PHASES OF $\mathcal{N} = 1$ THEORIES AND FACTORIZATION OF SEIBERG–WITTEN CURVES*

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In this talk I review the structure of vacua of $\mathcal{N} = 2$ theories broken down to $\mathcal{N} = 1$ and it's link with factorization of Seiberg–Witten curves. After an introduction to the structure of vacua in various supersymmetric gauge theories, I discuss the use of the exact factorization solution to identify different dual descriptions of the same physics and to count the number of connected domains in the space of $\mathcal{N} = 1$ vacua.

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1. Introduction

Supersymmetric gauge theories serve as a laboratory for studying nonperturbative physics of gauge theories. Thanks to the huge simplifications and constraints on the possible behavior of these theories due to the additional (super-)symmetry present, it is possible to obtain exact results and obtain fully nonperturbative answers to at least some questions.

Remarkable progress has been made in exploring the structure of vacua of various supersymmetric gauge theories. Since the early progress on $\mathcal{N} = 1$ SYM theories in the eighties (see *e.g.* the review talk [1] and references therein), further elaborated by emphasizing the constraints of holomorphicity in the nineties [2] a very complete picture was obtained in the case of $\mathcal{N} = 2$ theories [3, 4] bringing together such mathematical structures as algebraic curves and integrable systems. More recently string theory constructions led to progress in studying $\mathcal{N} = 2$ theories broken down to $\mathcal{N} = 1$ through an addition of a superpotential for the adjoint superfield in the original $\mathcal{N} = 2$ theory [5], and a remarkable link with random matrix theory [6]. Later this link was understood purely in field theoretic terms [7–9].

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Later [10, 12] it was found that the structure of the $\mathcal{N}=1$ vacua where the gauge group is broken down to $\mathrm{U}(N_1)\times\mathrm{U}(N_2)$ is surprisingly complicated and they lie in connected domains, whose number jumps widely from one value of $N_\mathrm{c}=N_1+N_2$ to another.

In this talk I would like to give an introduction to these issues and show how one can study the structure of these vacua a.k.a. "the phases of $\mathcal{N}=1$ theories" in a systematic fashion using an exact general solution for the factorization of Seiberg–Witten curves in the above situation [13].

The plan of this talk is as follows. First I would like to present the structure of vacua for a range of SYM starting from pure $\mathcal{N}=1$ through $\mathcal{N}=2$ to $\mathcal{N}=2$ broken down to $\mathcal{N}=1$. Then I discuss the interrelation between the $\mathcal{N}=1$ vacua and factorization of Seiberg–Witten curves, in Sec. 4 I review the solution of the factorization problem given in [13] and proceed to apply it to determine connected domains of $\mathcal{N}=1$ vacua. I close with a summary and outlook.

2. The vacua of supersymmetric gauge theories

In this section I will briefly review the structure of the vacua of some supersymmetric gauge theories starting from the simplest case and ending on the theories which are the focus of this talk.

2.1. Pure $\mathcal{N}=1$ Supersymmetric Yang–Mills (SYM)

Pure $\mathcal{N}=1$ SYM is the simplest supersymmetric gauge theory. Apart from the ordinary Yang–Mills field, it contains just a minimally coupled fermion in the *adjoint* representation. This fermion is called "the gaugino" or "gluino". The SYM Lagrangian is just

$$L=-\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu}+\frac{1}{g^2}\bar{\lambda}^a iD\lambda^a+i\frac{\theta}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a\mu\nu}\,, \tag{1}$$

where we included also the topological θ-term.

Let us now explore the symmetries of this theory. On the classical level (1) is invariant under a U(1) R-symmetry which only transforms the gaugino

$$\lambda^a\rightarrow e^{i\alpha}\lambda^a\,. \tag{2}$$

This symmetry is broken on the quantum level by an anomaly. However, it may be compensated by a shift in the θ angle: $\theta\rightarrow\theta+2N_\mathrm{c}\alpha$. Then since a shift of θ by 2π gives the same theory one is left with a residual discrete $\mathbb{Z}_{2N_\mathrm{c}}$ symmetry.

