by $1/N^2$. This can be readily seen *e.g.* from the SVZ sum rule [8] analysis at $\theta \neq 0$, for instance, for highly excited two-gluino bosons one can estimate

$$\delta m_n^2 \sim \frac{m\,\Lambda^{-2}}{g^2}\, N_c^{-2}\, \mathrm{Re}\,\langle \mathrm{Tr}\,\lambda^2\rangle\,, \tag{4}$$

where n is the number of the radial excitation, $n \gg 1$. CP odd quantities which might be generated at $\theta \neq 0$ are $O(1/N)$.

The impact of $\theta \neq 0$ becomes of order $O(N^0)$ only if $\delta\theta \sim 2\pi N$.

At the same time, even though the bosonic spectra are planar equivalent, the vacuum energy densities are not equal. The vacuum energy density[2]

$$\mathcal{E} = m g^{-2}\,\langle \mathrm{Tr}\,\lambda^2\rangle \sim N^2 \tag{5}$$

is sensitive to the overall number of the fundamental degrees of freedom. It is obvious that we cannot demand the equality of the vacuum energies in the parent/daughter theories. Equation (5) is fully compatible with the fact that the m dependence of the composite boson masses is identical in the parent and daughter theories. This can be seen from OPE for the two-point function

$$\left\langle g^{-2}\,\mathrm{Tr}\,\bar\lambda_{\dot\alpha}\lambda_\alpha,\ g^{-2}\,\mathrm{Tr}\,\bar\lambda_{\dot\beta}\lambda_\beta \right\rangle\,, \tag{6}$$

which scales as N^2. The mass correction to the above two-point function is given by (5). The relative weight of the leading (unit) operator and the mass correction term is N-independent.

Returning to the θ-dependence, the coincidence of the parent/daughter boson spectra can be maintained provided

$$\theta_D = \frac{1}{2}\,\theta_P\,. \tag{7}$$

As was mentioned, for $\theta = O(1)$ the vacuum angle effects in the spectra are irrelevant as they are suppressed by $1/N^2$. However, one can consider

[2] Tr is normalized in such a way that $\mathrm{Tr}\,(T^a T^b) = \delta^{ab}$ where T^a are color generators in the fundamental representation.

$\theta \sim CN2\pi$ where C is small numerically but not parametrically[3]. Then Eq. (7) follows from holomorphic dependence of appropriate quantities on complexified coupling constant which is dictated, in turn, by supersymmetry of the parent/daughter theories. I will further comment on this issue in Sec. 7.

The θ term has no impact whatsoever on perturbation theory. It is not seen at all in perturbation theory. Therefore, perturbative proofs of planar equivalence have nothing to say regarding this aspect. On the other hand, non-perturbative quantities, such as the gluino condensate, do carry a θ dependence which imposes the above identification of the parent/daughter vacuum angles. The Dashen points [9] in the parent/daughter theories do not match each other, as a consequence of a mismatch in the vacuum multiplicities. It is worth emphasizing that *(i)* in discussing the θ evolution we have to stick, at $\theta \geq \pi$, to a "wrong" (quasi-stable) vacuum which will ensure a smooth evolution; *(ii)* the Dashen phenomenon is then irrelevant in the leading in N approximation.

Besides particle excitations both theories have domain-wall excitations. The tensions of the elementary walls and their multiplicities scale as N and are, therefore, different in the parent/daughter theories. A common factor here is that all domain walls interpolating between vacua with distinct values of the gluino condensate are stable.

4. A refinement of the proof of planar equivalence for orientifold daughter

A certain aspect in the previous analysis of non-perturbative planar equivalence between the SYM parent and orietifold daughter was treated at an intuitive level. This gap is closed in a refined proof [5] making use of the fermion loop expansion. The equivalence extends to θ effects, *e.g.* the topological susceptibility — a feature which is certainly lost in the case of the orbifold daughter. This is in one-to-one correspondence with the fact that the vacuum structure of the orientifold daughter at $N \to \infty$ is identical to that of the parent theory. In particular, there is an exact matching of the Dashen transitions.

5. The role of Z_2 invariance in the orbifold theory

The Lagrangian of the orbifold theory,

$$\mathcal{L} = -\frac{1}{4g^2}\left[\left(G^a_{\mu\nu}G^{\mu\nu\,,a}\right)_e + \left(G^a_{\mu\nu}G^{\mu\nu\,,a}\right)_m\right] + \frac{1}{g^2}\left[\overline{\lambda^{i_e}_{j_m}}\,(i\not{D}\lambda)^{i_e}_{j_m} + \overline{\lambda^{i_m}_{j_e}}\,(i\not{D}\lambda)^{i_m}_{j_e}\right], \tag{8}$$

[3] If $C \sim 1$ one looses the vacuum quasi-stability.

has an obvious discrete Z_2 symmetry with respect to the interchange $e \leftrightarrow m$. (Note that in Fig. 1 two Weyl spinors,

$$\lambda^{i_e}_{j_m} \qquad \text{and} \qquad \overline{\lambda^{j_m}_{i_e}}\,,$$

are combined in one Dirac spinor. For a while I will omit the subscript D in the gauge coupling. One should remember, however, that $g_D^2 = 2g_P^2$, see Eq. (1).)

A crucial physical question is whether or not this Z_2 symmetry is spontaneously broken. If it is dynamically broken, the number of vacua is doubled. As a manifestation of the discrete symmetry breaking, domain walls must emerge, which interpolate between the vacua related by the broken Z_2. The corresponding order parameters are Z_2 odd. For historical reasons, the Z_2 odd sector of the theory is referred to as a "twisted sector".

