

SEARCH FOR THE NUCLEAR HYPER-DEFORMATION: MOTIVATIONS AND NEW STRATEGIES*

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We present arguments suggesting, in contrast to commonly accepted way of thinking, that the mechanisms of the super- and of the hyper-deformation in nuclei, and thus the physics motivations behind, are different. Consequently the research strategies of the nuclear *hyper-deformation* as opposed to *super-deformation* should be adapted appropriately, and certainly changed with respect to what seems to be the present day attitude. New results of the corresponding calculations are illustrated and strategies better adapted to the present day instrumental sensitivity are formulated.

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1. Introduction: super-deformation — historical remarks

Our present-day knowledge concerning the nuclei at large elongations comes primarily from the discrete spectroscopy studies of the nuclear *super-deformation*. Such studies began with the nuclei around ^{132}Ce and ^{152}Dy . Theoretical predictions of the existence of the whole island of super-deformed nuclei in the $A \sim 150$ mass region, where the successful experimental discoveries started in 1986, have been formulated already in 1985 with the help of the results of the mean-field calculations, Ref. [1]. In the latter article, the deformed Woods–Saxon Hamiltonian and the “universal” parametrisation of Ref. [2] have been employed. The first confirmation through the discovery of the yrast SD-band in the ^{152}Dy nucleus, came in 1986, Ref. [3]. Already at that time *i.e.* long before the discoveries of hundreds of the SD-bands

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in several dozens of nuclei, the *systematic abundance scheme* of the SD mechanism throughout the Periodic Table has been predicted by theory using the notion of an approximate pseudo SU_3 symmetry of the nuclear mean-field in 1987, Ref. [4]. Today, nearly two decades later, an excellent confirmation of this abundance scheme exists; it is founded by numerous experiments (*cf.* Ref. [5] for an extensive collection of the experimental results).

Below we would like to discuss in some detail the following two issues:

1. *What did the experimental studies of the super-deformation bring to the theoretical understanding of the extreme deformation phenomena?* and
2. *How can/should we use this knowledge in the hyper-deformation studies?*

2. Super-deformation from a perspective: What did we learn?

The rank of importance of scientific discoveries is always an issue of a debate and the “final judgement” usually evolves in time. After nearly twenty years of the high-spin super-deformation studies we do not hesitate to place the issue of the numerical values of deformation parameters at the last place on our list of importance¹. What are then these “super-important” pieces of knowledge in relation to the nuclear super-deformation?

We will list them first and give a short description together with a few comments in the following sections (2.1–2.6). Here let us emphasise:

1. On the theory side, the discovery, and on the experimental side, the confirmation of the so-called pseudo SU_3 symmetry, a fundamental symmetry of the nuclear mean-field, and the related multiplet structure, see Ref. [4] and references therein. This fundamental feature is relatively seldomly addressed, especially in the experimental literature. Below we focus on its implications that bypass the frontiers of physics of super-deformed nuclei. In contrast, the remaining items are related directly to superdeformation;
2. The discovery of the so-called “identical-band” structures and their origin;
3. The discovery of the additivity of the quadrupole moments and related weakening of the pairing correlations at high spins and deformations.

¹ Let us stress that our statement does not imply at all that the deformation parameter values are unimportant; in fact they are used below to arrive at, in our opinion, the discoveries that go deeply into the quantum mechanics of an atomic nucleus and the knowledge about its mean-field, bypassing by far the “primitive geometry” arguments. We wish to emphasise their role as means rather than goals in themselves.

2.1. Fundamental symmetries of the nuclear mean-field

The prediction, fully confirmed today, that *the super-deformed nuclei should be abundant in nature* as well as the direct prediction of *where in the Periodic Table the whole nuclear regions with this mechanism should occur* was based on the symmetry arguments, and more precisely, on the SU_3 group properties. The latter have been used as guide-lines of a formulation of an approximate symmetry of the realistic nuclear mean field. Here we would not like to enter into any mathematical details presenting instead the most important qualitative aspects.

We begin with a few comments about the misleading nomenclature used traditionally in the literature related to the symmetries in question. Indeed, the Elliot's SU_3 model arriving early in the evolution of our knowledge about an atomic nucleus gave an elegant description of collective rotation arising in the many-nucleon systems. A different aspect of the SU_3 formalism, that aims at expressing discrepancies between the harmonic-oscillator model of the nuclear mean-field ("pure" SU_3 symmetry) and the realistic realisations of this field in the presence of the spin-orbit interaction ("pseudo" SU_3 symmetry) has been introduced by a number of authors, *cf.* Ref. [6]. Since the name " *SU_3 -model*" has already been reserved in the literature following Elliot, this new realisation was given a bit unfortunate adjective pseudo².

In the early times of rather naive ideas about the nuclear mean-field, the analytically soluble harmonic oscillator was an attractive nuclear model. Its single particle spectra have the form of characteristic multiplets: degenerate at the spherical form of the potential, the single-particle energies split characteristically with increasing deformation, *cf.* Fig. 1 (right). Here we would like to turn the reader's attention to the most important in the present context a point, usually ignored in the illustrations seen in the literature, *viz.* the *residual degeneracies* still present when deformation increases. They are symbolised by the increasing thickness of full lines contained in the figure. Indeed, the spherical harmonic oscillator manifests simultaneously *two* symmetries that are of interest for us at this point. The universal one (present in all spherical potentials) implies the "magnetic" *i.e.* the $(2j+1)$ -degeneracies of the single j -shells. An extra one implies the degeneracy of all the states in a given spherical N -shell *i.e.* for $\ell = N, N-2, \dots, 1$ or 0 for all the associated positive j values given by $j = \ell + \frac{1}{2}$ and/or $j = \ell - \frac{1}{2}$.

