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FINITE TEMPERATURE QUANTUM CORRELATIONS IN SU(2)$_c$ QUARK STATES AND QUANTUM SPIN MODELS

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The entanglement at finite temperatures is analyzed by using thermal models for colored quarks making up the hadron physical states. We have found that these quantum correlations entirely vanish at $T_c \geq m_q/\ln(1.5)$. For temperatures larger than T_c the correlations are classical. We have also worked out the entanglement for the transverse Ising spin chain. In dependence on both temperature T and transverse field λ we can identify a certain region, where the quantum effects are likely to dominate the system. We suggest the mutual information as a quantitative measure for the correlations in the ground state.

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1. Introduction

One of the consequences of existing finite entropy at zero temperature is the understanding of Gibbs paradox, which represents one of very old thermodynamical problems. We now know that one has to take into consideration the distinguishability for better understanding finite entropy in the ground state. On the one hand, we find that the change in the entropy, when non-identical particles have been mixed at fixed particle number and volume, goes to zero for vanishing temperature. This is known as the Nernst's heat theorem. On the other hand, for identical particles the entropy goes to a temperature-independent value $\mathcal{O}(n)$ proportional to the number of mixed

states at zero temperature. It is obviously independent on all thermodynamical quantities other than the number of particles itself. This is usually stated under the name of the third law of thermodynamics and dates back to the beginning of last century. Surprisingly, these two cases are often mixed, and it is supposed that in all systems the entropy goes to zero for $T \to 0$. This might be well founded from the accommodativeness to approximate S to zero for zero temperature.

There are many physical systems, in which one has to avoid this approximation. For instance, Schrödinger expected that the ground state of a gas of atoms encloses 2^N degenerate configurations in its structure, which must then yield an entropy of $N \ln 2$. The first application of these ideas to the quark matter has been introduced in [1]. It has been found that the $\mathrm{SU}(2)_c$ color symmetry for each of the colored quarks in the ground state gives an entropy $\ln 2$. This has been extended to the models at finite temperatures in [2]. Such an entropy can be seen as a reflection of the probabilities of quark mixing maneuver inside the mesons. Therefore, at zero temperature the confined quarks seem to be continuously tousled objects.

A further example for the physical systems, in which a finite entropy exists at zero temperature is the recent lattice QCD result for the equation of state of quark–antiquark [3]. It has been found that the entropy takes the constant value $2 \ln 3$ for $T \to 0$. This is obviously well compatible with the evaluation of the entropy of $\mathrm{SU}(3)_c$ colored singlet state based on the models given in [2]. There are also other applications, *e.g.*, the equation of state of the hadronic matter at very low temperatures [4], the quark-pair condensates at very high densities and low temperatures and the compact stellar objects [5].

In this paper we study the quantum correlations and their impact of systems with $\mathrm{SU}(2)_c$ color symmetry at finite temperatures. We suggest the concurrence as a measure for the thermal effects on these correlations. Our aim is to evaluate a temperature, up to which the quantum correlations can survive. In Section 3 we look at the transverse Ising model. The conclusions and outlook are given in Section 4.

2. $\mathrm{SU}(2)_c$ quarks states

Using the models proposed in [2] based on the Boltzmann weighting for the finite temperature states of colored quarks we postulate that the total density matrix in the ground state of a meson singlet-state of $\mathrm{SU}(2)_c$ color symmetry in the Hilbert–Schmidt space $C^2 \otimes C^2$ could be written as

$$\rho_{AB} = \frac{1}{4}\left(I \otimes I - \alpha e^{-\frac{\epsilon(p)}{T}}(\sigma_z \otimes I + I \otimes \sigma_z) - \left(1 - e^{-\frac{\epsilon(p)}{T}}\right)\sum_{i=1}^{3} \sigma_i \otimes \sigma_i\right), \tag{1}$$

where I stands for the identity operator, $\epsilon(p) = \sqrt{m^2 + p^2}$ is the single particle relative energy, $\sigma^i; i = x, y, z$ are the standard Pauli matrices and the quantity $\alpha e^{-\epsilon(p)/T}$ describes the evolution of the color as a function of the temperature T. α represents the free parameter in this model. Clearly, for $\alpha = 1$ we go back to the expression for the thermal reduced density matrix given in [2] in equation (10). However, in order to fulfill the condition that the total density matrix has a non-negative value, we first suppose that