If the above Z_2 is not broken, the spectrum of the theory in each vacuum can be classified with regards to Z_2. For instance, Z_2 even particles do not mix with Z_2 odd, all domain walls of the unbroken theory are Z_2 symmetric, and so on.

The fate of nonperturbative planar equivalence between the orbifold theory and its supersymmetric parent is inseparable from the fate of Z_2. As was shown in [10–12], if Z_2 is unbroken, perturbative planar equivalence extends to the nonperturbative level. In the opposite case of the dynamical Z_2 breaking, planar equivalence is not expected to survive at the nonperturbative level. A shift of the vacuum energy from zero ensues: the vacuum energy density is expected to become negative, see Secs. 6, 8 and 10. Other immediately observable consequences refer to the particle spectrum. Multiple (parity/spin) degeneracies which would be inherited from supersymmetric Yang–Mills under planar equivalence, will be lifted.

To see that this is indeed the case suffice it to note that if the twisted scalar field[4]

$$T \equiv \left(\mathrm{Tr}\, G_e^2 - \mathrm{Tr}\, G_m^2\right) \tag{9}$$

develops a Z_2-breaking vacuum expectation value, while its pseudoscalar counterpart

$$\tilde{T} \equiv \left(\mathrm{Tr}\, G_e\tilde{G}_e - \mathrm{Tr}\, G_m\tilde{G}_m\right) \tag{10}$$

[4] Here and below the normalization of traces is such that

$$\mathrm{Tr}\, G^2 = \sum_{a=1}^{4N^2} G^a_{\mu\nu}\, G^{\mu\nu\ a}\,, \qquad \mathrm{Tr}\,(G^2)_e = \sum_{a=1}^{N^2} \left(G^a_{\mu\nu}\, G^{\mu\nu\ a}\right)_e\,,$$

and so on.

does not, this will be transmitted to the untwisted sector *e.g.* through a term

$$\delta\mathcal{L} = \frac{1}{N}\,\tau^2\,\sigma\,, \tag{11}$$

where τ is a meson for which the interpolating field is T, while σ is the dilaton (the corresponding interpolating operator is $S = \mathrm{Tr}\,G_e^2 + \mathrm{Tr}\,G_m^2$). A vacuum expectation value $\langle\tau\rangle \sim N$ will entail a shift in $\langle\sigma\rangle \sim N$, which will lead, in turn, to a shift in the σ mass of order $O(N^0)$, not accompanied by a corresponding shift in the mass of the untwisted pseudoscalar meson.

Thus, understanding dynamics governing the Z_2 symmetry of the orbifold model is a key to solving the issue of nonperturbative planar equivalence in the case at hand. What does today's theory tell us on that?

6. The mode of Z_2 implementation

String theory prompts us [13, 14] that in the non-supersymmetric (or $\mathcal{N} = 0$) orbifold daughter of $\mathcal{N} = 4$ SYM theory, the Z_2 symmetry is spontaneously broken above a critical value of the 't Hooft coupling. The orbifold field theory under consideration can be described by a brane configuration of type-0 string theory [15]. Type-0 strings contain a closed-string tachyon mode in the twisted sector. The tachyon couples [14] to the twisted field (9) of the $\mathrm{SU}_e(N)\times\mathrm{SU}_m(N)$ gauge theory. The prediction of string theory [14] is that the perturbative vacuum at $\langle T\rangle = 0$ is unstable. In the *bona fide* vacua a condensate of the form

$$\left\langle \mathrm{Tr}\,G_e^2 - \mathrm{Tr}\,G_m^2 \right\rangle = \pm\Lambda^4 \tag{12}$$

must develop.

Of course, a long way lies between the above string construction and the orbifold field theory specified in Fig. 1 or Eq. (8) *per se*. Therefore, it is natural to address the issue of the spontaneous Z_2 breaking directly in field theory. In Ref. [16] (see also [11], v.1) low-energy theorems were suggested as a tool for proving nonequivalence of the orbifold daughter theories to the parent SYM theory. These theorems become instrumental under the assumption of exact coincidence between the corresponding vacuum condensates. However, as explained in Sec. 3, the vacuum condensate coincidence is not necessary, generally speaking. The above-mentioned low-energy theorems reflect not only the vacuum structure — they are potentially sensitive to the number of fundamental degrees of freedom. This aspect was pointed out in [12]. In passing from the orbifold theory to its parent the number of fundamental degrees of freedom doubles.

Relaxing the requirement of exact coincidence makes the low-energy theorems uninformative: allowing for unequal condensates one concludes that these theorems cannot prove or disprove the Z_2 symmetry breaking.

Another argument suggested in [11] is based on the domain wall dynamics. If Z_2 was unbroken and NPO conjecture valid, the domain walls in the orbifold theory that are inherited from SYM theory would be stable. Apparently, this is not the case. To discuss the issue in more detail I will have to briefly review what is known of the vacuum structure in the orbifold theory (Sec. 7).

