INTERACTION BETWEEN TOPOLOGICAL DEFECTS AND RADIATION *

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The spectral structure of linearization around soliton in the ϕ^4 and s-G models is presented. Negative radiation pressure in ϕ^4 model is discussed and analytical calculation presented in the second order. The production of topological defects forced by radiation coupled to the internal degree of freedom of soliton is studied. The fractal boundary for this creation is also described.

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1. Introduction

Topological defects (or topological solitons) are usually static localized solutions of nonlinear partial differential equation which smoothly interpolate between two disconnected vacua. They arise usually in a context of effective field theories [1, 2] (*i.e.* Ginzburg–Landau theory). They play important role in many branches of modern physics such as condensed matter physics (vortices in liquid helium or superconductors), domain walls in cosmology [3] or ferromagnets, all kinds of topological defects in liquid crystals. Some of the topological defects reveal many similarities to particles (kinks in 1+1 d, 2+1 d vortices or 't Hooft–Polyakov monopoles). They can interact with each other or external fields [4] and radiation. They can also be created or annihilated. Skyrme model had some successes in the theory of hadrons. There is also a nice example of sine-Gordon equation and its duality to quantum theory of certain particles (Thirring's model). Some of the topological defects, such as solitons in sine-Gordon model, are very well understood thanks to the complete integrability of the equations. There

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exists Bäcklund's transform which is a sort of nonlinear superposition role for the solitons. Thanks to this transform we can construct an exact (!) solution of the nonlinear partial equation with interacting solitons. This is true only for very few equations. Although we can usually find the static (or boosted) solutions quite easily or small perturbations around vacuum (which can be dealt with using ordinary perturbation theory) the evolution of more complicated systems can still give some surprising results. In non-integrable equations each interaction between solitons results in radiation and dissipation of the energy. We cannot separate these two sectors. In this paper we will show how radiation (small perturbation around solitonic solution) can effect the evolution of these highly nonlinear objects. Some of the results presented in this paper can also be found in more details in [5] and [6].

2. Perturbed soliton in ϕ^4 and sine-Gordon models

Let us consider real scalar field in 1+1 d described by the equation:

$$\ddot{\phi} - \phi'' + \frac{\partial U(\phi)}{\partial \phi} = 0\,. \tag{1}$$

Depending on the potential $U(\phi)$ we have different models. In case when U has at least two equal minima (also called vacua) the model possesses solutions called topological defects. The most often studied models are sine-Gordon: $U_{\rm sG} = 1 - \cos\phi$, and ϕ^4: $U_{\phi^4}(\phi) = \frac{1}{2}\left(\phi^2 - 1\right)^2$. Sine-Gordon possesses infinite number of vacua $\phi_{\rm v} = 2k\pi$, where k is integer, and ϕ^4 has only 2 minima $\phi_{\rm v} = \pm 1$. In both models one can easily find static solutions which interpolate between two (neighboring in s-G case) vacua $\phi_{\rm s,sG} = 4\arctan e^x$ $\phi_{{\rm s},\phi^4} = \tanh x$. These solutions are called also topological solitons or kinks. The most significant difference between these two models is integrability of sine-Gordon. Thanks to Bäcklund transform one can find an *exact* analytical solution with any number of interacting solitons. This is impossible in case of ϕ^4 where the only known exact solutions (apart the trivial ones $\phi = \pm 1$) are static or Lorentz boosted kinks. Nevertheless in both cases the interaction between soliton and radiation is not fully understood. Let us add some small perturbation $\xi(x,t)$ to the static kink solution [5, 7]. After plugging $\phi = \phi_{\rm s} + \xi$ to the Eq. (1) we obtain:

$$\ddot{\xi} - \xi'' + V(x)\xi + N(\xi, x) \equiv \ddot{\xi} + \hat{L}\xi + N(\xi, x) = 0\,, \tag{2}$$

where

$$V(x) = U''(\phi_{\rm s}(x)) = \begin{cases} -\frac{2}{\cosh^2 x} & \text{for} \quad \text{s-G} \\ 4 - \frac{6}{\cosh^2 x} & \text{for} \quad \phi^4 \end{cases}\,, \tag{3}$$

and $N(\xi, x)$ is remaining nonlinear part of Taylor's expansion (if Taylor's expansion exists, which is not true for example for compactons [8]). We substitute $\xi(t,x) = e^{i\omega_k t}\eta_k(x)$ and find the eigenvalues and the eigenfunctions of the operator $\hat{L} = -\frac{d^2}{dx^2} + V(x)$. The spectra can be divided into three groups