² In some traditions/languages this adjective refers usually to "lower quality" *e.g.* poor ideas or "false" appearances. In crystallography for instance pseudo-symmetries refer to "false" (non-existing) symmetries. In nuclear physics this term turns out to represent probably one of the most intriguing approximate symmetries on the sub-atomic scale, see the text around Figs. 1 and 2.

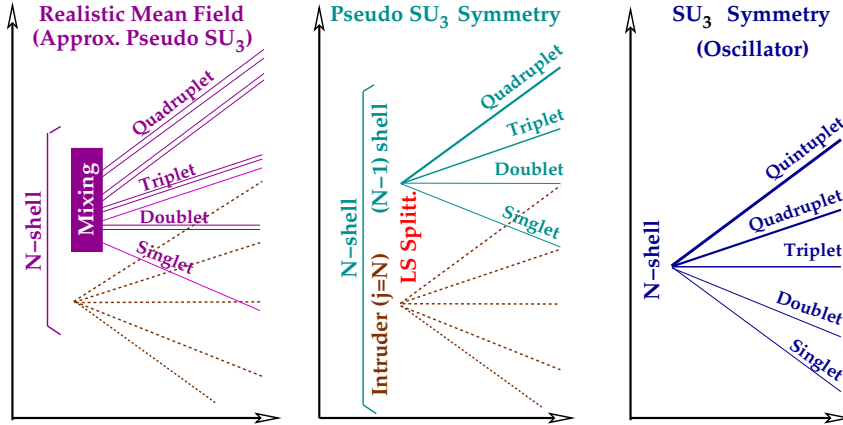


Fig. 1. A schematic representation of three types of symmetries discussed in the text. For simplicity the structure of one main shell (N -shell) is shown: here $N = 4$. Right: Pure harmonic oscillator *i.e.* pure SU_3 symmetry. All the j -shell orbitals are superposed with one another so that we see only 5 curves although in reality there are 15. Superpositions are such that the degeneracies increase with energy. Middle: Spin-orbit interaction taken into account by a down-shift of the highest- j orbital that becomes an “extruder” orbital. Out of the $N = 4$ shell we obtain now pseudo- $N \equiv \tilde{N} \equiv N - 1$ shell with the degeneracy pattern characteristic for the “real” ($N = 3$) shell plus the extruder orbital that does not overlap with the other ones anymore. This scheme is called by definition *pseudo SU_3 symmetry*. Left: Realistic situation; extruder multiplet remains separated from all remaining orbitals, however there is no exact degeneracy of the j -orbitals at zero deformation. At small deformations there is a rearrangement of the orbital structure taking place marked with the word “Mixing”. Already at small deformations a new scheme emerges: an approximate pseudo SU_3 symmetry, where the \tilde{N} orbitals lie close to each other without being strictly degenerate.

In reference to the illustrations in Fig. 1 let us set $N = 4$ to facilitate the presentation. We have 5 single- j orbitals now, *viz.* $j = \frac{9}{2}, \frac{7}{2}, \frac{5}{2}, \frac{3}{2}$ and $\frac{1}{2}$. In particular, the highest $j = j_{\max} = N + \frac{1}{2} = \frac{9}{2}$ shell gives $j_{\max} + \frac{1}{2} = N + 1 = 5$ single-particle orbitals in total. The residual degeneracies take the form of a doublet, a triplet, a quadruplet and a quintuplet in this case. They arise because of the particular overlaps between these single j -shell orbitals: the $m = \frac{3}{2}$ member of the $j = \frac{9}{2}$ orbital coincides with the member $m = \frac{1}{2}$ of the $j = \frac{7}{2}$ orbital; the members $m = \frac{5}{2}$, $m = \frac{3}{2}$ and $m = \frac{1}{2}$ of the orbitals $j = \frac{9}{2}$, $j = \frac{7}{2}$ and $j = \frac{5}{2}$, respectively, coincide forming a triplet, *etc.*

In order to simulate the presence of the spin-orbit (LS) splitting, it has been noticed long ago that an arbitrary phenomenological shift of the highest $j = j_{\max}$ orbital improves considerably the resemblance of the simple

oscillator based spectra to experiment. This situation is referred to as the pseudo SU_3 symmetry, sometimes as pseudo-oscillator symmetry; the j_{\max} orbital is called “extruder”, as being expelled from its original N -shell and/or “intruder” taking into account that its new position will be among the levels belonging to the $(N - 1)$ -shell below.

The realistic situation represented by the solutions to the realistic nuclear mean-field models resembles closely the pseudo SU_3 symmetry case, except that instead of contributing to an exact degeneracy, the members of the $(\tilde{N} = N - 1)$ -shell split into multiplets composed of close-lying levels³, *cf.* Fig. 1 left, thus representing the *approximate pseudo SU_3 symmetry*.