Concluding this section, it is instructive to outline a possible scenario of the development of the tachyonic mode coupled to the twisted operator (9). Let us give a mass term m to the fermion field in (8),

$$\mathcal{L}_m = -m\, g^{-2}\, \bar{\Psi}\, \Psi\,. \tag{13}$$

This mass term is obviously Z_2 invariant. We will consider m as a free parameter, keeping the dynamical scale Λ fixed. Then, at $m/\Lambda \to \infty$ the fermion field can be integrated out leading to two disconnected SU(N) gauge theories, electric and magnetic. At finite but large values of m, there is a weak connection between the electric and magnetic theories which can be described by a (local) operator $m^{-4}\,\mathrm{Tr}\, G_e^2\,\mathrm{Tr}\, G_m^2$. The mass-squared matrix of the electric/magnetic scalar glueballs takes the form

$$\mathcal{M}^2 = \begin{pmatrix} \mu_e^2 & \alpha^2 \\ \alpha^2 & \mu_m^2 \end{pmatrix}, \tag{14}$$

where $\mu_e^2 = \mu_m^2 = \mathrm{const}\,\Lambda^2$ and α is a small parameter proportional to m^{-2}. The Z_2 invariance of the theory manifests itself in the fact that $\mu_e^2 = \mu_m^2 \equiv \mu^2$. The eigenvalues of $\mathcal{M}^2$ are $\mu^2 \pm \alpha^2$. The corresponding eigenstates are built of mixtures of the electric and magnetic gluons, with Z_2 parity $+1$ and -1, respectively.

Now, let us diminish m moving towards Λ. This enhances interaction between the electric and magnetic sectors, which no longer can be described by a local operator. If at $m = 0$ the transition matrix element α^2 is larger than the diagonal ones μ^2, a negative eigenvalue in the twisted sector emerges. At a certain critical value of m,

$$m_* \sim \Lambda\,,$$

the Z_2-odd glueball becomes massless, while further decrease of m from m_* to zero makes the corresponding channel tachyonic causing condensation of the operator (9) and a radical vacuum restructuring signifying spontaneous breaking of the Z_2 invariance.

One can illustrate the very same statement in a slightly different language of effective Lagrangians. Indeed, if one approaches the critical value m_*

from the large m side one can describe the vacuum structure by the effective Lagrangian of the type [17]

$$\mathcal{L} = S_e \ln \frac{S_e}{e} + S_m \ln \frac{S_m}{e} + \eta S_e S_m , \tag{15}$$

where $S_{e,m} = \mathrm{Tr}\, G^2_{e,m}$, and I put $\Lambda = 1$. The above Lagrangian is explicitly Z_2 invariant. Of course, it is valid only at $\eta \ll 1$, where the vacuum solution is Z_2 invariant too, $S_e = S_m \approx 1$. Assume that, at a qualitative level, Eq. (15) can be extrapolated to $\eta \sim 1$. Then, at $\eta = e$ the vacuum solution is still Z_2 symmetric, $S_e = S_m = e^{-1}$, but the mass eigenvalue corresponding to $S_e - S_m$ vanishes. Further increase of η leads to Z_2-asymmetric vacuum solutions while the Z_2-symmetric extremum is no more minimum of the potential.

7. Vacuum structure of the orbifold daughter at a glance

The gauge group of the orbifold theory is a direct product of two SU(N)'s. Correspondingly, it has two vacuum angles conjugated to two distinct non-contractible cycles in the space of fields. We will introduce these two vacuum angles as follows:

$$\begin{aligned} \mathcal{L}_\theta \;=\; & \frac{\theta_D}{32\pi^2}\left[\left(G^a_{\mu\nu}\tilde{G}^{\mu\nu\,,a}\right)_e + \left(G^a_{\mu\nu}\tilde{G}^{\mu\nu\,,a}\right)_m\right] \\ & + \frac{\vartheta}{32\pi^2}\left[\left(G^a_{\mu\nu}\tilde{G}^{\mu\nu\,,a}\right)_e - \left(G^a_{\mu\nu}\tilde{G}^{\mu\nu\,,a}\right)_m\right] . \end{aligned} \tag{16}$$

They refer to non-twisted and twisted sectors of the theory, respectively. Since the parent theory has no twisted sector, the NPO conjecture requires $\vartheta = 0$. Let us set $\vartheta = 0$ for the time being, and focus on θ_D.

The order parameter of the parent theory marking its $2N$ vacua, the gluino condensate, is mapped onto the fermion condensate $\bar{\Psi}\frac{1}{2}(1-\gamma_5)\Psi$ in the daughter theory. Note that this operator is Z_2 invariant; hence, its nonvanishing vacuum expectation value is insensitive to the spontaneous breaking of Z_2.

Following the standard line of reasoning, one can conclude that the fermion condensate does develop, and has N distinct values,

$$\left\langle \bar{\Psi}\frac{1}{2}(1-\gamma_5)\Psi \right\rangle = \text{const.}\, N\, \Lambda^3 \exp\left(i\,\frac{2\pi\, k + \theta_D}{N}\right) , \quad k = 1, 2, \ldots, N\,. \tag{17}$$

The N-valuedness of the fermion condensate is in one-to-one correspondence with the dependence on θ_D/N which, in turn, follows from the consideration of the chiral anomaly. Thus, the fermion condensate marks N distinct chirally asymmetric sectors.

In the absence of the fermion mass term, $m = 0$, the vacuum angle θ_D is physically unobservable. Indeed, at $m = 0$ the axial current is classically conserved. The chiral anomaly then allows one to completely rotate away the vacuum angle θ_D. No physically measurable quantity can depend on it. In particular, the vacuum energy is θ_D independent. Only if $m \neq 0$, the vacuum angle θ_D becomes observable.

Note that the dependence of the fermion condensate on θ_D indicated in Eq. (17) and the relation between the parent and daughter vacuum angles (7) are compatible with the θ dependence of the gluino condensate in SU($2N$) SYM theory,

$$\langle \lambda^a_\alpha \lambda^{a,\alpha} \rangle = -6(2N)\, \Lambda^3 \exp\left(i\, \frac{2\pi k + \theta_P}{2N} \right) , \qquad k = 1, \ldots, 2N\,. \tag{18}$$

The g_P^2/g_D^2 ratio, see Eq. (1), also matches. Thus, the fermion condensate of the orbifold theory could have been projected from the parent theory provided that the NPO conjecture was valid.

