CHAIN OF IMPACTING PENDULUMS AS NON-ANALYTICALLY PERTURBED SINE-GORDON SYSTEM*

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We investigate a mechanical system consisting of infinite number of harmonically coupled pendulums which can impact on two rigid rods. Because of gravitational force the system has two degenerate ground states. The related topological kink, likely the simplest one presented in literature so far, is a compacton, that is it has strictly finite extension. In the present paper we elucidate the relation of such system with sine-Gordon model. Also, solutions describing waves with large amplitude, and an asymptotic formula for the width of the kink are obtained.

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1. Introduction

Theoretical studies of classical mechanical systems with large number of degrees of freedom are relevant to many branches of condensed matter physics (not to mention engineering sciences), where quantal aspects of microscopic world are practically not visible. Moreover, since the seminal work of Fermi, Pasta and Ulam [1] it is clear that such systems can also provide useful models on which concepts of theoretical physics can be tested or illustrated. For example, studies of anharmonic lattices give insights into nonlinear transport phenomena, see *e.g.*, [2], or mechanisms of thermal conductivity, see, *e.g.*, [3]. Another example: sine-Gordon solitons, which appear in many places including quantum field theory (the duality with Thirring model), have beautiful realisation in a system composed of harmonically coupled pendulums, see, *e.g.*, [4].

In recent papers [5, 6] we have investigated a mechanical system, described below in Sec. 2, which has infinite number of degrees of freedom. It exhibits spontaneous symmetry breaking and related to it topological

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kink and antikink. Its main attractive feature is extraordinary simplicity of the kink. It has strictly finite extension equal to π in a certain limit, and in that limit the kink is represented by the piece of sine function in the interval $[-\pi/2,\ \pi/2]$. To the best of our knowledge, it is the simplest topological defect presented in literature so far. Furthermore, time evolution of the system during the symmetry breaking transition can be described analytically. For these reasons, the system is quite interesting as a testing ground for various ideas of theory of topological defects. Physics of such objects poses many very interesting albeit hard to solve dynamical questions, see [7,8] for overviews.

The purpose of the present paper is to supplement our previous investigations [5,6]. In next section we briefly recall the model and certain results in order to make the paper self-contained. We also introduce a folding transformation which maps trajectories of certain auxiliary, simpler model into trajectories of our system. Sec. 3 is devoted to large amplitude waves — the ones having small amplitudes were investigated in [6]. The two types of waves involve very different kinds of motion. In Sec. 4 we calculate the width of the kink in a limit when it increases to infinity. Connection of our system with the one considered in connection with the sine-Gordon soliton is elucidated in Sec. 5. We show that from mathematical viewpoint the unfolded version of our model can be regarded as non-analytically perturbed sine-Gordon system. In particular, the sine-Gordon soliton and breather provide approximate solutions of our model. Nevertheless, our system and the sine-Gordon system are very different from physical viewpoint. Finally, in Sec. 6 we point out several dynamical problems which can be posed within our model and which, in our opinion, deserve a careful investigations.

2. The model and the folding transformation

We consider one dimensional model with the Lagrangian

$$L = \tfrac{1}{2}(\partial_\tau \phi)^2 - \tfrac{1}{2}(\partial_\xi \phi)^2 - V(\phi)\,, \tag{1}$$

where the field potential $V(\phi)$ has the form

$$V(\phi) = \begin{cases} \cos\phi - 1 & \text{for} \quad |\phi| \le \phi_0\,, \\ \infty & \text{for} \quad |\phi| > \phi_0\,, \end{cases} \tag{2}$$

see Fig. 1. Here ϕ_0 is a constant, $\pi > \phi_0 > 0$. Thus, all values of the real field ϕ are restricted to the interval $[-\phi_0,\ \phi_0]$. The potential has two symmetric minima at $\phi = \pm\phi_0$, and it is not continuous at them. This model describes an infinite set of harmonically coupled pendulums in a homogeneous gravitational field: ξ is a dimensionless coordinate along a wire to which the pendulums are attached, τ is a dimensionless time variable, and

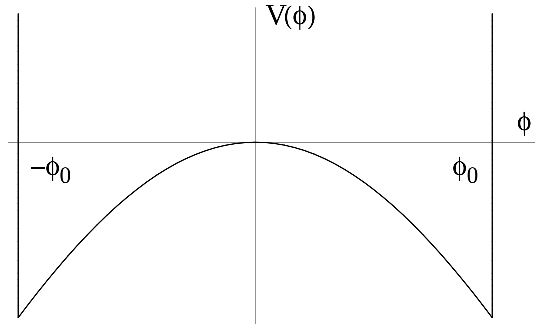

Fig. 1. The potential $V(\phi)$.