$$-\frac{1}{2} \leq \alpha \leq \frac{1}{2}\,. \tag{2}$$

For $\alpha = 0$, the mixed state in the whole system will be colorless for all temperatures. At $T = 0$ equation (1) gives the expected density matrix for the pure colored singlet state in SU(2)$_c$ ground state. As introduced above, Eq. (1) is also able to reproduce the finite temperature behavior, where we expect no quantum correlations to exist. There is no direct description of a kind of quantum transition available by means of Eq. (1). One should notice that ρ_{AB} satisfies the condition Tr $\rho^2 \leq 1$.

In order to compute the entanglement of such states at finite temperatures, we use the concurrence formalism [6], which is defined as

$$\mathcal{C} = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}\,, \tag{3}$$

where λ's are the square roots of the eigenvalues of $\rho_{AB}(\sigma_y \otimes \sigma_y \rho^{\star}_{AB} \sigma_y \otimes \sigma_y)$ in decreasing order and $\rho^{\star}_{AB}$ is the corresponding complex conjugation in the computational basis $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$ straightforward algebra we find that

$$\mathcal{C}(T) = \begin{cases} 1 - (1 + \frac{\sqrt{1-4\alpha^2}}{2}) e^{-\frac{\epsilon(p)}{T}} & \text{if } T \leq T_{\rm c} \\ \\ 0 & \text{if } T \geq T_{\rm c} \end{cases}\,, \tag{4}$$

where $\frac{\epsilon(p)}{\ln(3/2)} \leq T_{\rm c}$.

The value of α can then be fixed through the hypothetical assumption that the transition temperature $T^d_{\rm c}$ from *confined* hadron gas to the *deconfined* quark gluon phase at zero chemical potential would have the same value as the one for the transition from *entangled* to *non-entangled* SU(2)$_c$ colored states. In this case

$$\alpha^2 = \frac{1}{4}\left(1 - 4\left(e^{\frac{m_q}{T^d_{\rm c}}} - 1\right)^2\right)\,. \tag{5}$$

From the lattice QCD simulations we know that for two quark flavors the transition temperature at $\mu_q = 0$ is ~ 173 MeV. If we take the mass threshold

$m_q(\mu)$ of light quarks as $\epsilon(p)$ [8] where μ in this case is a renormalization group scale and for simplicity we set $m_q(\mu(T)) = 5\,\mathrm{MeV}$, we find that α at which $T_\mathrm{c} = T_\mathrm{c}^d$ equals 0.499.

As shortly explained above, the entanglement transition corresponds to $\mathcal{C} = 0$. Therefrom, we can estimate T_c for the simple case of $\alpha = 0$ and $m_q = 5$ MeV, as

$$T_\mathrm{c} \sim 12.5\,\mathrm{MeV} \ll T_\mathrm{c}^d\,. \tag{6}$$

This is the lower value of T_c. In Fig. 1 we draw $\mathcal{C}$ for the two-site correlations in $\mathrm{SU}(2)_c$ quark states as a function of T for different values of α. We find that $T_\mathrm{c} \ll T_\mathrm{c}^d$ for $\alpha = 0$, and $T_\mathrm{c} \to \infty$ for $\alpha \to 0.5$.

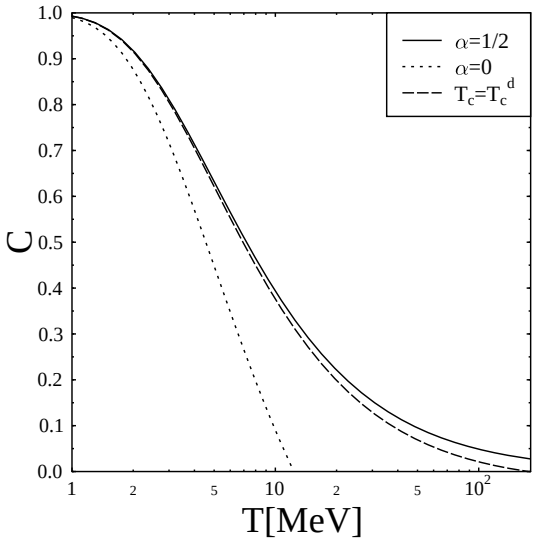

Fig. 1. Concurrence $\mathcal{C}$ as a function of T for two-site correlations in SU(2) quark states with $m_q = 5$ MeV. For $\alpha = 0$, the 'critical' temperature takes the value $T \sim 12.5$ MeV (dotted line). For $\alpha \to 1/2$ we find that $T_\mathrm{c} \to \infty$ (solid line), whereas for $\alpha = 0.499$ we get $T_\mathrm{c} = T_\mathrm{c}^d$ (dashed line).