As an example, I depicted the chiral condensates of the parent/daughter theories at

$$\theta_D = \pi\,, \qquad \theta_P = 2\pi\,, \tag{19}$$

in Fig. 3. $P_{0,\pm 1}$ are the vacua of the SYM theory, while $D_{\pm 1}$ are the vacua of the orbifold theory. Since the vacua reflect the discrete chiral symmetry breaking, $2N$ vacua of SYM theory are degenerate, and so are N vacua of the orbifold theory. Whether or not they are degenerate between themselves, depends on the validity of planar equivalence.

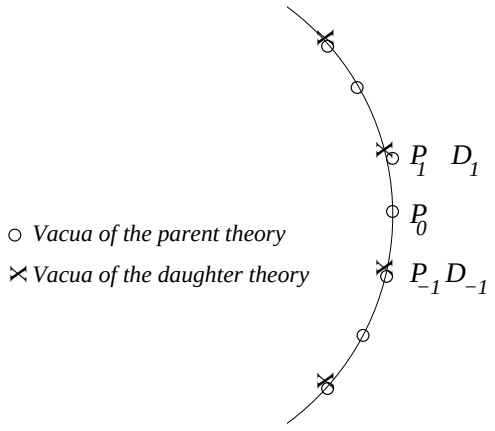

Fig. 3. The vacuum structure in the SU($2N$) SYM theory and its SU(N)×SU(N) orbifold daughter. Shown is the complex plane of the order parameters, the gluino condensate $-\langle \lambda^a_\alpha \lambda^{a,\alpha} \rangle$ and the fermion condensate $-\left\langle \bar{\Psi} \frac{1}{2} (1 - \gamma_5) \Psi \right\rangle$, respectively.

Introducing $m \neq 0$ one lifts the vacuum degeneracy. For instance, for real and positive m the vacua $P_{\pm 1}$ are excited (quasistable) because

$$\mathcal{E}_{P_{\pm 1}} > \mathcal{E}_{P_0}\,.$$

At the same time, the daughter theory has two-fold degeneracy,

$$\mathcal{E}_{D_{+1}} = \mathcal{E}_{D_{-1}}\,,$$

a phenomenon well-known at $\theta = \pi$, the so-called Dashen phenomenon [9]. As was explained in Sec. 3, θ-dependent effects are suppressed by $1/N$.

The fermion condensate (17) is a good order parameter for the chiral symmetry breaking. It cannot serve, however, as an order parameter for the Z_2 breaking. In the orbifold theory with the spontaneously broken Z_2 the fermion condensate does not differentiate those vacua which are connected to each other by Z_2 because it is Z_2-even. We must supplement (17) by a Z_2-odd expectation value of (9). This vacuum expectation value (VEV) is dichotomic. The fermion condensate (17) in conjunction with $\langle T\rangle = \pm\Lambda^4$ fully identifies each degenerate vacuum of the orbifold theory. If spontaneous breaking of the discrete chiral symmetry produces N vacua, this number is doubled in the process of Z_2 breaking. Somewhat symbolically, the corresponding vacuum structure is presented in Fig. 4. The angular coordinate represents the phase of (17), while the radial coordinate can take two distinct values representing the dichotomic parameter $\langle T\rangle$.

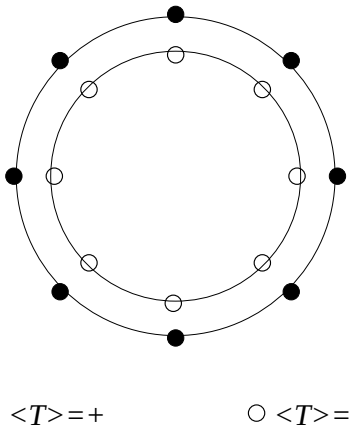

Fig. 4. The vacuum structure of the SU(8)×SU(8) orbifold theory reflecting spontaneous breaking of the Z_2 symmetry.

The orbifold theory has a remarkable feature: because of its proven *perturbative* planar equivalence to SYM theory, the vacuum energy density — certainly a Z_2 symmetric parameter — can and does play the role of the order

parameter for Z_2 breaking. The vacuum energy density is proportional to the vacuum expectation value of the Z_2-even gluon operator $\langle \mathrm{Tr}\, G_e^2 + \mathrm{Tr}\, G_m^2 \rangle$. Indeed, this operator is related, in turn, to the total energy-momentum tensor of the theory,

$$\theta_\mu^\mu = -\frac{3N}{32\pi^2} \sum_{\ell=e,m} \left(G_{\mu\nu}^a \, G_{\mu\nu}^a \right)_\ell \, . \tag{20}$$

Ever since the gluon condensate was introduced in non-Abelian gauge theories [8] people tried to identify it as an order parameter. Nobody succeeded. The orbifold theory is the one where it is an order parameter, albeit in a special sense.

If Z_2 is unbroken, the orbifold theory is perfectly equivalent at $N \to \infty$ to SYM theory, and then $\langle \mathrm{Tr}\, G_e^2 + \mathrm{Tr}\, G_m^2 \rangle$ reduces to $\langle \mathrm{Tr}\, G_{\mathrm{SYM}}^2 \rangle$. The latter condensate vanishes due to supersymmetry of the parent theory. Hence, the Z_2 symmetric vacua in the daughter theory would have vanishing vacuum energy density in the leading order in N.