$\phi(\xi,\tau)$ is equal to the angle between the upward vertical direction and the arm of the pendulum located at the point ξ and time τ. The pendulums can move only in planes perpendicular to the wire. The restriction

$$-\phi_0 \le \phi \le \phi_0 \tag{3}$$

is enforced by two rigid rods parallel to the wire. We assume that the pendulums bounce elastically from the rods. For a more detailed description of this system see [5].

Time evolution of $\phi(\xi,\tau)$ is governed by the following equation

$$\left(\partial_\tau^2 - \partial_\xi^2\right)\phi = \sin\phi\,, \quad \text{if} \quad |\phi| < \phi_0\,, \tag{4}$$

and by the elastic bouncing condition

$$\partial_\tau\phi(\xi,\tau) \to -\partial_\tau\phi(\xi,\tau)\,, \quad \text{if} \quad \phi(\xi,\tau) = \pm\phi_0\,. \tag{5}$$

Hence, the time evolution of the velocities $\partial_\tau\phi$ is in general not continuous. The vertical upward position, that is $\phi = 0$, is unstable. The stable positions are given by $\phi = \pm\phi_0$. Energy of the system is conserved. Typical motion of a pendulum consists of smooth "flights" between impacts with the rigid rods.

In the case $\phi_0 \ll 1$ we may replace $\sin\phi$ by ϕ, and instead of Eq. (4) we then have

$$\left(\partial_\tau^2 - \partial_\xi^2\right)\phi = \phi\,, \quad \text{if} \quad |\phi| < \phi_0\,. \tag{6}$$

Physically, the kink is obtained when the pendulums gradually turn from one rod to the other one along the wire. Its center is identified with the point ξ_1 such that $\phi(\xi_1) = 0$ — the corresponding pendulum points in the upward direction. Static kink is represented by a time-independent solution $\phi_{\rm c}(\xi)$ of Eq. (4) or Eq. (6) which interpolates between $-\phi_0$ and $+\phi_0$, and such that $\phi_{\rm c}$ and $\partial_\xi\phi_{\rm c}$ are continuous when $\phi_{\rm c} \to \pm\phi_0$. Such solutions have particularly simple form in the case of Eq. (6): for example, the kink located at $\xi_1 = 0$ is given just by

$$\phi_{\rm c}(\xi) = \begin{cases} -\phi_0 & \text{for} \quad \xi \leq -\frac{\pi}{2}\,, \\ \phi_0 \sin\xi & \text{for} \quad -\frac{\pi}{2} \leq \xi \leq \frac{\pi}{2}\,, \\ \phi_0 & \text{for} \quad \xi \geq \frac{\pi}{2}\,. \end{cases} \tag{7}$$

Anti-kink is represented by $\phi_{\overline{c}}(\xi) = -\phi_{\rm c}(\xi)$. Translations or "Lorentz" boosts give, respectively, shifted or moving kink or anti-kink. As discussed in [5], the kink has strictly finite extension for any $\phi_0 < \pi$. Therefore, it is a simple example of so called compactons which were found for modified Korteweg–deVries models [9–11], and in a system of pendulums with non-trivial anharmonic coupling between them [12].

Condition (5) is quite troublesome because it introduces discontinuity in phase space trajectories of pendulums. For this reason, often it is convenient to "unfold" the model. By this we mean passing to a new model with a new field $\underline{\phi}(\xi,\tau)$ such that $\partial_\tau\underline{\phi}$ has continuous time evolution. $\underline{\phi}$ can take arbitrary real values. The relation between ϕ and $\underline{\phi}$ has the following form. If

$$\underline{\phi} \in [-\phi_0 + 2k\phi_0, \phi_0 + 2k\phi_0]\,, \tag{8}$$

where k is an integer, then

$$\phi(\xi,\tau) = \begin{cases} \underline{\phi}(\xi,\tau) - 2k\phi_0 & \text{if} \quad k \ \text{ is even}\,, \\ 2k\phi_0 - \underline{\phi}(\xi,\tau) & \text{if} \quad k \ \text{ is odd}\,. \end{cases} \tag{9}$$

The impacts on the rods occur when $\underline{\phi} = \phi_0 + 2l\phi_0$ with integer l. Relation (9) is depicted in Fig. 2.