The density matrices given in Eq. (1) have some particular properties. For specific values of T and α (see Fig. 2) they belong to the states with high entanglement, the so-called Maximally Entangled Mixed States (MEMS) [9]. This would suggest that the thermal evolution of the colored quark will maximize the entanglement at given degree of mixing.

Moreover, it has been shown in [10] that in an antiferromagnetic ring for Heisenberg model (one-dimensional with boundary conditions) with an even number of spin-1/2 the maximum entanglement is equal to the entanglement of the ground state (minimum energy) with zero z component of the spin. The transverse spin models will be studied in Section 3.

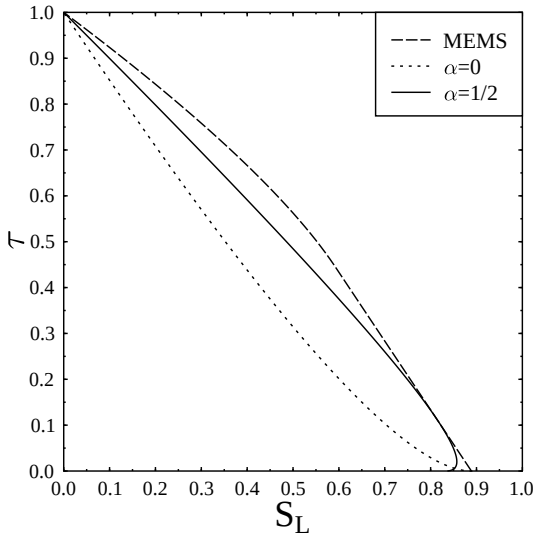

Fig. 2. $\tau \equiv \mathcal{C}^2$ as a function of the linear entropy $S_L = 4/3(1 - \mathrm{Tr}\rho^2)$ for the Maximally Entangled Mixed States (MEMS) and for the entropy calculated for $\alpha = 0$ and $\alpha = 1/2$, respectively.

In Fig. 2 we plot $\tau \equiv C^2$ *versus* the linear entropy $S_L = \frac{4}{3}(1 - \mathrm{Tr}\rho^2)$. Clearly, we notice that the states with the properties given in Eq. (1) typically do not maximize the entanglement at a given degree of state mixing. These are the states with $\alpha = 0$. But for the special case $\alpha \to 0.5$ such states are very close to minimization of the entanglement at all degrees of mixing. For $S_L = 0$, *i.e.*, for a pure state, the results for MEMS and all values of α coincide.

As mentioned in Section 1, the effects of the quantum entropy of SU(2)$_c$ of colored quark states on the bag pressure have been discussed in [4]. This can be also analyzed here by computing the color entropy of single-quark in one meson. In Fig. 4 we show the results for different values of α. In the case $\alpha = 0$ we notice that the quark quantum entropy $S_{q,2}$ is a constant for all temperatures. As given in [4] this case corresponds to excluding the quantum entropy from the equation of state. The reason for this result is clear, since for $\alpha = 0$ all states are composed of completely random *sub*states mixed together with the singlet states. In both groups the single-site entropy takes maximum (the net color is zero). Though for $\alpha = 0$ there is no color polarization for each quark (local polarization is zero) and the overall color is also zero, but the colors of single-quark are still correlated. For temperatures below T_c the correlations is a mixture of quantum and classical ones. However, for $T > T_\mathrm{c}$ the correlations are completely classical. To