2.2. Physical significance of the fundamental unitary symmetries

Let us begin by reminding the reader of the importance and generality of the above observations given the fact that the mean-field theory of the nucleonic interactions plays a role of the microscopic *reference theory* — analogous to that of the Standard Model in the theory of interactions among the elementary particles. Several more specific, advanced theories can be built after having introduced the single-nucleonic mean-field solutions as a basis. It is, therefore, of fundamental importance to the whole field of the nuclear structure physics to *discover, examine and use the consequences* of the underlying symmetries, even if they are approximate.

In fact, the most fundamental unitary symmetry associated with the microscopic theories of the nucleus is inherent to the general form of the N -body nuclear *Hamiltonian* constructed out of the two-body interactions

$$\begin{aligned}\hat{H} &= \sum_{\alpha\beta} t_{\alpha\beta} \hat{c}_\alpha^+ \hat{c}_\beta + \frac{1}{2} \sum_{\alpha\beta} \sum_{\gamma\delta} \langle \alpha\beta | \hat{V} | \gamma\delta \rangle \hat{c}_\alpha^+ \hat{c}_\beta^+ \hat{c}_\delta \hat{c}_\gamma \\ &= \sum_{\alpha\beta} [t_{\alpha\beta} + \sum_{\gamma} \langle \alpha\gamma | \hat{V} | \beta\gamma \rangle] \hat{N}_{\alpha\beta} - \sum_{\alpha\beta} \sum_{\gamma\delta} \langle \alpha\beta | \hat{V} | \gamma\delta \rangle \hat{N}_{\alpha\delta} \hat{N}_{\beta\gamma}, \quad (1)\end{aligned}$$

where

$$\hat{N}_{\alpha\beta} \equiv \hat{c}_\alpha^+ \hat{c}_\beta \quad \text{and} \quad [\hat{N}_{\alpha\beta}, \hat{N}_{\gamma\delta}] = \delta_{\beta\gamma} \hat{N}_{\alpha\delta} - \delta_{\alpha\delta} \hat{N}_{\gamma\beta}, \quad (2)$$

for $\alpha, \beta, \gamma, \delta = 1, 2, \dots, n$. The commutation relations in Eq. (2) coincide with those of the generators of the special unitary group in n dimensions thus implying that the nuclear physics Hamiltonians are bi-linear forms of

³ In fact, the levels in question split into doublets and singlets: if the total number of levels in the multiplet is even, the corresponding multiplet is composed of doublets (with no singlet), if the corresponding number is odd, then we have a number of doublets and one singlet. This structure is reminiscent of the so-called pseudo SU_2 symmetry, associated with the SU_2 group, a subgroup of SU_3 . Here we are not going to discuss this particular aspect of the unitary symmetry.

the SU_n generators. This allows to express the solutions to the Schrödinger equation with the Hamiltonian in Eq. (1) in terms of the irreducible representations of the unitary groups, construct observables other than the energy in terms of the generators and, more generally, employ the group representation theory to the description of the many-body systems.

2.3. Special role of the SU_3 -based symmetries

The nuclear *mean-field* Hamiltonians are usually some particular cases of the expression in (1), while the SU_3 group is one of numerous subgroups of the SU_n group. This observation provides the symmetry-oriented mathematical background and encourages the use of the group-theoretical concepts within the nuclear mean-field theory. The particular SU_3 and pseudo SU_3 subgroups and related symmetries imply the existence of the characteristic multiplet structures as discussed in Sec. 2.1. From the physics point of view, however, the very fact that the nuclear single-particle multiplets exist in the realistic spectra and that they take the form represented in Fig. 1, is an intriguing result next to a miracle.

Indeed, let us recall that the nuclear mean field is an auxiliary potential (operator) corresponding to the averaging of the nucleon–nucleon interactions over many occupied single-nucleonic configurations. At small elongations, the geometrical characteristics such as the nucleonic probability distribution in space and the overlaps with the other distributions over which the averaging is performed are very different from those at moderate deformation and still very different from those corresponding to the super-deformed shapes. Yet, the small energy-spread of the multiplets (doublets, triplets, quadruplets *etc.* as presented in Fig. 1, left) is nearly independent of the elongation! This result in itself is remarkable⁴. It signifies the existence of the symmetry of the *two-body interactions* that “forces” an approximate deformation independence of the energy-spread of each of the multiplets.

At first, one could think that the pseudo-oscillator Hamiltonian is merely a small modification of the underlying oscillator Hamiltonian. In reality, an “innocent” shift of the intruder orbitals has very important consequences.

⁴ It is sometimes argued (incorrectly) that the multiplets are there “because the nuclear potential is almost that of the harmonic oscillator so that there is no other possibility anyway”. Such a statement is wrong for at least two important reasons. Firstly, the conceptual one: the mean field potential is a complicated functional resulting from the averaging over a big number of quantum states and the related configurations and if at the end it resembles any simple-looking function it is either incidental or a result of a symmetry. Secondly, the numerical one: it is well established but seldomly spoken that in contrast to some beliefs, the realistic single-particle wave functions are very different from the harmonic-oscillator ones, *cf. e.g.* Fig. 2 in Ref. [8], despite the fact that the energies group themselves in resemblance to the pseudo-oscillator multiplets.