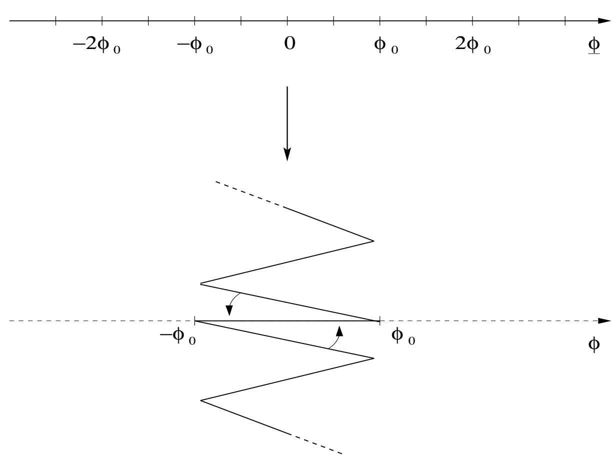

Fig. 2. The folding transformation.

Evolution of $\underline{\phi}$ is governed by the equation

$$(\partial_\tau^2 - \partial_\xi^2)\underline{\phi} = -\frac{d\underline{V}(\underline{\phi})}{d\underline{\phi}}\,, \tag{10}$$

where $\underline{V}(\underline{\phi})$ is given by the following formula: if $\underline{\phi}$ belongs to the interval (8), then in the case of Eq. (4)

$$\underline{V}(\underline{\phi}) = \cos(\underline{\phi} - 2k\phi_0) - 1\,,$$

(see Fig. 3) and

$$\underline{V}(\underline{\phi}) = -\tfrac{1}{2}(\underline{\phi} - 2k\phi_0)^2 \tag{11}$$

in the case of Eq. (6).

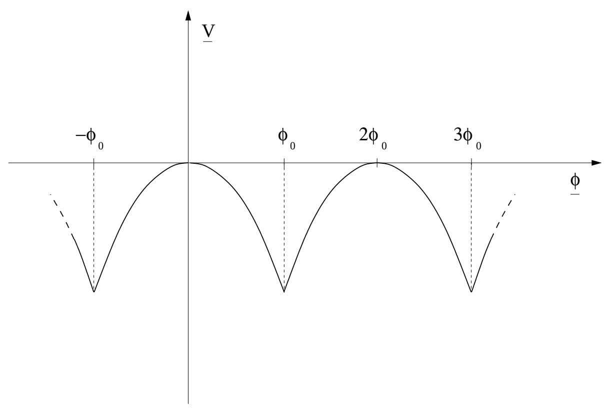

Fig. 3. The potential $\underline{V}(\underline{\phi})$.

Time evolution of $\underline{\phi}$ and of its first derivatives $\partial_\tau\underline{\phi}$, $\partial_\xi\underline{\phi}$ is continuous. It follows from transformation (9) that condition (5) is automatically satisfied.

3. Waves of impacting pendulums

Let us first consider uniform motions of the pendulums, *i.e.*, the ones such that ϕ does not depend on ξ. In this case Eq. (6) is reduced to

$$\frac{d^2\phi}{d\tau^2} = \phi\,, \tag{12}$$

if

$$|\phi| < \phi_0\,.$$

The energy integral of motion corresponding to Eq. (12) has the form

$$\dot{\phi}^2 - \phi^2 = c_0\,. \tag{13}$$

In the case the constant c_0 is positive we may write $c_0 = u^2$, where $u > 0$ is the absolute value of the velocity of pendulums at the vertical upward position ($\phi = 0$). This class of trajectories describes the pendulums impacting simultaneously on one barrier rod with the velocity $(u^2 + \phi_0^2)^{1/2}$, elastically bouncing back from it and moving over the upward positon and further, until they impact on the other rod, from which they are reflected again — the motion is periodic in time. The pertinent solutions of Eq. (12) have the form