It turns out that in this theory the gaugino condenses $\langle\lambda\lambda\rangle = \Lambda^3$. This vacuum expectation value (VEV) breaks down $\mathbb{Z}_{2N_c}$ down to $\mathbb{Z}_2$. The remaining N_c transformations move between N_c vacua characterized by

$$\langle\lambda\lambda\rangle = \Lambda^3 e^{i\frac{k}{N_c}}\,, \qquad\qquad k = 0\ldots N_c - 1\,. \tag{3}$$

So a $U(N_c)$ SYM has N_c discrete vacua labeled by $k = 0\ldots N_c - 1$. Around each such vacuum the $SU(N_c)$ part of the theory will develop a mass gap, and one will be left with a massless $U(1)$.

Analogous reasoning for a SYM theory with gauge group $U(N_1)\times U(N_2)$ leads to $N_1 N_2$ vacua labeled by two integers $k_1 = 0\ldots N_1 - 1$ and $k_2 = 0\ldots N_2 - 1$. The massless degrees of freedom will be $U(1)^2$.

2.2. $\mathcal{N}=2$ SYM and Seiberg–Witten curves

The $\mathcal{N}=2$ SYM is just the $\mathcal{N}=1$ SYM together with an adjoint chiral superfield (containing an adjoint scalar ϕ, and fermionic partner). The vacuum condition is just that ϕ can be diagonalized, so one has an N_c dimensional moduli space of vacua. These can be parametrized by N_c complex parameters

$$u_p \equiv \left\langle \frac{1}{p}\mathrm{Tr}\,\phi^p \right\rangle = \frac{1}{p}\sum_{i=1}^{N_c} x_i^p\,, \qquad\qquad p = 1\ldots N_c\,. \tag{4}$$

Seiberg and Witten found [3, 4] that all the low energy properties of the theory are encoded in the geometry of a certain Riemann surface called the Seiberg–Witten curve. I will now review this some features of this construction.

Around a generic vacuum (a generic point in the moduli space) one will have $U(1)^{N_c}$ low energy effective theory. Its couplings are encoded in the geometry of the associated Seiberg–Witten curve

$$y^2 = P^2_{N_c}(x;\{u_p\}) - 4\Lambda^{2N_c}\,, \tag{5}$$

where Λ is the scale of the theory while $P_{N_c}(x;\{u_p\})$ is a polynomial of order N_c given by

$$P^2_{N_c}(x;\{u_p\}) = (x - x_1)(x - x_2)\ldots(x - x_{N_c})\,. \tag{6}$$

A lot of other physical properties of the theory are encoded in the Seiberg–Witten curve (5). In particular there are $N_c - 1$ species of monopoles which are generically massive. They become massless at special points in the moduli space of vacua where the u_p's are tuned so that the right-hand side of the

Seiberg–Witten curve (5) has double zeros. We say then that the Seiberg–Witten curve *factorizes*. I will consider below the case when all but one species of monopoles become massless. Then the SW curve can be written as

$$y^2 = P^2 - 4\Lambda^{2N_c} = F_4(x)\, H^2_{N_c-2}(x)\,, \tag{7}$$

where $F_4(x)$ is a polynomial of degree 4. The factorization problem which we will consider below is to find the $\{u_p\}$'s for which the SW curve looks like (7).

A final piece of information that one can extract from the Seiberg–Witten curve (5) is the knowledge of all VEV's for $\mathrm{Tr}\,\phi^k$. This is in fact nontrivial as for $k > N_c$ one has instanton corrections to the classical VEV's (equal to $\sum_i x_i^k$). The answer is conveniently expressed in terms of the meromorphic 1-form

$$\omega = \frac{d}{dx} \log\left(P + \sqrt{P^2 - 4\Lambda^{2N_c}}\right) dx = \frac{P'}{\sqrt{P^2 - 4\Lambda^{2N_c}}}\, dx\,. \tag{8}$$

Then the VEV's are given by

$$\left\langle \mathrm{Tr}\,\phi^k \right\rangle = \mathrm{Res}_{\,x=\infty}\, x^k\, \omega\,. \tag{9}$$

2.3. $\mathcal{N} = 2$ broken down to $\mathcal{N} = 1$

The final theory whose vacuum structure I would like to describe is the $\mathcal{N} = 2$ theory which is broken down to $\mathcal{N} = 1$ through the addition of a superpotential term to the action:

$$\int d^4x \int d^2\theta\, W_{\mathrm{tree}}(\Phi)\,, \tag{10}$$

where W_{tree} is a polynomial

$$W_{\mathrm{tree}}(\Phi) = \sum_p g_p \frac{1}{p} \mathrm{Tr}\, \Phi^p\,. \tag{11}$$

This term breaks $\mathcal{N} = 2$ to $\mathcal{N} = 1$ and, moreover, modifies the vacuum structure of the theory. Here we will just look at the classical picture leaving the discussion of the full quantum description to the next section.