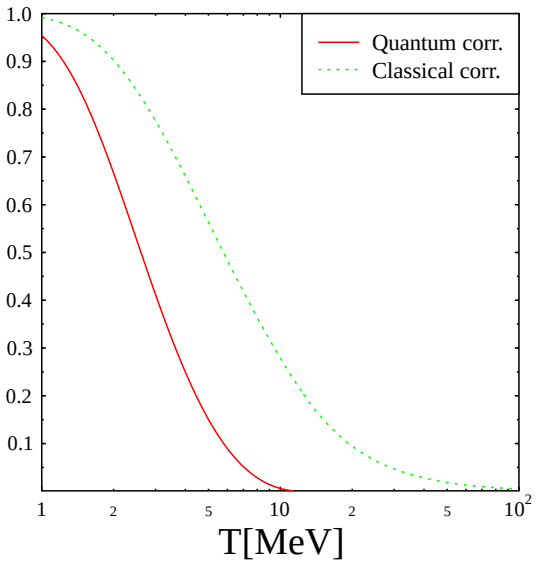

Fig. 3. The relative entropy of entanglement is depicted as a function of T (for $\alpha = 0$ and $m_q = 5$ MeV). The quantum correlations are given in Eq. (7). The classical correlations are the difference between the total correlations from the von Neumann mutual information and the quantum correlations. At T_c the quantum correlations entirely vanish, whereas the classical ones remain finite for all temperatures.

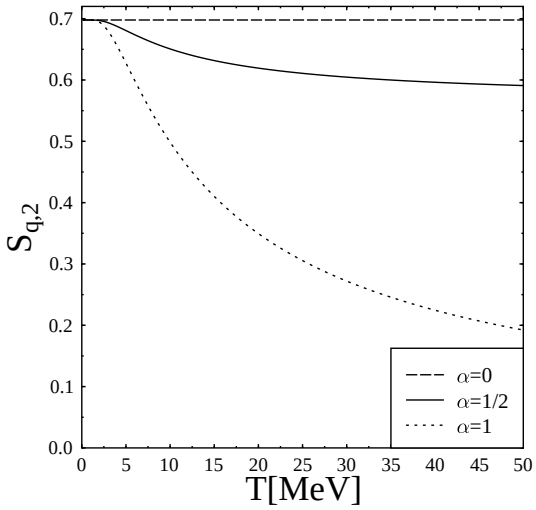

Fig. 4. Single-site color entropy as a function of the temperature T (see text for more details).

illustrate these features we plot in Fig. 3 the entanglement measured by the relative entropy from the entanglement, which is defined as [11]

$$E_r = \min_{\rho^* \in \mathcal{D}} S(\rho \| \rho^*)\,. \tag{7}$$

For completeness, we mention that the classical correlations are defined as the difference between the total correlations measured by the von Neumann mutual information and the quantum correlations measured by the relative entropy from the entanglement [12].

$$\Psi(\rho) = S(\rho \| \rho_A \otimes \rho_B) - \min_{\rho^* \in \mathcal{D}} S(\rho \| \rho^*), \tag{8}$$

where $\mathcal{D}$ is a set of all separable states in the Hilbert space, in which the *sub*spaces A and B are also defined here. ρ_A and ρ_B are the corresponding reduced density matrices, respectively. The upper limit of temperatures, at which both quantum and classical correlations exist, is also shown in Fig. 1 as dotted line ($\alpha = 0$). As expected, at T_c the quantum correlations are zero whereas the classical ones remain finite. That is also the case at higher temperatures.

3. Transverse Ising model

In this section we would like to investigate the quantum correlations (entanglement) for some spin models in which strong correlations exist. Then compare the results with the results of the SU(2)$_c$ colored quark states given in the previous section. The general category of all spin models goes under the name of the Heisenberg model. For transverse Ising model with N sites the Hamiltonian reads

$$\mathcal{H} = -\sum_{i=0}^{N-1} (\lambda \sigma_i^x \sigma_{i+1}^x + \sigma_i^z), \tag{9}$$

where σ^i, $i = \{x, y, z\}$ are the standard Pauli matrices and λ is the transverse field. It should not be confused with the eigenvalues given in Eq. (3). At zero temperature it is expected that this system goes through a quantum phase transition QPT (from para- to ferro-magnetic phase transition). $\lambda < 1$ corresponds to a para-magnetic and $\lambda > 1$ to a ferromagnetic phase. Unlike the ordinary phase transition, which occurs at finite temperatures, the fluctuations in QPT are fully quantum and the transition is due to the changes in the ground state wave functions. At the critical point long-range correlations are expected. This kind of correlations is also expected near the transition point in the *classical* thermal phase transition. However, as