1. translational zero modes $\eta_{\rm t} = \phi'_{\rm s}$: $\phi_{\rm s}(x+\delta x) \sim \phi_{\rm s}(x) + \delta x \phi'_{\rm s}(x)$, $\omega_{\rm t} = 0$,
2. discrete oscillational modes (there is none for s-G and only one for ϕ^4 but in general there can be any finite number)
$$\eta_{{\rm d},\phi^4}(x) = \frac{\sinh x}{\cosh^2 x}, \qquad \omega_{\rm d} = \sqrt{3}\,,$$
3. scattering modes:
$$\eta_q(x) = \begin{cases} e^{iqx}(iq - \tanh x)/\sqrt{q^2+1} & \text{for} \quad \text{s-G} \\ \frac{e^{iqx}\left(3\tanh^2 x - 1 - q^2 - 3iq\tanh x\right)}{\sqrt{(q^2+1)(q^2+2)}} & \text{for} \quad \phi^4 \end{cases},$$
where $q^2 = \omega^2 - m^2$, $\omega^2 > m^2$ and $m_{\rm sG} = 1$, $m_{\phi^4} = 2$.

It is very significant that scattering modes in these models have no reflection part. This is not a general feature. In fact it is quite rare, but spectra for $V(x)$ given in a form:
$$V(x) = -\frac{N(N+1)}{\cosh^2 x}$$
are reflectionless for all integer N [9]. There are exactly N bounded modes. One of them is a translational mode and the rest of them are oscillational. In the following section we will include the second order to show how scattering modes (*i.e.* radiation) interact with the translational mode forcing the kink to accelerate. In the next section we will consider excitation of the discrete mode by the radiation. As a result, for large amplitudes, the creation of new defects will occur.

3. Radiation pressure

As we have mentioned in the previous section, the scattering modes both in s-G and ϕ^4, unlike in most models, have no reflection part. The wave traveling from $+\infty$ has exactly the same amplitude after transition through the kink with only a phase shift. In other words, the kinks are transparent but still they refract the radiation. In this (linear) order, that is when we put $N = 0$, the kink cannot move when is exposed to the radiation, the energy and momentum conservation laws prohibit that. In models which do not posses the reflectionless spectrum solitons are pushed by radiation as ordinary particles which reflect the radiation. In order to examine what happens to the kinks exposed to the radiation one must go beyond that order and take into account the higher powers of incoming wave amplitude. We will consider the ϕ^4 case. Analogous calculation can be carried out for sine-Gordon with very similar result. It would be interesting to use integrability of the s-G model to describe the effect. When we include the first nonlinear term our equation takes the form:

$$\ddot{\xi} + \hat{L}\xi + 6\phi_{\rm s}\xi^2 = 0\,. \tag{4}$$

Let us look for solutions given by the power series:

$$\xi = A\xi^{(1)} + A^2\xi^{(2)} + \cdots,$$

where $A\xi^{(1)}(x,t) = \frac{1}{2}A\eta_q(x)e^{i\omega t}$+c.c. is a wave traveling from $+\infty$ with an amplitude A (η_q is normalized so that $\lim_{x\to\pm\infty}|\eta_q^2| = 1$). In the order $\mathcal{O}(A^2)$ equation (4) has the form:

$$\ddot{\xi}^{(2)} + \hat{L}\xi^{(2)} = -6\phi_{\rm s}\xi^{(1)2} = -\frac{3}{2}\phi_{\rm s}\left(e^{2i\omega t}\eta_q^2 + 2\eta_q\eta_{-q} + e^{-2i\omega t}\eta_{-q}^2\right)\,. \tag{5}$$