When the Z_2-symmetric point becomes unstable, the Z_2-asymmetric vacua must have a negative energy density. Equation (20) implies then that in the genuine vacua

$$\langle \mathrm{Tr}\, G_e^2 + \mathrm{Tr}\, G_m^2 \rangle > 0 \tag{21}$$

at order $O(N^2)$. In this way the gluon condensate acquired the role of a Z_2 breaking order parameter, much in the same way as $\langle \mathrm{Tr}\, G_{\mathrm{SYM}}^2 \rangle$ is the order parameter for supersymmetry breaking in SYM theory.

8. Domain-wall-based argument for Z_2 breaking

In this section we analyze the domain wall dynamics in the Z_2 orbifold theory. Since domain walls are "QCD D-branes" [18] a similarity between the wall dynamics and D-brane dynamics is clear.

Why domain walls? As well-known, domain walls are physical manifestations of spontaneously broken discrete symmetries. Since our consideration aims at exploring the Z_2 breaking in the orbifold daughter theory, addressing domain walls is an adequate maneuver.

To begin with, let me recapitulate the domain wall topic in the parent theory. SU($2N$) SYM theory has BPS domain walls [19] that carry both tension σ and charge Q (per unit area), with $\sigma = Q$. The expressions for the tension and charge can be written as follows [20]:

$$\sigma \;=\; \frac{3(2N)}{32\pi^2} \int\limits_{\mathrm{wall}} dz\, \mathrm{Tr}\, G^2 \, , \tag{22}$$

$$Q \;=\; \frac{3(2N)}{32\pi^2}\int\limits_{\rm wall} dz\,{\rm Tr}\,G\tilde{G}\,, \tag{23}$$

where z is the direction perpendicular to the wall plane. Equation (22) is a consequence of the scale anomaly. The walls interpolating between the adjacent vacua (*e.g.* P_0 and P_1 in Fig. 3) are called elementary, or 1-walls. One can consider bound states of the elementary walls too. These walls interpolate between the vacua i and $i+k$ with $k>1$ and are referred to as k-walls. For instance, the wall interpolating between P_{-1} and P_1 in Fig. 3 is a 2-wall. At $N=\infty$ it is marginally stable, since the tension of the 2-wall is twice the tension of the 1-wall. Although elementary walls do interact via the exchange of glueballs, there is an exact cancellation between the contribution of even- and odd-parity glueballs [20] at $N=\infty$. From the world-sheet theory standpoint, the no-force result is due to the Bose–Fermi degeneracy on the wall. I will return to the world-sheet theory shortly, after a brief remark regarding generalizations of Eqs. (22) and (23) in the orbifold daughter theory.

For the tension and charge of the orbifold theory domain walls one can write [5]

$$\sigma_D \;=\; \frac{3N}{32\pi^2}\int\limits_{\rm wall} dz\,{\rm Tr}\,G_e^2+\frac{3N}{32\pi^2}\int\limits_{\rm wall} dz\,{\rm Tr}\,G_m^2\,, \tag{24}$$

$$Q_D \;=\; \frac{3N}{32\pi^2}\int\limits_{\rm wall} dz\,{\rm Tr}\left(G\tilde{G}\right)_e+\frac{3N}{32\pi^2}\int\limits_{\rm wall} dz\,{\rm Tr}\left(G\tilde{G}\right)_m\,. \tag{25}$$

It is suggestive to think of the domain walls in the orbifold theory as of marginally bound states of fractional "electric" and "magnetic" domain walls, with the following tensions and charges:

$$\begin{aligned}
\sigma_e \;&=\; \frac{3N}{32\pi^2}\int dz\,{\rm Tr}\,G_e^2\,, \qquad & \sigma_m &= \frac{3N}{32\pi^2}\int dz\,{\rm Tr}\,G_m^2\,,\\
Q_e \;&=\; \frac{3N}{32\pi^2}\int dz\,{\rm Tr}\,(G\tilde{G})_e\,, \qquad & Q_m &= \frac{3N}{32\pi^2}\int dz\,{\rm Tr}\,(G\tilde{G})_m\,.
\end{aligned} \tag{26}$$

Assuming unbroken Z_2's *i.e.* $\sigma_e=\sigma_m$, we would get

$$\sigma_{e,m}=\frac{1}{2}\left(\sigma_e+\sigma_m\right)\,, \tag{27}$$

[5] In SYM theory such integrals are well-defined since $\langle G^2\rangle$ vanishes in any supersymmetric vacuum. In the orbifold theory this is not necessarily the case. The integrals in Eq. (24) must be properly regularized.

i.e., a fractional amount of tension, in full analogy with fractional D-branes. Then we would have to conclude that, say, at $k = 2$ two parallel electric domain walls do not interact at $N = \infty$. The same would be valid for the magnetic walls. Unfortunately, the world-sheet theory in the case at hand does not support this conclusion.

I will again start from SU($2N$) SYM theory. The world-sheet theory for k-walls in $\mathcal{N} = 1$ gluodynamics was derived by Acharya and Vafa [21]. It was shown to be a (2+1)-dimensional U(k) theory with level-$2N$ Chern–Simons term and it was shown to have (2+1)-dimensional $\mathcal{N} = 1$ supersymmetry. The action of the theory is

$$S = \int d^3x \left\{ \mathrm{Tr}\left(-\frac{1}{4e^2} F^2 + \frac{2N}{16\pi} \varepsilon^{ijk} A^i F^{jk} + \frac{1}{2} (D_i \Phi)^2 \right) + \text{fermions} \right\} . \quad (28)$$

All fields in the action, including the fermion fields, transform in the adjoint representation of U(k). For definiteness, we will consider a minimal case $k = 2$.