the system has a zero temperature and with the assumption that the ground state is non-degenerate, the system, as a whole, is expected to build up a pure state. It follows that the correlations, which are in principle the accessible signatures for QPT, are obviously stemming from the long-range entanglement in the ground state. Through the change in the correlation functions which will be reflected as a fundamental change in the entanglement of the ground state, we get a distinct signature for probing QPT.

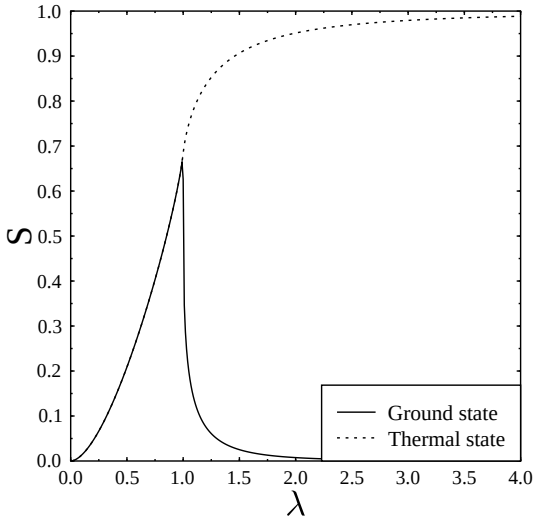

Fig. 5. Zero temperature single-site entanglement S of the ground state of transverse Ising model (solid line). The dashed line shows the corresponding results for the thermal ground state.

Before we study the two-site entanglement we want to know the role of λ in describing the entanglement. To illustrate this we start in Fig. 5 with the single-spin quantum entropy [16] which in turn provides a measure of the entanglement of one spin with the rest of the chain. This we achieve by evaluating the entropy of the single-site density matrix ρ_A of the ground state after tracing out all other spins (see Appendix). In transverse Ising model we notice that S varies from zero at $\lambda = 0$ to a maximum value at $\lambda = 1$ (Fig. 5). For higher values of λ, S declines, rapidly, and for $\lambda \to \infty$, S reaches the asymptotic value zero. For the thermal state ($\rho_i = \lim_{T\to 0} \mathrm{Tr}_{j\neq i} e^{-\beta\mathcal{H}}/Z$) we can also apply the von Neumann prescription for calculating the entropy. We get here the same behavior as in the ground state entropy for $0 \leq \lambda \leq 1$. But for $\lambda > 1$, we notice that S increases until it reaches the asymptotic value 1 for $\lambda \to \infty$. This is due to the equal mixing of spin thermal states so we can conclude that the single-spin entropy in the thermal ground state does not measure the entanglement in

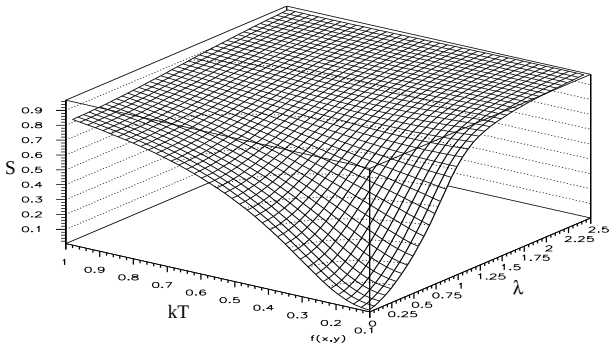

Fig. 6. Single-site thermal entanglement S as a function of T and λ in transverse Ising model.