We can separate the equation by substituting $\xi^{(2)} = e^{2i\omega t}\xi_{+2}^{(2)} + \xi_0^{(2)} + e^{-2i\omega t}\xi_{-2}^{(2)}$. Let us take a closer look at the equation for $\xi_0^{(2)}$:

$$\ddot{\xi}_0^{(2)} + \hat{L}\xi_0^{(2)} = -3\phi_{\rm s}\eta_q\eta_{-q}\,. \tag{6}$$

When we integrate both sides of the above equation with the translational mode we will obtain second time derivative of the translational mode on the l.h.s. of the equation which is from definition an acceleration of the kink (we also use that $\hat{L}\eta_{\rm t} = 0$):

$$\frac{\left\langle \ddot{\xi}_0^{(2)} \middle| \eta_{\rm t}\right\rangle}{\langle\eta_{\rm t}|\eta_{\rm t}\rangle} = -a = -3\frac{\langle\phi_{\rm s}\eta_q\eta_{-q}|\,\eta_{\rm t}\rangle}{\langle\eta_{\rm t}|\eta_{\rm t}\rangle}\,. \tag{7}$$

As we mentioned before the scattering modes have no reflected part, that results in vanishing the r.h.s. of the above equation. In other words there is no acceleration of the kink in this order. Fortunately we can find the exact solution of the time-independent equation:

$$\hat{L}\xi_0^{(2)} = -3\phi_{\rm s}\eta_q\eta_{-q}\,. \tag{8}$$

Namely,

$$\xi_0^{(2)} = -\frac{9}{2}\frac{\tanh x}{(q^2+1)(q^2+4)\cosh^2 x} - \frac{3}{4}\frac{x(q^2-2)}{(q^2+4)\cosh^2 x} - \frac{3}{4}\tanh x. \qquad (9)$$

We also need to solve the equation for $\xi_{\pm 2}^{(2)}$

$$\left(\hat{L} - 4\omega^2\right)\xi_{\pm 2}^{(2)} = -\frac{3}{2}\phi_{\rm s}\eta_{\pm q}^2\,. \qquad (10)$$

We are in comfortable situation because we already know the solutions of homogeneous part of the equation. We can construct the Green's function and integrate it with the r.h.s. of the equation. For large x we can express the solution in a form convenient for asymptotic approximation:

$$\begin{aligned}\xi^{(2)}(x\to\infty) \;=\; -\frac{3}{2W}\Bigg(&\eta_{-k}(x)\int_{-\infty}^{\infty} dy\;\eta_k(y)\eta_q^2(y)\phi_{\rm s}(y)\\ &-\eta_{-k}(x)\int_{x}^{\infty} dy\;\eta_k(y)\eta_q^2(y)\phi_{\rm s}(y)\\ &+\eta_k(x)\int_{x}^{\infty} dy\;\eta_{-k}(y)\eta_q^2(y)\phi_{\rm s}(y)\Bigg) + \text{c.c.}\,,\end{aligned} \qquad (11)$$

where $k = \sqrt{4\omega^2 - 4}$ is a wave vector for frequency 2ω and $W = -2ik$ is Wronskian. We can express the solution for $x \ll -1$ in a similar fashion. The above integrals are not difficult to calculate, and we can write the asymptotic form of the solution:

$$\xi_{\pm\infty}^{(2)}(x) = b(q, \pm k)\cos(2\omega t \mp kx \pm \delta_1) \pm c\cos(2\omega t + 2qx \pm \delta_2)\,, \qquad (12)$$

where

$$b(q,k) \;=\; -\frac{3}{2}\pi\frac{q^2+4}{q^2+1}\sqrt{\frac{q^2+4}{k^2+1}}\frac{1}{k\sinh\left(\frac{2q+k}{2}\pi\right)}\,, \qquad (13)$$

$$c \;=\; \frac{1}{8}\,. \qquad (14)$$

In this order energy and momentum are not conserved so we need to consider the whole system kink+radiation using the asymptotic forms of the solution already obtained. Let us put the kink inside a large box. The energy flowing from the r.h.s. of the kink into the box averaged over a period is equal to:

$$\frac{dE_r}{dt} = \frac{1}{2}A^2 q\omega + A^4\left(-k\omega b^2(q,+k) + 2q\omega c^2\right)\,, \tag{15}$$

and energy escaping from the box:

$$\frac{dE_l}{dt} = -\frac{1}{2}B^2 q\omega + A^4\left(-k\omega b^2(q,-k) + 2q\omega c^2\right)\,. \tag{16}$$