$$\phi(\tau) = \pm u \sinh(\tau - \tau_0)\,. \tag{14}$$

At the time τ_0 all pendulums are in the vertical position. They fall onto one of the rods at the time τ_q such that

$$u \sinh(\tau_q - \tau_0) = \phi_0\,. \tag{15}$$

Hence,

$$\tau_q - \tau_0 = \operatorname{arsinh}(\phi_0/u) = \ln\left(\frac{\phi_0}{u} + \sqrt{1 + \frac{\phi_0^2}{u^2}}\right)\,. \tag{16}$$

The period T_0 of this motion is given by the formula

$$T_0 = 4(\tau_q - \tau_0)\,. \tag{17}$$

The phase space picture of trajectories of this type is presented in Fig. 4 by the curve A. Because the velocity changes its sign at the moment of the impact, the trajectory consists of two disconnected parts.

The curves B and C in Fig. 4 represent two different sets of trajectories such that the pendulums do not pass through the vertical position ($\phi = 0$): the pendulums bounce from one rod, raise a little bit and fall on the same rod. Also this motion is periodic in time. The corresponding solutions of Eq. (12) have the form

$$\phi(\tau) = \pm \phi_m \cosh(\tau - \tau_0)\,, \tag{18}$$

where ϕ_m is a constant such that $0 < \phi_m < \phi_0$. In this case, $c_0 = -\phi_m^2$ in formula (13). At the time τ_0 pendulums reach their highest position, $\phi(\tau_0) = \pm\phi_m$. They fall on the rod after the time interval $\tau_q - \tau_0$, where

$$\cosh(\tau_q - \tau_0) = \frac{\phi_0}{\phi_m}\,. \tag{19}$$

Hence,

$$\tau_q - \tau_0 = \ln\left(\frac{\phi_0}{\phi_m} + \sqrt{\frac{\phi_0^2}{\phi_m^2} - 1}\right)\,. \tag{20}$$

The period is equal to $T_0 = 4(\tau_q - \tau_0)$. Waves generated from these solutions were discussed in [5].

There are solutions of Eq. (12) for which $c_0 = 0$:

$$\phi_\pm(\tau) = A\phi_0 \exp[\pm(\tau - \tau_0)]\,, \tag{21}$$

where $A = \pm 1$. The solution ϕ_+ describes pendulums which at the time τ_0 start to move up from one of the rods with the initial velocity $A\phi_0$ and reach the upward position at $\tau = +\infty$. The other solution describes the reverse process. These motions are not periodic ($T_0 = \infty$), and they separate the previous three classes of trajectories. In Fig. 4 they are represented by the dashed lines.

Finally, the thick dot in the center of Fig. 4 denotes the trivial, unstable trajectory $\phi = 0$, and the thick dots at $\phi = \pm\phi_0$ the stable equilibrium positions.

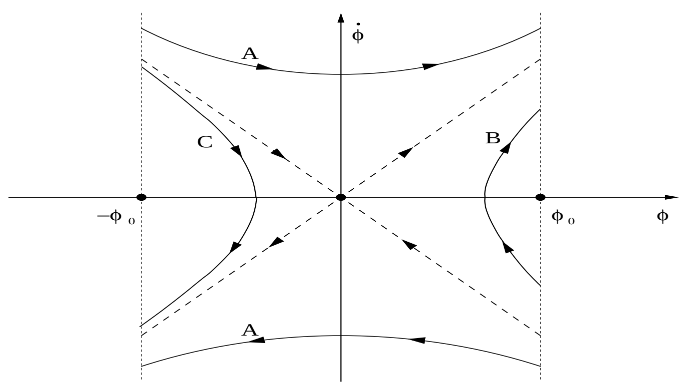

Fig. 4. Phase space trajectories of the uniformly impacting pendulums.