Firstly, the deformed single-particle wave-functions, solutions to the problem with the spin-orbit shifts, are *complicated mixtures* of the basis wave functions generated by the *appropriately deformed* harmonic oscillator potential, *cf.* reference contained in the footnote 4. Secondly, the whole pattern of the pseudo-oscillator single-particle shell-gaps is totally different as compared to the pure oscillator case. Indeed, Fig. 2 illustrates the dramatic differences between the SD shell-gap positions within the approximate pseudo SU_3 symmetry (right) and the full SU_3 symmetry (left). On the right, the intruder orbitals have been placed at their empirical positions (“shifted intruder orbitals”, pseudo SU_3 symmetry) rather than at the positions required by the pure harmonic oscillator (pure SU_3 symmetry). Referring, as an example, to the neutron SD shell closures that correspond to the Rare Earth nuclei of $A \sim 150$ mass range, the existence of the whole series of the SD shell-gaps *i.e.* 80, 82, 84, 86 and 88 of comparable sizes in the pseudo-symmetry case deserves noticing. The latter is opposed to the shell closures with the SD magic numbers 80 and 110. Similarly, in the case of protons, the two big gaps at $Z = 64, 66$ instead of the gaps at either 60 or 80 are present.

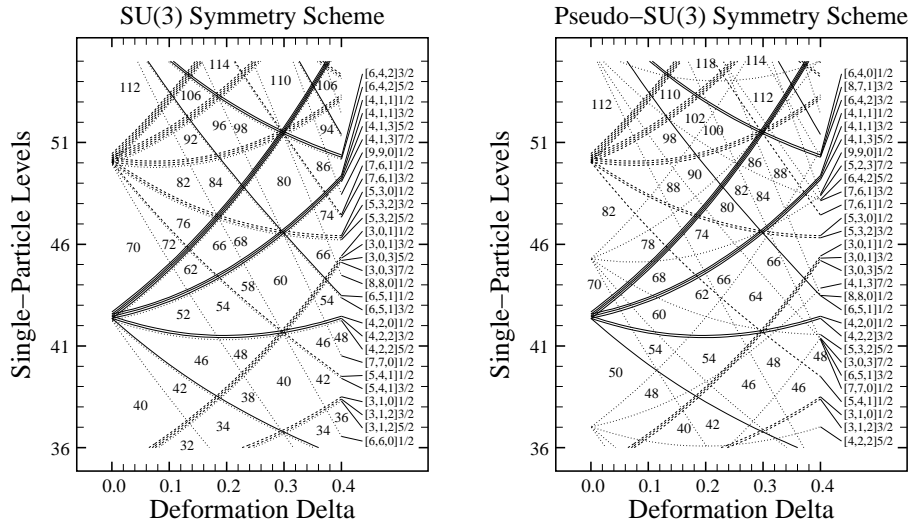


Fig. 2. The harmonic-oscillator single-particle spectrum, left, with the explicitly marked degeneracies of the multiplets (*cf.* Fig. 1, right), compared with the pseudo-oscillator spectrum, right. We say that the multiplet scheme represented on the left results from the SU_3 symmetry of the underlying mean-field Hamiltonian (harmonic-oscillator Hamiltonian); in analogy, the one represented on the right is the result of the pseudo SU_3 symmetry. In the latter case the intruder orbitals are shifted downwards by the amounts $\sim \langle \hat{\ell} \cdot \hat{s} \rangle$, the empirical spin-orbit splittings.

Summarising: The two most important aspects deserve noticing. Firstly, the degeneracies of the respective multiplets in the two schemes differ by one unit, and, secondly, the positions of the strongest gaps are totally different. Indeed, in the harmonic oscillator case these positions are fixed at $\delta = 0.3$, the latter corresponding to the axis ratio $a : b = 2 : 1$. In the pseudo-oscillator (pseudo SU_3 symmetry) case, there are chains of large gaps, e.g. at 80, 82, 84, 86 and 88, corresponding to increasing deformations. The experiment fully confirms the pseudo SU_3 scheme in contrast to the harmonic oscillator (SU_3 scheme) as discussed in the following section.

2.4. The SU_3 -based symmetries and relation to experiment

Not only are the shell-gap sizes totally different according to the two schemes but also their positions along the horizontal axis represent a totally different pattern, with increasing gap-associated axis-ratios in the case of the pseudo SU_3 case and fixed $3 : 2$ and $2 : 1$ axis ratios in the case of the SU_3 oscillator. The realistic Strutinsky type calculations [13] predicted the total energy equilibrium deformations in full agreement with the smoothly rising trend of the pseudo SU_3 symmetry case.

The experiment confirms fully the pseudo SU_3 scheme and equally radically contradicts the SU_3 oscillator scheme with its $2 : 1$ axis-ratio arguments. The life-time measurements, *cf.* Ref. [14], give, for the “central” super-deformed $^{152}_{66}\text{Dy}_{86}$ nucleus the quadrupole moment

$$Q_0^{\text{exp}} = (17.5 \pm 0.2)eb \quad \leftrightarrow \quad a : b \approx 1.75 \neq 2 : 1. \quad (3)$$