In the same way we calculate the momentum balance:

$$\frac{dP_r}{dt} = \frac{1}{2}A^2 q^2 + \frac{1}{2}A^4\left(k^2 b^2(q,+k) + 4q^2c^2\right), \tag{17}$$

$$\frac{dP_l}{dt} = -\frac{1}{2}B^2 q^2 - \frac{1}{2}A^4\left(k^2 b^2(q,-k) + 4q^2c^2\right). \tag{18}$$

If the kink is not moving at an initial time then $\dot{E}_r + \dot{E}_l = 0$ and $\dot{P}_r + \dot{P}_l = -F$, where F is a force acting on the kink. The above system of equations can be solved easily and the resulting force is equal to:

$$F = \frac{1}{2}A^4 k\left[(k-2q)b^2(q,-k) - (k+2q)b^2(q,k)\right]. \tag{19}$$

Since $k - 2q$ is always positive and $b(q,-k)$ is always larger than $b(q,k)$ (which for large q is negligible) the force is always positive. That means the kink will accelerate towards the source of radiation. We have found the *negative radiation pressure!* On the l.h.s. of the kink there is a surplus of momentum which pushes the kink towards the radiation.

We have calculated the stationary case as if the radiation came from infinity. We have neglected the boundary conditions which can distort this picture a little, but in numerical simulations of the whole partial equation the effect is clearly visible. The calculation presented above is consistent with calculation the projection onto the translational mode in $\mathcal{O}(A^4)$ order (similar as in Eq. (7)).

For large amplitudes the higher orders can give different results. From the amplitude above about $A = 0.26$ (the boundary is smeared) the kinks start to behave normally and are pushed by radiation.

It is also worth mentioning that, in numerical simulations, the negative pressure survives when we perturb the equation by adding the term $\varepsilon(\phi^2-1)^3$ or $\varepsilon(\phi^2-1)^4$ to the potential even for large values of $\varepsilon \approx 1$. In the ϕ^4 model there exists a solution almost periodic in time, called bion, oscillon or pseudo-breather, which corresponds to bound state of kink and antikink. Although its stability is not yet understood it is a quite long living state, loosing its energy due to the radiation in very small amounts. We have found numerically that such objects also experience the negative pressure. The same is with breathers in s-G model.

4. Fractals

In the previous section we have shown how soliton accelerates when is exposed to radiation coming from one direction. We have calculated energy and momentum balance far away from the kink. We have neglected the processes which took place in the vicinity of the kink. Let us now consider much more symmetric problem when kink is exposed to the radiation from both sides.

As showed in [7], when the internal degree of freedom (oscillational mode) is excited it radiates due to the nonlinear coupling to the scattering modes (similar problem using different methods was investigated in [11] and [12]). But if the excitation is large enough, a creation of two antikinks can occur and a kink and antikink can be radiated to both sides leaving an antikink in the middle. We can induce this process by exciting the oscillational mode by radiation. It turns out that for large amplitudes of incoming waves we can force the creation [6].

Let us now first take a look at the second order equation for perturbation around the kink, Eq. (4), and substitute in the first order $\xi^{(1)}(x,t) = \frac{1}{2}\left(\eta_q(x) + \eta_{-q}(x)\right)\cos\omega t$ (a wave coming from both sides).