In the orbifold daughter, the world-sheet theory becomes, by virtue of the orbifold procedure, a $\mathrm{U}_e(1) \times \mathrm{U}_m(1)$ gauge theory with a neutral scalar field and "bifundamental" fermions. The same conclusion on the world-sheet theory can be obtained directly through a consideration of type-0 string theory similar to that of Acharya and Vafa. In this case the world-sheet action is

$$\begin{aligned} S \;=\; \int d^3x \Bigg\{ & \sum_{\ell=e,m} \left(-\frac{1}{4e^2} F_\ell^2 + \frac{N}{16\pi} \varepsilon^{ijk} A_\ell^i F_\ell^{jk} + \frac{1}{2} (\partial_i \Phi_\ell)^2 \right) \\ & + \bar{\Psi} \left(\Phi_e - \Phi_m \right) \Psi + \dots \Bigg\} . \end{aligned} \quad (29)$$

The occurrence of the Yukawa coupling $\bar{\Psi} \left(\Phi_e - \Phi_m \right) \Psi$ in the daughter theory, with no counterpart in the parent one, is the fact of a special importance.

One can interpret the above expression as follows. The daughter wall is a sum of the electric and magnetic walls that interact with each other via the bifundamental fermions. The electric branes can be separated from the magnetic branes as is seen from the fact that the Yukawa term $\bar{\Psi} \left(\Phi_e - \Phi_m \right) \Psi$ in the action (29) can make the bifundamental fermion massive. The vacuum expectation values

$$\langle \Phi_e \rangle = v_e \, , \qquad \langle \Phi_m \rangle = v_m \, , \quad (30)$$

which can be chosen to be real are in one-to-one correspondence with the wall separation. If $v_e \neq v_m$ a mass μ for the world-sheet fermions is generated,

$$\mu = v_e - v_m \, . \quad (31)$$

At $\mu \to \infty$ the fermions decouple — we have two decoupled U(1) theories. The world-sheet theory on the separated electric (or magnetic) domain walls is just a bosonic U(1) gauge theory with a level-N Chern–Simons term. It is not supersymmetric. There is no reason for the wall tension non-renormalization and the no-force statement.

The above conclusion can be backed up by a calculation of the wall repulsion [11,22]. Needless to say, this repulsion is in contradiction with the NPO conjecture.

9. Back to the bulk theory

If the orbifold theory has Z_2-odd vacua, the tachyon field potential must have minima away from the origin, as shown in Fig. 5, *cf.* the last paragraph in Sec. 7. String theory gives us a hint that the point $T = 0$ is unstable. Field theory allows us to say that the potential $V(T)$ is bounded from below since the regime of large expectation values is fully controlled by semiclassical dynamics.

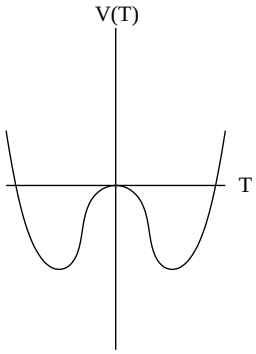

Fig. 5. The tachyon field potential.

From the field-theoretic standpoint it is clear that the only possibility open is that in the *bona fide* vacuum $\langle T \rangle \sim \Lambda^4$. Non-stabilization of tachyons would mean $\langle T \rangle \gg \Lambda^4$, which is ruled out.

In the parent SYM theory with the gauge group SU($2N$), there are $2N$ vacua, with the gaugino condensate as an order parameter, see Fig. 3. The domain walls interpolate between these $2N$ various vacua. In the daughter theory the situation is more complicated. Z_2 breaking implies that each vacuum of the N "false" perturbative vacua splits into two, see Fig. 4.

A scenario of the wall inheritance from the parent to daughter theory we have in mind is as follows. We first pretend that the daughter theory is planar equivalent to SYM, and that the Z_2 symmetry is unbroken. Start from a 2-wall in the parent theory. It will be inherited, as a minimal wall in the daughter theory. This is seen from Fig. 3. We may consider *e.g.*

the wall connecting D_{-1} and D_1 in the daughter (this is a minimal wall in the daughter), *versus* the wall connecting P_{-1} and P_1 in the parent (this is a 2-wall in SYM theory).

In the parent theory two 1-walls comprising the 2-wall do not interact with each other (at $N = \infty$). If we consider them on top of each other, the world-volume theory has U(2) gauge symmetry. However, nobody precludes us from introducing a separation. Then we will have U(1) on each 1-wall, U(1)×U(1) altogether. The tension of each 1-wall is 1/2 of the tension of the 2-wall, it is well-defined and receives no quantum corrections. The fact that the world-volume theory on each 1-wall is supersymmetric is in one-to-one correspondence with the absence of quantum corrections. In the daughter theory the minimal wall splits into one electric and one magnetic repelling each other. (The electric one connects D_{-1} with the would-be vacuum which is a counter-partner of P_0, the magnetic one connects the would-be vacuum which is a counter-partner of P_0 with D_1).

How can one visualize this situation?

In the parent theory we have degenerate minima at all points P_i. In the Z_2 broken orbifold theory these minima become saddle point (still critical points, but unstable). Near every second saddle point two minima develop. Of course, the walls that would be inherited from SYM are all unstable, with tachyonic modes. 1-walls are transformed into electric/magnetic walls of the orbifold theory, which are still unstable and, in fact, decay. Each of them separately could decay only into a "twisted wall" connecting white and adjacent black true vacua. The "untwisted" electric+magnetic wall can decay into a minimal stable wall of the daughter theory which connects two neighboring black vacua or two neighboring white vacua.