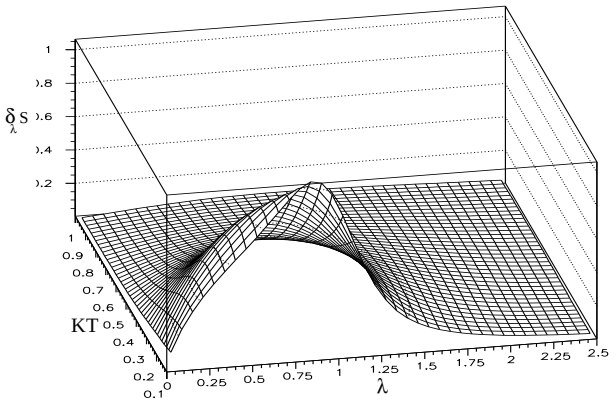

Fig. 7. First derivative of single-site thermal entanglement $\partial_\lambda S$ as a function of T and λ.

the limit $\lambda \to \infty$. It gives merely the degree of mixing. For the first time, we study here also the single-site thermal quantum entropy and its derivative with respect to λ. It should be understood that in this case the quantum entropy is not a measure for the entanglement, however one can expect that the change in the quantum entropy could, in turn, reflect the change in the entanglement. In Figs. 6 and 7 we show the results of $kT - \lambda$ plane. We notice that when $\lambda = 1$ both functions are approximately maximum (for all temperatures). As expected, the first derivative of the quantum entropy for high temperatures is always peaked at $\lambda = 1$. This region is not fully compatible with the thermal concurrence published in [13]. More precisely, when we compare our results (Figs. 6 and 7) with Fig. 6 in [13], we see that in the latter the concurrence increases for decreasing λ for high temperatures!

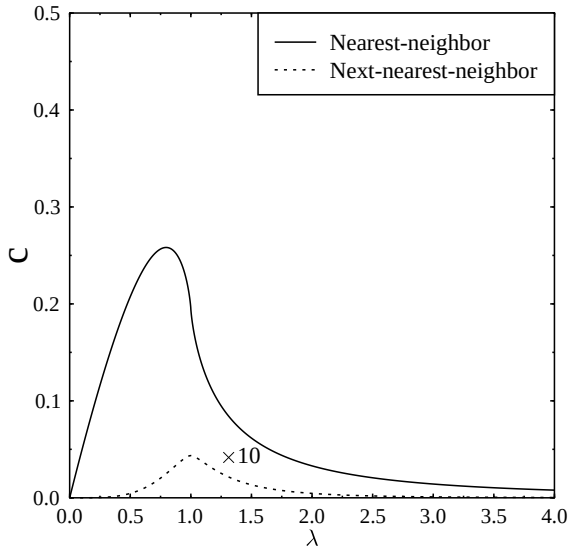

Fig. 8. The entanglement measured by the concurrence of nearest- and next-nearest-neighbor two spins at zero temperature depicted as a function of the transverse field λ. The next-nearest-neighbor concurrence is very small, so for better visualization are multiplied here by factor 10.

The two-site entanglement, as its name indicates, measures how the two spins separated by a distance r are entangled. This measure is to be accomplished by evaluating the concurrence of the entanglement of formation of two-site density matrix after tracing out all other spins in the chain (see Appendix). The two-site entanglement for the nearest-neighbor and the next-nearest-neighbor spin-pairs at zero temperature for the transverse Ising model shown in Fig. 8. The length of entanglement ξ apparently vanishes for all other pairs, since the concurrence gets negative. Therefore, the true non-local quantum part is a short-range one, whereas the classical part is a long length one (diverging for an infinite system). We now would like to compare the classical with the quantum correlations. We use the same formalism as in the previous section, however, since it is not possible to directly evaluate the relative entropy for the two-site density matrix, we provide upper and lower bounds to estimate the quantum correlations and afterwards we extract the classical ones. The upper bounds are calculated from the entanglement of formation

$$E_r^{\max} = h\left(\frac{(1+\sqrt{1-\mathcal{C}^2})}{2}\right), \tag{10}$$

where $h(x) = -x\log x - (1-x)\log(1-x)$ is the Shannon entropy. The lower

bound reads [15]

$$E_r^{\min} = (\mathcal{C}-2)\ln\left(1-\frac{\mathcal{C}}{2}\right) + (1-\mathcal{C})\log(1-\mathcal{C})\,. \tag{11}$$

In Fig. 9 the bounds on classical and quantum correlations are depicted as a function of λ. We see that the quantum correlations are smaller than the classical ones. The latter are merely produced by the entanglement in the ground state. This suggests to study of the mutual information as well as the single- and two-site entanglement [16, 17], because the mutual information combines all correlations (classical and quantum) which are created by the entanglement [17].