We want to force the creation of kinks by exciting the internal degree of the kink. Therefore, we take

$$\xi(x,t) = A\xi^{(1)} + A_{\mathrm{d}}(t)\eta_{\mathrm{d}}(x) + \eta_\perp(x,t)\,, \tag{20}$$

where $\eta_\perp$ is orthogonal to η_{d}. If we assume that both A_{d} and $\eta_\perp$ are of order $\mathcal{O}(A^2)$ then the first contribution to A_{d} from $\eta_\perp$ will be of order $\mathcal{O}(A^3)$. We will limit ourselves only to the second order. In order to isolate the time evolution of the oscillational mode we simply project the whole equation upon this mode. We obtain:

$$\ddot{A}_{\mathrm{d}} + \omega_{\mathrm{d}}^2 + A^2\alpha(q)\left(1+\cos 2\omega t\right) + \beta(q)AA_{\mathrm{d}}\cos\omega t + \gamma A_{\mathrm{d}}^2 = 0\,, \tag{21}$$

where

$$\alpha(q) \;=\; \frac{9\pi}{64N^2}(8q^4+34q^2+17)\left(1-\operatorname{sech} q\pi\right)\,, \tag{22}$$

$$\beta(q) \;=\; 3\pi q^2\frac{q^4+2q^2-8}{8N\sinh\frac{q\pi}{2}}\,, \tag{23}$$

$$\gamma \;=\; \frac{9\pi}{16}\,. \tag{24}$$

Here $N=\sqrt{(q^2+1)(q^2+4)}$ is the normalization.

The projected equation has some obvious limitations but it also carries certain features of the whole system. We have neglected the coupling to the radiation $\eta_\perp$ and backreaction onto the discrete mode, but we can clearly see the interaction between the radiation A and oscillational mode $A_{\rm d}$. We think it is sufficient in order to give some qualitative predictions on how the kink's internal degree would behave.

On the other hand, we can find the numerical solution of the whole partial differential equation (1) and by projection onto the oscillational mode we can find its amplitude. Then we can compare the solution of the simplified equation with the solution of the partial differential equation. For small amplitudes there is quantitative agreement, but for larger amplitude only qualitative similarity remains. Nevertheless both of these solutions possess a very important feature. Suppose that we have excited the oscillational mode so much that the creation of two defects had occurred. After radiating kink and antikink to both sides only an antikink remains in the middle. The projection onto the oscillational mode is equal to

$$A_{\rm d} = \langle \phi - \phi_{\rm s} | \eta_{\rm d} \rangle = \langle -2\phi_{\rm s} | \eta_{\rm d} \rangle = -\frac{\pi}{2}\,. \tag{25}$$

If during the numerical simulation $A_{\rm d}$ jumps from 0 to $-\pi/2$ it is a clear evidence that the creation had occured. In figure 1 we present the example for the creation process.

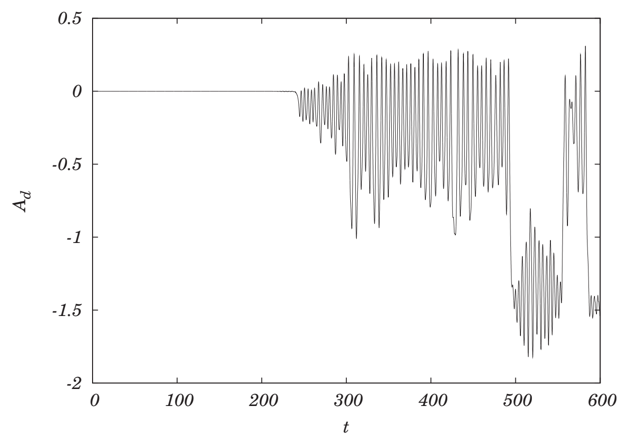

Fig. 1. The excitation of the discrete mode for $A = 0.5$ and $\omega = 3.5$. We can see the creation of kink–antikink pairs for $t \approx 500$, $t \approx 560$ and $t \approx 590$.

Figure 2 shows the minima of $A_{\rm d}$ calculated numerically using the full partial differential equation for different amplitudes A and frequencies ω of the incoming radiation. If the amplitude of radiation is small enough the creation is not possible and $A_{\rm d}$ oscillates around 0 with some small amplitude (smaller than $\pi/2$). The minima during the time evolution are represented by different shades. The black points correspond to the creation. One can see there is no real threshold for the creation. The boundary is presumably fractal. In figure 3 we present the dependence log of the number of boxes containing boundary upon the −log of the size of boxes. The slope of the fitted straight line should give so called box dimension. It is 1.69 ± 0.02 which is more than 1. This supports the hypothesis that the structure has really fractal properties.