10. Why non-perturbative non-equivalence is natural?

I this section I will try to illustrate why a shift of the vacuum energy from zero is expected in the orbifold theory. Needless to say this can only happen if perturbative planar equivalence gives place to non-equivalence at the non-perturbative level. The issue to be discussed here is the vacuum angle dependence, see Eq. (16). In this section I will treat N as a fixed parameter assuming that transition to $N \to \infty$ is smooth, as is the case in pure Yang–Mills theory.

As was mentioned, physical quantities do not depend on θ_D, as this angle can be rotated away. A weak dependence appears if $m \neq 0$, but we will be interested in the limit $m \to 0$. For our present purposes θ_D is irrelevant and can be set at zero.

Unlike θ_D, the second vacuum angle, ϑ, cannot be rotated away: the only axial current of the theory is Z_2 even while the ϑ term in Eq. (16) is Z_2 odd.

Thus, physics must be ϑ dependent even at $m \to 0$. Of course, at the end of the day we want to focus on the $\vartheta = 0$ sector. Nothing precludes us, however, from dealing with $\vartheta \neq 0$ sectors at intermediate stages of our consideration. Knowledge of pure Yang–Mills theory and Yang–Mills theory with massless quarks can be used as a reference frame and a guiding principle.

In pure Yang–Mills theory the vacuum angle reflects a non-trivial topology in the space of fields and the possibility of tunneling [23], a nonperturbative effect which makes the vacuum energy θ dependent and decreases the vacuum energy at $\theta = 0$. Instantons exemplify the tunneling trajectories [24]. Massless quarks suppress instantons (and any other field configurations with nonvanishing topological charges), freeze tunneling and make physics (including the vacuum energy) θ independent. Likewise, in SYM theory instanton does not contribute to the vacuum energy because of the gluino zero modes (an instanton-antiinstanton configuration could contribute but it has a vanishing topological charge and is topologically unstable.)

In the orbifold theory we have two topological charges. Massless bifundamental fermions do suppress tunneling in the direction conjugate to θ_D. That's why physics cannot depend on this parameter. However, the orbifold theory exhibits a new phenomenon: topologically stable instanton-antiinstanton pairs, connected through fermion zero modes, see Fig. 6. The stability is due to the fact that they belong to distinct gauge factors. Therefore, although the overall topological charge (electric + magnetic) vanishes (all fermion zero modes are contracted), still instanton$_e$ cannot annihilate antiinstanton$_m$. The "twisted" topological charge, conjugate to ϑ, is the difference between the electric topological charge and the magnetic one. Non-trivial topology and tunneling with regards to the twisted topological charge is not suppressed by massless fermions.

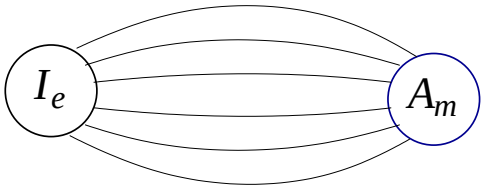

Fig. 6. Topologically stable instanton-antiinstanton pairs in the orbifold theory. Instanton belongs to the electric SU(N) while antiinstanton to the magnetic SU(N).

That's why physics does depend on ϑ. With regards to ϑ effects, the orbifold theory is expected to be similar to pure Yang–Mills, with no massless quarks. The instanton$_e$-antiinstanton$_m$ pair plays the role of the instanton in pure Yang–Mills. In particular, the vacuum energy $\mathcal{E}$ becomes a function of ϑ (more exactly, ϑ/N), and, if so, there is absolutely no reason for $\mathcal{E}(\vartheta) = 0$ at $\vartheta = 0$.

In fact, one is expected to find "vacuum families", of the type described by Witten [25] (see also [7]): a group of $\sim N$ quasistable "vacua" entangled in the process of ϑ evolution and interchanging their position each time ϑ reaches $k\pi$ where k is integer[6]. The issue of the dynamical Z_2 breaking in this language is formulated as follows: at $\vartheta = 0$ each vacuum family contains two degenerate stable vacua connected by a Z_2 transformation. At generic $\vartheta \neq 0$ the Z_2 symmetry of the action is explicitly broken by the ϑ term in Eq. (16).

11. Conclusions

Examples of cross-fertilization between string theories and gauge field theories are abundant. The topic of planar equivalence between supersymmetric and non-supersymmetric gauge theories emerged in this way. In the recent years it produced quite a few spectacular results and stimulated various activities in diverse directions. Two classes of non-supersymmetric models were identified as daughter theories: orbifold and orientifold. Planar equivalence is valid for both at the perturbative level.

In this talk I tried to summarize recent nonperturbative analyses of the orbifold theories. It was found, beyond reasonable doubt, that the Z_2 symmetry of the Z_2 orbifolds is the key to nonperturbative planar equivalence. If it is not dynamically broken, planar equivalence must extend to the nonperturbative level. The opposite is also true: spontaneous breaking of Z_2 entails a nonvanishing vacuum energy and a failure of planar equivalence. I discussed arguments in favor of nonperturbative nonequivalence such as domain wall dynamics and ϑ dependence. Unfortunately, there is no ironclad proof of the statement. At a certain point, low-energy theorems seemed to provide such a proof. It turned out, however, that they may or may not be relevant since they are sensitive not only to the vacuum structure of the parent/daughter theories, but also to the number of the fundamental degrees of freedom which is different in the parent/daughter theories.