The axis ratio $a : b$, very different from $2 : 1$, comes from the realistic total energy calculations predicting the equilibrium deformation at $\alpha_{20} = 0.61$ and $\alpha_{40} = 0.11$ and at the same time the correct, within, a few percent, Q_0^{calc} moment. It is perhaps worth mentioning here that the relation between the quadrupole moment and the quadrupole deformation is not a direct one, moreover, not even a unique one. Indeed, various combinations of the deformation parameters entering the definition of the deformed nuclear potential may result in the same Q_0^{calc} . In particular, in the microscopic calculations, the nuclear quadrupole moment is a sum of contributions coming from all the occupied orbitals: the orbitals down-sloping in function of the quadrupole deformation contribute typically positive- and up-sloping, negative contributions to the total quadrupole moment. In this sense one may speak about the additive contributions, the term that certainly does not apply to the deformation, yet a certain correlation does persist. This can be seen in another interesting experimental result for the so-called first and fourth SD bands in the $^{149}_{64}\text{Gd}_{85}$ nucleus for which

$$^{149}\text{Gd} [1] : Q_0^{\text{exp}} = (15.0 \pm 0.2) \text{ eb}, \quad (4)$$

$$^{149}\text{Gd} [4] : Q_0^{\text{exp}} = (17.5 \pm 0.5) \text{ eb}, \quad (5)$$

showing that a particle-hole configuration may contribute as much as $\sim 15\%$ of the total nuclear quadrupole polarisation. The Strutinsky calculations give indeed the smaller quadrupole deformation for the first and the larger for the fourth band (obviously without preserving the 15% ratio). The above results are compatible with the $a : b$ axis ratios even smaller than those in the case of the ^{152}Dy case showing again the non-applicability of the “2 : 1 axis ratio argumentation” together with the underlying SU_3 (oscillator) scheme.

We believe that the above symmetry aspect, pertinent for the whole nuclear mean-field theory and not thus for the super-deformation sub-field is indeed the most important global achievement of the super-deformation studies so far. The role of the *hyper-deformation* studies in this context can be seen as evident: *How far do the multiplet-structures extend in terms of the nuclear elongation, and more generally, on the way to fission?*

The presence (or not) of the systematic multiplet structures is a pre-determining factor for the nuclear stability, due to the relatively big shell structures accompanying the presence of the multiplets. This and related questions will be addressed in Sec. 3.

Summarising: The above results related to the measured quadrupole moments and first of all the results on numerous SD bands in practically all mass ranges, as predicted by the pseudo SU_3 scheme of Ref. [4], indicate that in nature the pseudo SU_3 scheme is indeed realised. The abundance of the SD nuclear bands and the measurements of the quadrupole moments that are in full quantitative agreement with the Strutinsky type realistic calculations confirm the realistic character of the multiplet structure illustrated in Figs. 1 and 2. Without such a multiplet grouping a systematic agreement with the observed quadrupole moments would not have been possible.

2.5. Additivity of the quadrupole moments and the vanishing pairing

The super-deformed nuclei in the Rare Earth region happen to provide a very special test ground for the nuclear structure studies, not only because one can examine the prominent multiplet structures and the unitary symmetry behind it as discussed above, but also because of the negligible pairing correlations. The way it appears, this aspect seems to be so far a unique feature of this particular nuclear mass range. Let us be more specific here since this important point does not seem to be appreciated in the literature to the extent it deserves.

The rotational properties of a great majority of deformed nuclei are known at-, and close to the ground-state configurations, where the pairing correlations are either dominating or still relatively strong. In contrast, in the Rare Earth SD nuclei the microscopic calculations suggest that the pairing correlations there are very weak. This is a result of the relatively large deformed shell-gaps of about 2 MeV on the average, in both proton and neutron single-particle spectra in the vicinity of $Z = 66$ and $N = 86$, and of the relatively strong Coriolis effects, the latter proportional to the rotational frequency. In the SD nuclei of the Rare Earth region one finds well separated (*i.e.* not perturbed by interactions with the neighbours) rotational bands extending over 15 transitions or even more. Because of the “purity” also other observables such as *e.g.* the expectation values of the multipole moments behave very smoothly in function of the rotational frequency what allows for a relatively certain theoretical identification of the experimental results that are averages over long frequency ranges there.

To formulate an argument let us use the ground-state expression for the expectation values of the multipole moments

$$Q_{\lambda\mu}(\text{conf.}) = \sum_{i \in \{\text{conf.}\}} 2v_i^2 \langle i | \hat{q}_{\lambda\mu} | i \rangle, \quad (6)$$

where the v^2 represents the pairing occupation probabilities (for the rotating nuclei the mathematical expression is slightly more complicated but the validity of the argument remains). At the vanishing pairing the non-trivial (non-zero) probabilities are equal to 1 what implies that the multipole moment of *e.g.* 1-particle 1-hole excitation of the ground-state configuration reads

$$Q_{\lambda\mu}(1p - 1h; \text{Nucl. } A) - Q_{\lambda\mu}(\text{g.s.}; \text{Nucl. } A) = [\langle p | \hat{q}_{\lambda\mu} | p \rangle - \langle h | \hat{q}_{\lambda\mu} | h \rangle], \quad (7)$$

since the bulk in the summations in Eq. (6) cancels out giving rise to an additivity rule⁵ valid in the vanishing pairing limit. We refer to it as an additivity of the first kind, valid in a given nucleus and under the assumption that the equilibrium deformations of the compared configurations are (at least nearly) the same.