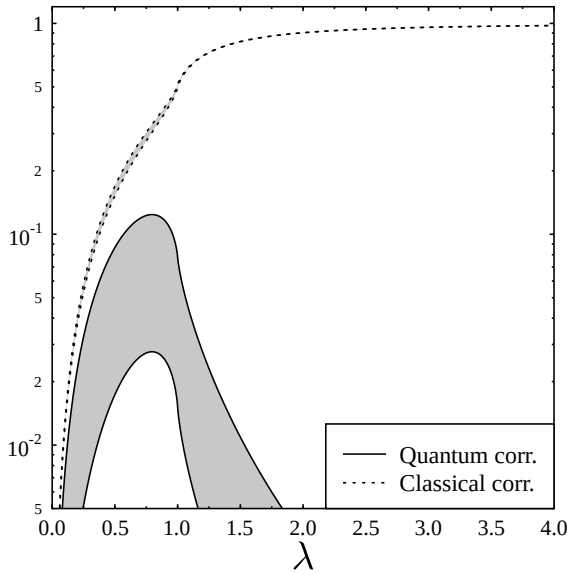

Fig. 9. Bounds on classical and quantum correlations for transverse Ising model are drawn *versus* the strength of the transverse magnetic field λ.

In [16] the scaling of concurrence has been studied. It has been found that for an infinite system the first derivative of the concurrence $\partial_\lambda \mathcal{C}$ is a diverging function at the critical point. It is important to see whether the mutual information has such a property. We show in Fig. 10 the first derivative of the mutual information $\partial_\lambda I$ (top curve). We see obviously for the first derivative of concurrence $\partial_\lambda \mathcal{C}$ (bottom curve) the same dependence as we obtained for $\partial_\lambda I$. This has been studied in details elsewhere [16]. As expected, at the critical point $\lambda = 1$ we notice that $\partial_\lambda I$ is singular and divergent. This divergence is a non-ambiguous signal for the change in the

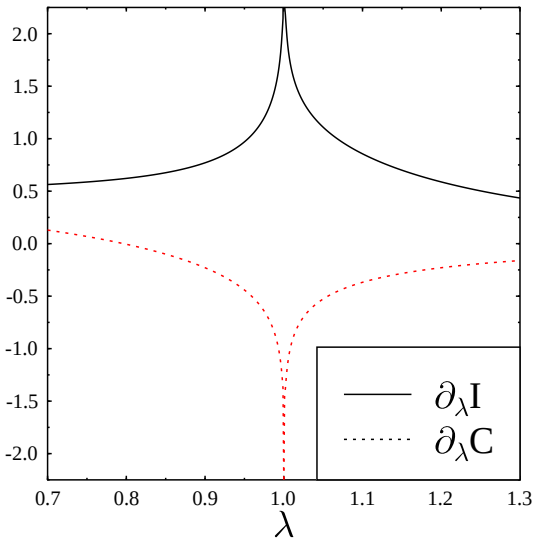

Fig. 10. First derivative of the quantum and classical correlations as functions of λ. For $\lambda < 1$ both derivatives are finite. They vanish for $\lambda \to \infty$. At the critical point ($\lambda = 1$) both functions are divergent, indicating radical change in the ground state structure.

ground state structure. Therefore, we expect that the mutual information can also be used to probe the entanglement in ground state.

Despite the color density matrix Eq. (1) (as discussed in the previous section) does not undergo QPT, it is still possible to compare the results with the transverse Ising model at zero temperature. In fact for $\lambda > 1$ the mixing due to the change of the temperature in colored system is compatible with the degeneracy mixing in the ground state. Clearly, for $\lambda > 1$ the relative entropies (Figs. 3, 8 and 9) in both systems are decreasing functions for increasing T and λ. However, the classical correlations are different in both systems (see Figs. 3 and 9). This is due to the tendency of the color system to be in a completely random state by loosing all kind of correlations at high temperatures.