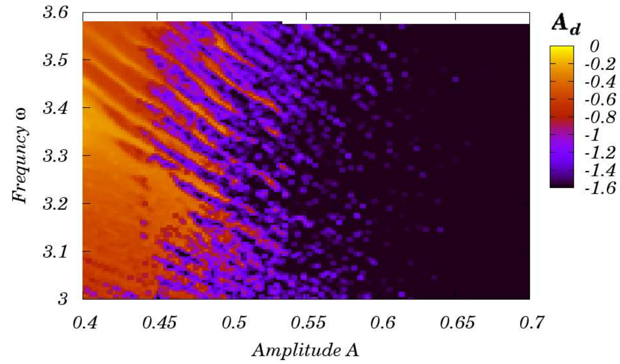

Fig. 2. Minima of A_d *vs* frequency ω and amplitude A of radiation coming from $L = 200$. Dark spots $A_\mathrm{d} < -\frac{\pi}{2} \approx -1.57$ represent creation process.

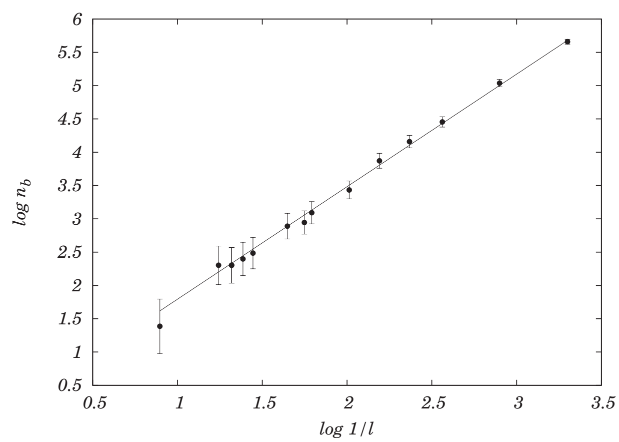

Fig. 3. Dependence $\log n_\mathrm{b}$ upon $\log 1/l$. The slope of the fitted line is 1.69 ± 0.02.

There is a question if the same features can be reproduced using our simplified equation (21). Similar calculations give the box dimension $1.56{\pm}0.01$ which is not very far from the value obtained from the whole partial equation. Having in mind all the simplifications we had done, the result seems to be quite good. This also proves that responsible for this fractal properties of the system is coupling between scattering modes and the oscillational one.

This is, of course, not an isolated example of the fractal structure in a context of topological defects. One of the nicest such structures was discovered and examined in [10]. In that paper it is argued that kink–antikink collision can proceed according to two scenarios. Kinks can be scattered back for (if the velocity is larger than 0.26), or annihilated (if velocity is smaller than 0.18) by forming pseudobreather called also bion or oscillon. In the region between 0.18 and 0.26 one can observe so called n-bounce windows, where kinks are bouncing on each other n times and wondering whether they want to annihilate or scatter back. Those windows create whole, very well defined, hierarchy which possesses some scaling indicating that the structure is fractal. The authors claim that responsible for that is coupling between oscillational modes of the colliding kinks.

5. Conclusions

In the first section we have compared two the most known field theory models in 1+1 d with respect to the spectral structure of the linearized equation around soliton solutions. In the next section we have shown that because of the reflectionless potential in the first linearized order the soliton exposed to the radiation should remain at rest. The second order calculation gave surprising result that the kinks in these two models would start accelerate towards the source of radiation. The calculations were confirmed by numerical simulations.

In the last section we have presented how the internal degree of freedom in the ϕ^4 model can be excited by external radiation and can lead to the creation of new defects. We found another surprising result that the boundary between solution with and without production of defects on amplitude-frequency plane is fractal. We have also presented simplified theory which gave similar results proving that the nonlinear coupling between radiation and oscillational mode is responsible for that structure.

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