The addition of the superpotential gives a condition for the classical vacua

$$W'_{\mathrm{tree}}(\phi) = 0\,. \tag{12}$$

This means that the eigenvalues of ϕ should be distributed among the extrema of W_{tree}. If all eigenvalues sit at a single extremum, $\mathrm{U}(N_c)$ is unbroken by the VEV of ϕ and at low energies we should have a $U(N_c)$ pure $\mathcal{N}=1$ SYM theory which leads to N_c discrete vacua and leaving a massless U(1).

If N_1 eigenvalues sit at one extremum and the remaining N_2 at another, the gauge group is broken down to a product $U(N_1)\times U(N_2)$ and one is left with $N_1 N_2$ vacua labeled by k_1, k_2. The massless degrees of freedom left after gaugino condensation will be a $U(1)^2$ gauge theory.

This classical picture will have to modified on the quantum level. In particular, as discussed above, one cannot treat the eigenvalues of ϕ as strictly classical quantities (*e.g.* the VEV's of higher powers of ϕ get instanton corrections). In the next section we will reanalyze what happens once we take into account our knowledge of the undeformed $\mathcal{N}=2$ theory encoded in the Seiberg–Witten curve.

3. $\mathcal{N}=1$ vacua and Seiberg–Witten curves

The new ingredient which one has to include in order to describe the vacua of $\mathcal{N}=2$ broken down to $\mathcal{N}=1$ are monopoles. If we neglect them then the F-flatness conditions for the vacuum cannot be met. *E.g.* if we just add a mass term for ϕ:

$$W_{\text{tree}} = \frac{1}{2} m \,\text{Tr}\, \phi^2 = m\, u_2 \,, \tag{13}$$

then varying w.r.t. u_2 gives $m=0$! The only way to obtain a solution is to realize that in the superpotential one has to include *all* relevant low energy degrees of freedom — in particular we have to include monopoles which can become massless, and thus have to be necessarily included in the low energy effective theory. So one has to consider instead[1]

$$W = m_{\text{monopole}}(u_2) M\tilde{M} + m\, u_2 \,. \tag{14}$$

Varying that superpotential we find that i) the value of u_2 (*i.e.* the point in the moduli space of the original $\mathcal{N}=2$ theory) has to be such that the monopole is massless and *(ii)* the monopoles condense. Recall from the previous section that the condition *(i)* means that the Seiberg–Witten curve has to *factorize*.

The above is true in general. The Seiberg–Witten curve corresponding to $\mathcal{N}=2$ theory broken down to $\mathcal{N}=1$ by the addition of a tree level superpotential has to factorize. In case of unbroken gauge group, all monopoles are massless, while for $U(N_c) \to U(N_1)\times U(N_2)$, all but one are massless and the SW curve has to look like (7). The nontrivial branch cuts of the curve intuitively correspond to the fact that at nonzero Λ, the eigenvalues

[1] Here we just quote for simplicity the SU(2) case.

no longer behave like classical quantities sitting exactly at the minima of the superpotential.

The above observation, together with the fact that at low energies the massless degrees of freedom are just $\mathrm{U}(1)^2$ led to a surprising possibility [10] that at nonzero Λ one could not really distinguish between certain breakings $\mathrm{U}(N_c) \to \mathrm{U}(N_1)\times\mathrm{U}(N_2)$ (in some specific discrete vacuum labeled by k_1 and k_2) and $\mathrm{U}(N_c) \to \mathrm{U}(N_1')\times\mathrm{U}(N_2')$ in the vacuum labeled by k_1' and k_2'. These could be interpreted as dual descriptions of the same physics *i.e.* of the same underlying Seiberg–Witten curve. Moreover, it was found [10] that once one changes the deforming tree-level superpotential the vacua move around but *remain* within a certain number of connected domains (see Fig. 1). In [10] an analysis was performed of the structure of these domains up to $N_c = 6$ by direct factorization of the Seiberg–Witten curves. In this talk I would like to review a general solution of the factorization problem valid for any N_c and the resulting picture of the domains of $\mathcal{N} = 1$ vacua. But before I present further details I will review how are the parameters of the vacua N_1, k_1, k_2 linked to the factorized SW curve.