In the case of the *quadrupole deformation* that is of principal interest in this article, there emerges another type of additivity, referred to as of the second kind; it is defined below. Its occurrence is possible because in many situations and in particular here the single-particle expectation values of the quadrupole moments can be considered nearly constant in terms of the

⁵ Should the pairing correlations be significantly present, the pairing occupation coefficients will in general be configuration-dependent, cancellation of all but 1-particle and 1-hole contributions will not hold and the additivity in (7) will not hold anymore.

quadrupole deformation. Indeed, in some realisations of the nuclear mean-field theory one may demonstrate that on the average the *slopes* of the single-particle levels in function of the quadrupole deformation are proportional to the expectation values of the \hat{q}_{20} . Consequently, for the single particle levels that are nearly linear in function of the quadrupole deformation the single-particle contributions to the quadrupole moments are constant and to a good approximation independent of the actual deformation of the nucleus. Under these conditions we may formulate the additivity of the second kind:

$$Q_2(\text{Nucl. } A) - Q_2(\text{Nucl. } B) = \delta Q_2(p - h), \quad (8)$$

where the term on the right is composed of a few (additive) contributions associated directly to the differences in the single-particle configurations in the *two nuclei*, A and B . Under the conditions discussed here (no pairing, approximately constant slopes of the single particle levels in function of the quadrupole deformation) the contributions on the right-hand side may be considered (to a good approximation) independent of nuclei A and B and thus usable as a kind of universal building blocks for several SD Rare Earth nuclei.

Let us stress the very special conditions under which this additivity occurs:

1. The pairing correlations must be weak or negligible;
2. The nuclear configurations must be unperturbed by the interactions such as the level repulsion (neither in function of rotational frequency, nor in function of the quadrupole deformation);
3. The global dependence of the single-particle levels on the quadrupole deformations must be to a good approximation linear. All these conditions are met in the discussed context *i.e.* in many of the SD bands in the Rare Earth region, but not in other areas of the Periodic Table such as *e.g.* the SD Mercury and/or the SD Cerium nuclear regions; neither are these conditions present to such a broad extent anywhere else in the normally deformed nuclei.

Summarising: The mechanism of additivity discussed here has been examined in the theoretical analysis based on the self-consistent Hartree–Fock method and verified through comparison with experiment in Ref. [7]. It is, therefore, justified to claim that the super-deformation studies have revealed an existence of a unique, pair-less, broad nuclear range in the Rare Earth region where the rotational properties are particularly simple and where various nuclear structure models can be tested under unique conditions.

2.6. The identical band phenomenon

The so-called “identical-band mechanism” has been one of the biggest surprises in the short history of nuclear super-deformation. There exist several and not exactly equivalent ways of formulating the problem and in the following we select the one which is probably most directly related to the effective inertia properties.

As it is well known, an atomic nucleus as a quantum system *cannot be attributed any moment of inertia*. However, when the nuclear energy versus spin dependence is sufficiently smooth, a number of mathematical relations originating from the classical physics and using the notion of the moment of inertia can be introduced. In particular, taking as a guide-line the classical rigid-rotor relations and denoting the angular momentum by I we have

$$E_{\text{rot.}} = \frac{I^2}{2\mathcal{J}_{\text{rot.}}} \rightarrow \omega_{\text{rot.}} \equiv \frac{I}{\mathcal{J}_{\text{rot.}}} \leftrightarrow \omega_{\text{rot.}} = \frac{dE_{\text{rot.}}}{dI} \leftrightarrow \mathcal{J}_{\text{rot.}} = \frac{I}{\omega_{\text{rot.}}}. \quad (9)$$

We may introduce auxiliary quantities

$$\omega_{\text{nucl.}} \equiv \frac{E(I+2) - E(I)}{\Delta I} \approx \frac{dE(I)}{dI}, \quad (10)$$

where $\Delta I = 2$ in this case, and

$$\mathcal{J}^{(1)} \equiv \frac{I}{\omega_{\text{nucl.}}}, \quad (11)$$

that are strictly speaking not quantum mechanical observables but can conveniently be defined with the help of the quantum mechanical observables, the energy and the angular momentum of the system. Similarly, we may define

$$\mathcal{J}^{(2)} \equiv \frac{dI}{d\omega_{\text{nucl.}}}. \quad (12)$$

The two analogs of the classical moments of inertia, $\mathcal{J}^{(1)}$ and $\mathcal{J}^{(2)}$, called kinematical and dynamical moments, respectively. They are equal to each other only in the case of the rigid rotation. In general, due to the intrinsic structure of the system, the energy vs. spin relations in nuclei are not parabolic and the two moments are different.

The identical band mechanism is a mechanism associated with *pairs of neighbouring nuclei*, say B and C . Thus, simplifying the notation $\omega_{\text{nucl.}} \rightarrow \omega$, we will introduce

$$\mathcal{J}_B^{(2)} \equiv \left[\frac{dI}{d\omega} \right]_A \quad \text{and} \quad \mathcal{J}_B^{(2)} \equiv \left[\frac{dI}{d\omega} \right]_B. \quad (13)$$

Similarly we introduce the polarisations in terms of the $\mathcal{J}^{(2)}$ moments

$$\frac{\delta\mathcal{J}}{\mathcal{J}} \equiv \frac{\mathcal{J}_B^{(2)} - \mathcal{J}_A^{(2)}}{\mathcal{J}^{(2)}} \quad \text{with} \quad \mathcal{J}^{(2)} \equiv \frac{1}{2} \left(\mathcal{J}_A^{(2)} + \mathcal{J}_B^{(2)} \right). \quad (14)$$