In this sense, situation with the orientifold daughter theories is much more favorable. Nonperturbative planar equivalence certainly does hold for the orientifold theories. Why they are better than their orbifold cousins?

String theorists are familiar with this phenomenon. Type-II strings on orbifold singularities of the form C^3/Z_n, or type-0 strings always contain a tachyon in the twisted sector (and fractional branes).

For orientifold theories the situation is conceptually different. This nonsupersymmetric gauge theory has no twisted sector and, in particular, it does not contain fractional domain walls; hence, it is guaranteed that the theory inherits its vacua from the SUSY parent.

[6] This is in addition to N chiral sectors labeled by $\langle\bar{\Psi}\frac{1}{2}(1-\gamma_5)\Psi\rangle$. Note that the first crossover Dashen point is at $\vartheta = \pi/2$.

Similarly, the candidate for a string dual of the orientifold theory — Sagnotti's type-0′ model [26] — contains no tachyon since it was projected out by orientifolding.

The orientifold theory is closer to QCD. On the other hand, the orbifold theory has rich internal dynamics presenting, in a sense, a hybrid between QCD with massless quarks and pure Yang–Mills. Even though its planar equivalence to SYM theory is highly unlikely, it is an alluring target for future studies.

I am grateful to Adi Armoni and Sasha Gorsky for valuable comments on the manuscript. This work was supported in part by DOE grant DE-FG02 -94ER408.

REFERENCES

[1] G. 't Hooft, *Nucl. Phys.* **B72**, 461 (1974).

[2] M.J. Strassler, `hep-th/0104032`. This work builds on Refs. [3].

[3] S. Kachru, E. Silverstein, *Phys. Rev. Lett.* **80**, 4855 (1998); A.E. Lawrence, N. Nekrasov, C. Vafa, *Nucl. Phys.* **B533**, 199 (1998); M. Bershadsky, Z. Kakushadze, C. Vafa, *Nucl. Phys.* **B523**, 59 (1998); M. Bershadsky, A. Johansen, *Nucl. Phys.* **B536**, 141 (1998); Z. Kakushadze, *Nucl. Phys.* **B529**, 157 (1998); *Phys. Rev.* **D58**, 106003 (1998); M. Schmaltz, *Phys. Rev.* **D59**, 105018 (1999); A. Armoni, B. Kol, *J. High Energy Phys.* **9907**, 011 (1999).

[4] A. Armoni, M. Shifman, G. Veneziano, *Nucl. Phys.* **B667**, 170 (2003); *Phys. Rev. Lett.* **91**, 191601 (2003); *Phys. Lett.* **B579**, 384 (2004).

[5] A. Armoni, M. Shifman, G. Veneziano, *Phys. Rev.* **D71**, 045015 (2005).

[6] A. Armoni, M. Shifman, G. Veneziano, *"From Super-Yang–Mills Theory to QCD: Planar Equivalence and Its Implications,"* in *From Fields to Strings: Circumnavigating Theoretical Physics*, Eds. M. Shifman, J. Wheater and A. Vainshtein, World Scientific, Singapore 2004, Vol. 1, p. 353.

[7] M.A. Shifman, *Phys. Rev.* **D59**, 021501 (1999).

[8] M.A. Shifman, A.I. Vainshtein, V.I. Zakharov, *Nucl. Phys.* **B147**, 385; 448 (1979).

[9] R.F. Dashen, *Phys. Rev.* **D3**, 1879 (1971).

[10] P. Kovtun, M. Unsal, L.G. Yaffe, *J. High Energy Phys.* **0507**, 008 (2005).

[11] A. Armoni, A. Gorsky, M. Shifman, `hep-th/0505022`.

[12] P. Kovtun, M. Unsal, L.G. Yaffe, `hep-th/0505075`.

[13] I.R. Klebanov, A.A. Tseytlin, *Nucl. Phys.* **B546**, 155 (1999).

[14] I.R. Klebanov, A.A. Tseytlin, *J. High Energy Phys.* **9903**, 015 (1999); I.R. Klebanov, *Phys. Lett.* **B466**, 166 (1999).

[15] A. Armoni, B. Kol, *J. High Energy Phys.* **9907**, 011 (1999).

[16] A. Gorsky, M. Shifman, *Phys. Rev.* **D67**, 022003 (2003).

[17] A.A. Migdal, M.A. Shifman, *Phys. Lett.* **B114**, 445 (1982).

[18] E. Witten, *Nucl. Phys.* **B507**, 658 (1997).

[19] G.R. Dvali, M.A. Shifman, *Phys. Lett.* **B396**, 64 (1997), Erratum **B407**, 452 (1997).

[20] A. Armoni, M. Shifman, *Nucl. Phys.* **B670**, 148 (2003).

[21] B.S. Acharya, C. Vafa, `hep-th/0103011`.

[22] K. Zarembo, *Phys. Lett.* **B462**, 70 (1999); A.A. Tseytlin, K. Zarembo, *Phys. Lett.* **B457**, 77 (1999).

[23] R. Jackiw, C. Rebbi, *Phys. Rev. Lett.* **37**, 172 (1976); C.G. Callan, R.F. Dashen, D.J. Gross, *Phys. Lett.* **B63**, 334 (1976).

[24] A.A. Belavin, A.M. Polyakov, A.S. Shvarts, Y.S. Tyupkin, *Phys. Lett.* **B59**, 85 (1975).

[25] E. Witten, *Phys. Rev. Lett.* **81**, 2862 (1998).

[26] A. Sagnotti, `hep-th/9509080`; *Nucl. Phys. Proc. Suppl.* **56B**, 332 (1997).