The spatially homogeneous solutions discussed above can easily be generalised to solutions describing periodic in time and space waves just by performing a Lorentz-like boost. For example, in the familiar case of Klein–Gordon wave equation for a scalar field ψ, $\Box\psi = m_0^2\psi$, instead of Eq. (12) we have $d^2\psi/d\tau^2 = -m_0^2\psi$ with the solutions $\psi = A\exp(\pm i\, m_0\tau)$. Lorentz boost with velocity w, where $|w| < 1$, amounts to the substitution $\tau \to \zeta = \gamma(\tau - w\xi)$, where $\gamma = (1 - w^2)^{-1/2}$. In this way we obtain the plane wave with the frequency $\omega = \pm\gamma m_0$ and the wave vector $k = \pm m_0\gamma w$ such that $\omega^2 - k^2 = m_0^2$.

Applying the boost to the solutions of the class B or C we obtain waves which coincide with the waves discussed in our previous work [6] after the substitution $w = 1/v$. Waves obtained from solution A are combined piecewise from the functions

$$\phi(\tau,\xi) = \pm u \sinh(\zeta - \tau_0)\,, \tag{22}$$

see Fig. 5.

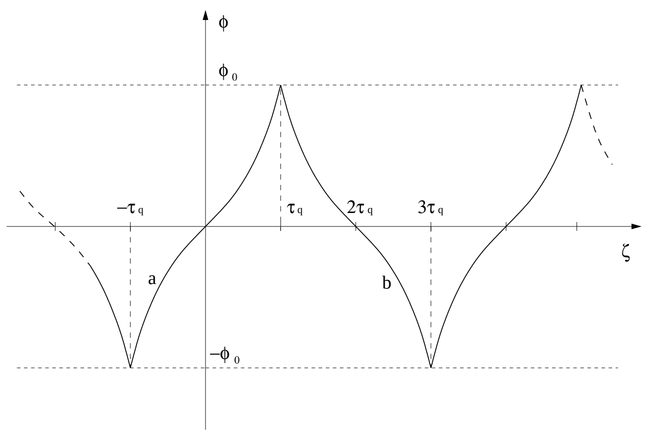

Fig. 5. The wave of impacting pendulums.

Here

$$\zeta = \frac{v\tau - \xi}{\sqrt{v^2-1}}\,, \tag{23}$$

τ_0 is a constant, and v is the phase velocity of the wave. It is assumed that $|v| > 1$. The group velocity is equal to the boost velocity $w = 1/v$. The wave length and the frequency are equal to

$$\lambda_0 = 4\sqrt{v^2-1}(\tau_q - \tau_0)\,, \qquad \omega_0 = \frac{v}{\sqrt{v^2-1}}\frac{1}{T_0}\,,$$

where T_0 is given by formula (17). Depending on the value of v, λ_0 can be any positive number, while $0 < |\omega_0| < 1/T_0$. When $v \to \infty$ we recover the uniformly bouncing pendulums: $\lambda_0 = \infty$, $\omega_0 = T_0^{-1}$.

The trick with boost can be applied if we have a solution which exists for all τ – otherwise we would obtain $\phi(\xi,\tau)$ on a part of the ξ axis only. For this reason, solutions (21) have to be combined together in order to cover the whole time axis. There are two possibilities which differ by overall sign factor:

$$\phi(\tau) = \begin{cases} \phi_0 \exp(\tau - \tau_0) & \text{for} \quad \tau \le \tau_0\,, \\ \phi_0 \exp(\tau_0 - \tau) & \text{for} \quad \tau \ge \tau_0\,, \end{cases}$$

and $-\phi(\tau)$. Boosting these solutions consists of replacing τ by ζ as above. We obtain a travelling wave in which the pendulum located at ξ_1 hits the rod at the time

$$\tau_1 = \frac{\sqrt{v^2-1}}{v}\,\tau_0 + \frac{\xi_1}{v}\,,$$

while all other pendulums either are raising (those at $\xi < \xi_1$) or falling down (those at $\xi > \xi_1$).

4. The width of the compacton when $\phi_0 \to \pi-$

In the particular case when $\phi_0 = \pi$ the two rods merge into one put below the wire which supports the pendulums, and the angle ϕ can take arbitrary value in the interval $[-\phi_0, \phi_0]$. As far as the static configurations are considered, such a system of pendulums coincides with the one corresponding to the sine-Gordon equation, where the rod is absent altogether. The point is that the least energy configurations of the systems coincide. Of course, for time-dependent configurations the two systems remain different because in our system the pendulums bounce from the rod, while in the sine-Gordon one they can move without encountering any obstacles.