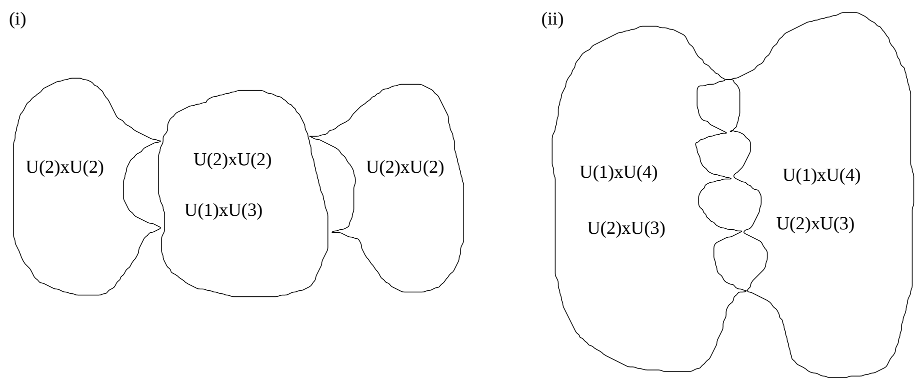

Fig. 1. Connected domains of $\mathcal{N} = 1$ vacua for *(i)* $N_c = 4$, *(ii)* $N_c = 5$. The labels denote different dual descriptions within each domain.

In the case of (7), there are two nontrivial compact cycles A and B. The periods of the meromorphic 1-form ω are then exactly N_1 and $\Delta_k \equiv k_2 - k_1$ ([10, 11] see also [12]):

$$\frac{1}{2\pi i}\oint_A \omega = N_1 \,, \qquad\qquad \frac{1}{2\pi i}\oint_B \omega = \Delta k \,. \tag{15}$$

The fact that one can have dual descriptions is easy to understand once one takes into account that the *choice* of A and B cycles is not unique and can be changed by a modular transformation. Therefore, N_1 and Δ_k can change while the underlying physics remains the same. However, in order to have a complete picture of the possible dual descriptions one has to incorporate the remaining discrete parameter k and look what sets of discrete parameters (N_1, Δ_k, k) lead to the same Seiberg–Witten curve. These sets will just be the different possible dual descriptions of the same physics. Each such set, on the other hand, corresponds to a connected domain of vacua[2] as in Fig. 1.

In the next section we will briefly present the solution to the factorization problem.

4. The solution to the factorization problem

The factorization problem is to find the set of $\{u_p\}$'s in the $\mathcal{N} = 2$ moduli space where the Seiberg–Witten curve factorizes as in (7). The case of complete factorization corresponding to unbroken gauge group and *all* monopoles being massless was solved by Douglas and Shenker [15] using special properties of Chebyshev polynomials. That case would correspond to the F_4 in (7) being substituted by a degree two polynomial F_2. Their solution, for each N_c depends on a discrete parameter $k = 0, \ldots N_c - 1$ which just labels pure $\mathcal{N} = 1$ vacua.

In the present case the situation is much more complicated. For each N_c the set of allowed labels (discrete parameters) increases (these are (N_1, k_1, k_2)) and new types of vacua tend to appear which cannot be related to those at smaller N_c (see [10] – these are the "Coulomb vacua" in the terminology of that paper).

The key to finding a solution to this problem are certain properties of the meromorphic 1-form ω

- For a factorized Seiberg–Witten curve of the form (7), ω defines a meromorphic 1-form on the *elliptic* curve

$$y^2 = F_4(x)\,. \tag{16}$$

- ω has residues $\pm N_c$ at infinity.
- ω has integer periods given by N_1 and Δk.

Once we know a meromorphic 1-form on the elliptic curve $y^2 = F_4(x)$ satisfying the above properties, we may reproduce the *full* Seiberg–Witten curve *i.e.* find all the $\{u_p\}$'s and the scale of the theory Λ from further properties of ω:

[2] This is so, since in addition to the discrete parameters we will have continuous parameters, varying which will fill out the domains.

- The $\{u_p\}$ is given by

$$u_p = \frac{1}{p} \operatorname{Res}_{x=\infty} x^p \omega \,. \tag{17}$$

- The scale of the theory can be obtained from

$$\left\{ \int_a^\infty \omega \right\}_{\rm reg} \equiv \lim_{x\to\infty} \left(\int_a^x \omega - N_{\rm c} \log x \right) = -\log \Lambda^{N_{\rm c}} \,, \tag{18}$$

where a is a branch point of the Seiberg–Witten curve.