To learn something about the above useful quantity in the nuclear context we need to model two aspects: the nuclear mean field and the self-consistency condition. Both requirements can be introduced with the help of the harmonic oscillator. Let us introduce the harmonic oscillator orbitals $|i\rangle \equiv |n_x(i), n_y(i), n_z(i)\rangle$ and the symbols

$$\Sigma_\kappa \equiv \sum_{i \in \{\text{conf.}\}} [n_\kappa(i) + \tfrac{1}{2}] \quad \text{for} \quad \kappa = x, y, z, \quad (15)$$

where $n_\kappa(i)$ are the numbers of occupied oscillator orbitals, thus specifying a single-particle many body configuration. One can demonstrate, Ref. [9], that the self-consistency condition associated with the configuration $\{\text{conf.}\}$ above takes the form

$$\Sigma_x \omega_x = \Sigma_y \omega_y = \Sigma_z \omega_z, \quad (16)$$

expressing the requirement that the shape of the modelled nucleonic density follows that of the harmonic oscillator potential. According to the same reference, the moment of inertia corresponding to the rotation about the \mathcal{O}_x -axis reads

$$\mathcal{J}^{(1)} = \frac{1}{2\omega_y\omega_z} \left[\frac{(\omega_y + \omega_z)^2}{\omega_y - \omega_z} [\Sigma_z - \Sigma_y] + \frac{(\omega_y - \omega_z)^2}{\omega_y + \omega_z} [\Sigma_z + \Sigma_y] \right]. \quad (17)$$

Adding a particle to the system, say associated with the state $|j\rangle$, will introduce obvious modifications

$$\begin{aligned} \Sigma_\kappa &\rightarrow \Sigma'_\kappa(j) \equiv \Sigma_\kappa + \delta\Sigma_\kappa(j), \\ \omega_\kappa &\rightarrow \omega'_\kappa(j) \equiv \omega_\kappa + \delta\omega_\kappa(j), \end{aligned} \quad (18)$$

where

$$\delta\Sigma_\kappa(j) \equiv n_\kappa(j) + \tfrac{1}{2}. \quad (19)$$

In the following we will limit our considerations to the axially-symmetric nuclei since both the super- and the hyper-deformed nuclei are expected to be axial. Taking into account the constant volume condition $\omega_x\omega_y\omega_z = \Omega = \text{const.}$, and introducing the elongation parameter defined as

$$\alpha \equiv \omega_\perp / \omega_z \quad \text{where} \quad \omega_x = \omega_y = \omega_\perp, \quad (20)$$

and furthermore, using the first order expansion in terms of small quantities $\delta\omega_\kappa$ and $\delta\Sigma_\kappa$, we obtain for a nucleus turning about the \mathcal{O}_x -axis

$$\frac{\delta\mathcal{J}}{\mathcal{J}} = \frac{1}{3\Sigma_\perp} \left[-\Sigma_\perp \frac{\delta\Omega}{\Omega} - \delta\Sigma_x + \frac{5-\alpha^2}{1+\alpha^2} \delta\Sigma_y + \frac{5\alpha^2-1}{\alpha(\alpha^2+1)} \delta\Sigma_z \right]. \quad (21)$$

One may show that $\delta\Omega/\Omega = -1/A$ and thus the volume-conservation contribution is always positive but decreasing with the nuclear mass.

The above relation deserves noticing for its clear and beautiful physical significance. First of all we may observe that adding the oscillator quanta in the x - and in the z -directions brings the inertia-polarisation contributions of always opposite signs. The appearance of the $\delta\Sigma_x$ term corresponds to an increase of the oscillations in the direction of the rotation axis and thus decreases the possibility for a corresponding particle to be present further away from that axis (*e.g.* moving towards the z -axis). Since the moment of inertia in question corresponds to the rotation *perpendicular to the elongation axis*, the less frequently the particle is to be found far from the x -axis, the smaller the corresponding moment of inertia. By the same token, increasing the number of quanta along the z -axis increases the probability of finding the particle further and further from the rotation axis and thus increases the inertia always. The most interesting situation arises in the case of the oscillations along the y -axis that is perpendicular to both the elongation and the rotation axes. Indeed, for small prolate deformations ($\alpha > 1$) the corresponding moment of inertia contribution is always positive, however, *exactly at the harmonic oscillator axis ratio 2 : 1* this contribution vanishes independently of the importance of the y -axis oscillation *i.e.* independently of $n_y(j)$. Increasing the elongation further *i.e.* for the axis ratios $\alpha > 2$ may only decrease the inertia polarisation $\delta\mathcal{J}/\mathcal{J}$ that may become negative.

Now we can define the “identical band mechanism” and at the same time explain its origin. Two rotational bands corresponding to two different nuclei (*e.g.* differing by one single nucleon) are called identical if the corresponding inertia polarisation vanishes *i.e.* the bands have the identical energy transitions. Indeed, this happens in numerous comparisons of the SD bands known experimentally: the transition energies differ on the level of a per-mille, and this over many, sometimes of the order of up to 20 transitions. Expression (21) gives the clue as far the mechanism is concerned⁶. For not very large deformations, say, $\alpha > 1$ but “reasonably” far from 2, to have the inertia polarisation as small as possible, the added j^{th} particle has to have no oscillations in the z -direction ($n_z(j) = 0$) and possibly strong oscillations

⁶ Some of results presented here have been discussed earlier by J. Dudek, Z. Szymański and T. Werner, preprint of the *Centre de Recherches Nucléaires*, Strasbourg, (CRN-PHTH/91-14).

in the x -direction ($n_x(j)$ possibly large). If the low-lying bands can be built on such configurations, *i.e.* if such orbitals are close to the Fermi level of the nucleus considered, the related inertia polarisations will be lower as compared to the inertia associated to other orbitals. However, if such orbitals are not available close to the Fermi level, the next good candidates are the ones with the oscillations along the y -axis since at large deformations, the coefficient in front of $\delta\Sigma_y$ will be small.