The soliton of sine-Gordon model has infinite extension because of its exponential tails. On the other hand, the compacton does not have tails for any $\phi_0 < \pi$. Therefore, its width $\xi_0(\phi_0)$ should be divergent when $\phi_0 \to \pi$ from below. In [5] we have obtained the following formula for the width

$$\xi_0(\phi_0) = \frac{\phi_0}{\sqrt{2}} \int\limits_0^1 d\lambda \frac{1}{\sqrt{\cos(\lambda\phi_0) - \cos\phi_0}}\,. \tag{24}$$

Substituting here $\phi_0 = \pi - \varepsilon$, $\varepsilon > 0$ and changing the integration variable to $\sigma = (\pi - \varepsilon)\lambda/2$ we obtain

$$\xi_0(\phi_0) = \int\limits_0^{\pi/2-\varepsilon/2} d\sigma \frac{1}{\sqrt{\cos^2(\varepsilon/2) - \sin^2\sigma}}. \tag{25}$$

Second change of integration variable, namely

$$\sigma \to s = \frac{\sin\sigma}{\cos(\varepsilon/2)}$$

yields the elliptic integral of the first kind

$$\xi_0(\phi_0) = \int\limits_0^1 ds \frac{1}{\sqrt{(1 - s^2\cos^2(\varepsilon/2))(1 - s^2)}}\,. \tag{26}$$

Thus,

$$\xi_0(\pi-\varepsilon) = K\left(\cos^2(\varepsilon/2)\right) . \tag{27}$$

Because

$$K(x) \cong \frac{1}{2}\ln\frac{16}{1-x} ,$$

for $x \to 1-$, see *e.g.* [13], we finally obtain that

$$\xi_0(\pi-\varepsilon) \cong \ln\frac{4}{\sin(\varepsilon/2)} , \tag{28}$$

when $\varepsilon \to 0+$. Thus, the width of the compacton diverges logarithmically when $\phi_0 \to \pi-$.

5. Equivalence with the non-analytically perturbed sine-Gordon system

The potential $\underline{V}(\underline{\phi})$ is periodic in $\underline{\phi}$, hence it can be written as Fourier series. Straightforward calculations show that potential (11) can be written in the form

$$\underline{V}(\underline{\phi}) = -\frac{\phi_0^2}{6}\left[1 + \frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi}{\phi_0}\underline{\phi}\right)\right] . \tag{29}$$

It is clear from Fig. 3 that the derivative $\underline{V}$ with respect to $\underline{\phi}$ does not exist at the points $\underline{\phi} = k\phi_0$, where k is an odd integer. At these points the Fourier series for $d\underline{V}(\underline{\phi})/d\underline{\phi}$ is divergent.

The constant and the $n = 1$ terms in formula (29) give the potential

$$\underline{V}^{(1)}(\underline{\phi}) = -\frac{\phi_0^2}{6} + \frac{2\phi_0^2}{\pi^2}\cos\left(\frac{\pi}{\phi_0}\underline{\phi}\right) , \tag{30}$$

which is analytic in $\underline{\phi}$. Let us introduce the new field

$$\psi(\sqrt{2}\tau, \sqrt{2}\xi) = \frac{\pi}{\phi_0}\,\underline{\phi}(\xi,\tau) + \pi\, . \tag{31}$$

Euler–Lagrange equation obtained from the potential $\underline{V}^{(1)}(\underline{\phi})$ can be written in the form

$$(\partial_\tau^2 - \partial_\xi^2)\psi(\tau,\xi) = -\sin\psi(\tau,\xi)\, , \tag{32}$$

which is identical with the sine-Gordon equation (on both sides we have replaced $\sqrt{2}\tau, \sqrt{2}\xi$ by τ, ξ).

The $n > 1$ terms in formula (29) give the non-analytic perturbation of the sine-Gordon potential (30),

$$\underline{V}^{\text{pert}}(\psi) = -\sum_{n=2}^{\infty} \frac{1}{n^2} \cos(n\psi)\,,$$

and of the sine-Gordon equation

$$(\partial_\tau^2 - \partial_\xi^2)\psi = -\sin\psi - \sum_{n=2}^{\infty} \frac{\sin(n\psi)}{n}\,.$$

The sum on the r.h.s. of the last equation is finite for all values of ψ, but it is not continuous at $\psi = 2k\pi$, where k is integer. This follows from the formula

$$\sum_{n=2}^{\infty} \frac{\sin(n\psi)}{n} = \frac{i}{2}\ln(-e^{i\psi}) - \sin\psi$$

and the fact that ln function has a cut along the negative part of the real axis.