The key difficulty in finding a solution is an explicit construction of a meromorphic 1-form with integer periods on an elliptic curve. In the conventional representation $y^2 = F_4(x)$ this seems impossible, however, it is very simple when one uses the representation of an elliptic curve as a torus *i.e.* a pararellogram with identified edges (see Fig. 2). Furthermore, one has to pick two points to represent the two infinities in the (y,x) representation. Then a meromorphic 1-form with prescribed poles at these points with residues $\pm N_{\rm c}$ is [14]

$$\omega = N_{\rm c} \frac{d}{dz} \log \frac{\theta(z-a_1)}{\theta(z-a_2)} + C \,, \tag{19}$$

where $\theta(z)$ is the Jacobi theta function. Without loss of generality we may set $a_1 = 0$. Then a_2 and C are uniquely determined from the periods, which in this representation of the elliptic curve are trivial to calculate using quasi-periodicity of the theta functions

$$\frac{1}{2\pi i} \int_A \omega \;=\; \frac{1}{2\pi i} \int_0^1 \omega = \frac{C}{2\pi i} \,, \tag{20}$$

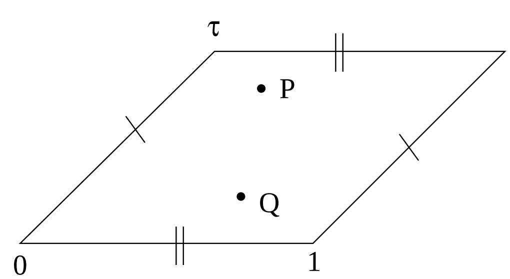

Fig. 2. The torus (elliptic curve) represented as a pararellogram with identified opposite edges. τ is the modular parameter, P and Q represent the infinities in the (x,y) picture. They are related to the a_i's appearing in (19) by $a_i + (1+\tau)/2$.

$$\frac{1}{2\pi i}\int\limits_{B}\omega \;=\; \frac{1}{2\pi i}\int\limits_{0}^{\tau}\omega = N_{\rm c}(a_1 - a_2) + \frac{C\tau}{2\pi i}\,, \tag{21}$$

and we easily get

$$C \;=\; 2\pi i N_1\,, \tag{22}$$

$$a_2 \;=\; \frac{N_1\tau - \Delta k}{N_{\rm c}}\,. \tag{23}$$

Now we are almost done. In order to reproduce the $\{u_p\}$'s and Λ we have to know how the original x coordinate depends on z. This can be done (see [13] for details) and the result is

$$x(z) = \frac{d}{dz}\log\frac{\theta(z-a_1)}{\theta(z-a_2)}\,. \tag{24}$$

Using this formula one can readily calculate the $\{u_p\}$'s and Λ from (17)–(18). In fact $x(z)$ is unique only up to a linear transformation $x \to \alpha x + x_0$. Using α we may set $\Lambda^{2N_{\rm c}}$ to its *given* physical value. We are, therefore, left with a continuous free parameter x_0 and a *discrete* one coming from the fact that we are still free to perform rescalings $x \to \alpha x$ with

$$\alpha = e^{2\pi i \frac{k}{2N_{\rm c}}}\,, \qquad\qquad k = 0\ldots 2N_{\rm c}-1\,, \tag{25}$$

which do not modify $\Lambda^{2N_{\rm c}}$.

Putting all of this together we see that our solution of the factorization problem is labeled by two complex continuous parameters τ and x_0 and three *discrete* ones: $(N_1, \Delta k, k)$ *i.e.* exactly the needed number to describe the expected $\mathcal{N}=1$ vacua.

5. Connected domains of $\mathcal{N}=1$ vacua

We can now ask the key questions about the structure of $\mathcal{N}=1$ vacua:

- What are the possible dual descriptions of the same physics?
- How many connected domains of vacua do we have for given $N_{\rm c}$?

As described at the end of Sec. 3, once we have an explicit construction of the factorized Seiberg–Witten curve labeled by the discrete parameters $(N_1, \Delta k, k)$, the above questions may be translated into the questions: *(i)* what *distinct* parameters $(N_1, \Delta k, k)$ lead to *identical* Seiberg–Witten curves? *(ii)* once we identify the dual descriptions, how many distinct Seiberg–Witten curves we are left with.