4. Conclusions and outlook

In the limit of the proposed *thermal* models for colored quarks making up the hadron physical states we have utilized the concurrence, in order to study the quantum correlations in $SU(2)_c$ colored quark states. The aim of this study is to see whether the quantum and/or the classical correlations can survive the deconfinement phase transition from hadron to quark–gluon plasma. Obviously, we have found a region, where an entanglement tran-

sition is highly expected. This gives an upper temperature limit $T_{\rm c}$, below which the quantum correlations are significant. At temperatures higher than $T_{\rm c}$ the quantum correlations entirely disappear allowing to dominate classical correlations.

We have also studied the correlations in another system, that has the same SU(2) symmetry, namely the transverse Ising model. Using the single-site entropy we have identified a certain region in the $T-\lambda$ plane, where the quantum effects dominate likely the behavior of this spin system. The classical correlations can be extracted from the knowledge of quantum and total correlations. We can conclude that the nature of quantum correlations is short-ranged, whereas that of classical correlations is long-ranged. We have also shown that the mutual information, which combines both types of correlations, is a possible candidate to study the structure of QPT. In a forthcoming work we shall address the mutual information and its impact in describing the correlations at finite temperatures in more detail. We also utilize the other models given in [2] for the SU(3)$_c$ quark states, which apparently is more complicated than SU(2)$_c$. We hope to compare the quantum phase transition in SU(3)$_c$ colored quark states with the possible transition in the interior of compact stellar.

Appendix

For the one-dimensional quantum Ising model with N spin-1/2 sites the one- and two-point correlation functions cab be calculated by using the operator expansion for the total density matrices in terms of the Pauli matrices.

$$\rho_i \;=\; \frac{1}{2}\sum_{k=0}^{3}\langle\sigma_k^i\rangle\sigma_i^k \tag{12}$$

$$\rho_{k,l} \;=\; \frac{1}{4}\sum_{i=0,j=0}^{3}\langle\,\sigma_k^i\sigma_l^j\rangle\,\sigma_k^i\otimes\sigma_l^j\,. \tag{13}$$

As in [14] we can replace the sums by integrals and afterwards calculate the different correlations as a function of the separations between arbitrary two-sites, $r=|k-l|$. Then for the thermodynamic limit $N\to\infty$ we find that

$$\langle\sigma_i^z\rangle \;=\; -\frac{1}{\pi}\int_0^{\pi}dk(1+\lambda\cos(k))\frac{\tanh(\beta\omega/2)}{\omega} \tag{14}$$

where $\beta=1/T$. For $T=0$ we have

$$\langle\sigma_i^x\rangle \;=\; \begin{cases} 0, & \lambda\le 1 \\ (1-\lambda^{-2})^{1/8}, & \lambda>1 \end{cases} \tag{15}$$

Using the definition:

$$C_r = \frac{1}{\pi}\int_0^{\pi} dk\cos(kr)(1+\lambda\cos(k))\frac{\tanh(\beta\omega/2)}{\omega} - \frac{\lambda}{\pi}\int_0^{\pi} dk\sin(kr)\sin(k)\frac{\tanh(\beta\omega/2)}{\omega}, \quad (16)$$

newpage the two-site correlation functions read [14, 16]

$$\langle\sigma_0^x\sigma_r^x\rangle = \begin{vmatrix} C_{-1} & C_{-2} & \dots & C_{-r} \\ C_0 & C_{-1} & \dots & C_{-r+1} \\ \vdots & \vdots & \ddots & \vdots \\ C_{r-2} & C_{r-3} & \dots & C_{-1} \end{vmatrix}, \quad (17)$$

$$\langle\sigma_0^y\sigma_r^y\rangle = \begin{vmatrix} C_1 & C_0 & \dots & C_{-r+2} \\ C_2 & C_1 & \dots & C_{-r+3} \\ \vdots & \vdots & \ddots & \vdots \\ C_r & C_{r-1} & \dots & C_1 \end{vmatrix}, \quad (18)$$

$$\langle\sigma_0^z\sigma_r^z\rangle = \langle\sigma_0^z\rangle^2 - C_rC_{-r} \quad (19)$$

It should be noticed that in Eq. (19) there is no factor 4 in the front of the first term as the case in [16]. As in [14] we find that the factor 4 should be multiplied by the magnetization m_z and not by the polarization $\langle\sigma_z\rangle$. Since $m_z = \langle\sigma_z\rangle/2$, we get a factor 1.

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