The replacement of $\underline{V}(\underline{\phi})$ by $\underline{V}^{(1)}(\underline{\phi})$ gives the sine-Gordon approximation to our system. The folding transformation applied to $\underline{V}^{(1)}(\underline{\phi})$ yields the potential

$$V^{(1)}(\phi) = -\frac{\phi_0^2}{6} + \frac{2\phi_0^2}{\pi^2}\cos\left(\frac{\pi}{\phi_0}\phi\right)\,,$$

where $-\phi_0 \leq \phi \leq \phi_0$. It is presented in Fig. 6. The vertical lines at $\phi = \pm\phi_0$ in Fig. 6 correspond to the two rods. The main difference between this potential and the original one, see Fig. 1, is that now the first derivative of

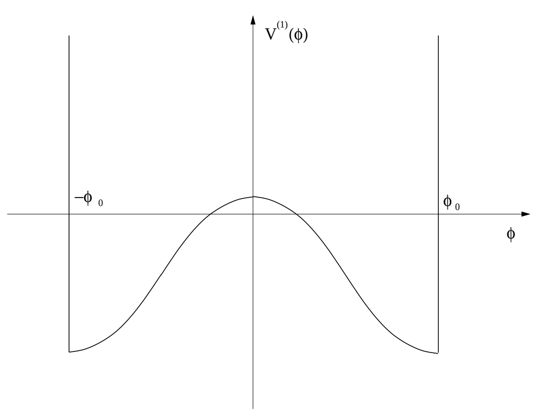

Fig. 6. The potential $V^{(1)}(\phi)$.

the potential vanishes at each of the two minima. This fact has profound influence on the shape of the soliton at large ξ: the soliton acquires the exponential tails.

It is clear that with the help of the sine-Gordon approximation and formula (9) we can utilise the well-known exact solutions of sine-Gordon equation in order to obtain approximate solutions of our system. For example, the sine-Gordon breather [4] gives

$$\underline{\phi}(\xi,\tau) = \frac{4\phi_0}{\pi} \arctan\left(\frac{\sin\left(\frac{\sqrt{2}\,p\,\tau}{\sqrt{1+p^2}}\right)}{p\,\cosh\left(\frac{\sqrt{2}\,\xi}{\sqrt{1+p^2}}\right)} \right) - \phi_0, \tag{33}$$

where $0 < p < \infty$ is a parameter. The folding transformation transforms this solution into the approximate trajectory of our system of pendulums: at the time $\tau = 0$ all pendulums are just bouncing from the rod at $\phi = -\phi_0$, next they move up. Some of them can cross the upward vertical position provided that $p \leq 1$, but they do not reach the other rod. All pendulums stop simultaneously at the time

$$\tau_m = \frac{\pi\sqrt{1+p^2}}{2\sqrt{2}p},$$

and they begin to slide back to the rod, from which they bounce again at the time $2\tau_m$. The motion is periodic in time. We do not know whether there exists breather in our original system with evolution equation (4) or (6).

6. Remarks

1. Our model has several attractive features: simplicity of differential equation (6); the simple analytical form (7) of the kink; neat physical realisation as the system of pendulums with their motions restricted by the two rigid rods. For these reasons we believe that the model can be useful for case studies of such interesting yet difficult to understand in detail phenomena as production of topological defects during symmetry breaking transitions (certain preliminary results in this direction were presented in [6]), scattering of kink on anti-kink, or interaction of kink with a boundary in analogy to investigations reported in [14].

2. Mechanical systems with impact have been investigated recently in connection with so called grazing bifurcation as a road to chaotic behaviour, see, *e.g.*, [15, 16] — the dynamics of such systems is far from trivial. Our system is of field theoretical type, that is it has infinite number of degrees of freedom, while most studies presented in literature so far are devoted to finite systems. It would be very interesting to find out whether the chaotic motions are present in our system and, if present, what is their role in dynamics of compactons.

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