Some analysis of the properties of the theta functions under modular transformations leads to the following identifications:

$$(N_1, \Delta k, k) \equiv (N_1, \Delta k - N_1, k)\,, \qquad N_1 \leq \Delta k\,, \tag{26}$$
$$(N_1, \Delta k, k) \equiv (N_1, \Delta k - N_1 + N_{\rm c}, k + N_1 + N_{\rm c} \bmod 2N_{\rm c})\,, \qquad N_1 > \Delta k\,, \tag{27}$$
$$(N_1, \Delta k, k) \equiv (N_{\rm c} - \Delta k, N_1, k - N_1 + N_{\rm c} \bmod 2N_{\rm c})\,, \quad \Delta k \neq 0\,, \tag{28}$$

and their inverses. Of course, the transformations of just N_1 and Δk under modular transformations are known from the outset, however, we need here to know exactly how does the third discrete parameter k enter these transformations.

So in order to answer the two questions here we have to start from the set $(N_1, \Delta k, k)$ with $1 \leq N_1 < N_{\rm c}$, $0 \leq \Delta k < N_{\rm c} - 1$ and $0 \leq k < 2N_{\rm c} - 1$ and generate orbits under the transformations (26)–(28) and their inverses. Each orbit will then correspond to a connected domain in the space of $\mathcal{N} = 1$ vacua, while the labels *within* a single orbit correspond to different possible dual descriptions of the same physics. In Table I we give the number of orbits (connected domains of $\mathcal{N} = 1$ vacua) for $N_{\rm c} \leq 22$.

TABLE I

Number of connected domains of $\mathcal{N} = 1$ vacua for $N_{\rm c} < 23$ calculated as the number of orbits under the identifications (26)–(28).

$N_{\rm c}$	No. of domains	$N_{\rm c}$	No. of domains
3	2	13	2
4	3	14	12
5	2	15	18
6	8	16	15
7	2	17	2
8	7	18	29
9	8	19	2
10	10	20	26
11	2	21	22
12	20	22	16

6. Summary and outlook

The structure of vacua in $\mathcal{N}=2$ theories broken down to $\mathcal{N}=1$ appears to be very complex and possesses a rich structure of a number of connected domains and specific dual descriptions which appear at the quantum level. In this talk I showed that the use of an explicit exact solution to the factorization problem may be an efficient tool to study this structure.

This work may be extended in various directions. One could add fundamental matter and/or consider different gauge groups (like orthogonal or symplectic), one might try to understand the identifications for all *three* discrete parameters geometrically in order to get more insight into the structure of the domains, which then might be generalized for higher genus.

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REFERENCES

[1] M.A. Shifman, *Int. J. Mod. Phys.* **A14**, 5017 (1999).
[2] N. Seiberg, `hep-th/9408013`.
[3] N. Seiberg, E. Witten, *Nucl. Phys.* **B426**, 19 (1994), Erratum **B430**, 485 (1994).
[4] N. Seiberg, E. Witten, *Nucl. Phys.* **B431**, 484 (1994).
[5] F. Cachazo, K.A. Intriligator, C. Vafa, *Nucl. Phys.* **B603**, 3 (2001).
[6] R. Dijkgraaf, C. Vafa, `hep-th/0208048`.
[7] F. Ferrari, *Nucl. Phys.* **B648**, 161 (2003).
[8] R. Dijkgraaf, M.T. Grisaru, C.S. Lam, C. Vafa, D. Zanon, *Phys. Lett.* **B573**, 138 (2003).
[9] F. Cachazo, M.R. Douglas, N. Seiberg, E. Witten, *J. High Energy Phys.* **0212**, 071 (2002).
[10] F. Cachazo, N. Seiberg, E. Witten, *J. High Energy Phys.* **0302**, 042 (2003).
[11] F. Cachazo, N. Seiberg, E. Witten, *J. High Energy Phys.* **0304**, 018 (2003).
[12] F. Ferrari, *Phys. Rev.* **D67**, 085013 (2003); F. Ferrari, *Phys. Lett.* **B557**, 290 (2003).
[13] R.A. Janik, *Phys. Rev.* **D69**, 085010 (2004).
[14] D. Mumford, *"Tata Lectures on Theta. I"*, Progress in Mathematics 28, Birkhauser, Boston 1983.
[15] M.R. Douglas, S.H. Shenker, *Nucl. Phys.* **B447**, 271 (1995).