In order that the above mechanism can manifest itself clearly it is important that the pairing correlations are small or vanishing. Indeed, only then by occupying the right orbital *i.e.* satisfying the discussed criteria can the corresponding contributions reach their maximum without being contaminated by numerous contributions from other orbitals.

*Summarising: The discovery of the identical bands in the SD nuclei points out indirectly to the lowering of the pairing correlations that does take place in many fast SD rotating nuclei. At the same time it offers a possibility of identifying the orbitals involved in the inertia polarisation. Even if the distinction between the orbitals differing by $\delta n_\kappa \sim 1$ or 2 could be difficult, the distinction between *e.g.* the orbitals with $n_z \sim N_{\max}$ and $n_z \sim 0$ offers a precious help in identifying the single-nucleonic structure underlying the SD phenomena.*

3. Before the hyper-deformation era: What should we expect?

In the preceding Section we have summarised what seemed to us being the most important scientific impact of the long term super-deformation studies in terms of its significance for the nuclear structure field. We have concentrated on the mechanisms or phenomena for whose discovery the superdeformation was essential (such as an existence of a pair-less island of fast rotating nuclei that may now serve as a unique test ground, Secs. 2.5–2.6) or the mechanism that is essential for all the deformed nuclei in the universe, *viz.* the presence of the pseudo SU_3 multiplets as a dominating feature of the nuclear deformed-shell model, Secs. 2.1–2.4. During the last 20 years there have been of course also other interesting studies related to the super-deformed nuclei, but slightly less important for the main lines of this article; they will not be discussed here due to the space limitations.

In the following we would like to review the expected similarities and differences in terms of the nuclear mechanisms when extrapolating the forms of behaviour from the super-deformed to the hyper-deformed nuclei.

3.1. Hyper-deformed nuclei: What are they like?

Let us begin by emphasising that we are confronted here with a long-standing conflict between the physical intuition and semantics. Intuitively, the hyper-deformed nuclei are expected to be those that are distinctly more

elongated as compared to the super-deformed ones. However, *there is no clear-cut criterion allowing to define a distinction between the super- and hyper-deformed nuclei in terms of shapes*. Today it is well known (*cf. e.g.* Fig. 3 in Ref. [8]) that the theoretical predictions qualify as “hyper-deformed” the nuclei with the quadrupole deformation $\alpha_{20} \sim 0.9$, *i.e.* the nuclei whose axis ratios are typically 2 : 1 (and not 3 : 1 as repeatedly stated in some older publications). Moreover, the axis-ratio parameter⁷ associated with almost all known super-deformed nuclei seldomly exceeds $1.7 \approx \sqrt{\pi}$ disqualifying the older, harmonic oscillator based axis-ratio criterion of being super-deformed ($a : b = 2 : 1$) or hyper-deformed ($a : b = 3 : 1$).

The axis-ratio not being any valid argument, a *half-way out of the trouble criterion* has been proposed in Ref. [11]: according to this criterion a super-deformed nucleus must have at least one intruder level occupied that comes from the $(N+2)^{\text{nd}}$ shell, N being the principal shell quantum number for the valence orbitals of the nucleus in question. (The arguments of this type apply for either neutrons, or protons, or both neutron and proton shells.) Similarly, the hyper-deformed nuclei should have occupied at least one intruder orbital from the $(N+3)^{\text{rd}}$ principal shell, the mega-deformed nuclei at least one intruder from the $(N+4)^{\text{th}}$ principal shell, *etc.* This is probably the “best criterion on the market” since it combines the occupation of the higher and higher intruder levels, a clear-cut mechanism with an important physical significance, with the fact that, on the average, those intruders come down in energy at higher and higher deformations. A weak point of this type of a criterion is that in principle it cannot be excluded that a strongly elongated nucleus exists without satisfying the above criterion (thus not being called super-deformed) nevertheless with the deformation possibly larger than the one of a “real” super-deformed nucleus in the neighbourhood.

After this short discussion combining the aspects of semantics and of physics, let us emphasize that the numerical values associated with the deformation parameters of the hyper-deformed nuclei play probably the least important role in the whole discussion. The increasing elongations in the studies of the nuclear configurations on the way to fission will most likely “automatically” reveal new surprises. To illustrate the richness of possibilities that may arise let us present a few theoretical predictions related to

⁷ A turmoil around the axis ratio arguments in nuclear structure physics increased even further when some publications started expressing the transformation from the measured quadrupole moment to a “deduced” deformation by dividing the former by a classical factor ZR^2 ; this procedure is unjustified according to the warnings of Ref. [7], yet used occasionally up to date. A turmoil grows to a little chaos when on top of that some low-order approximate expansion expressions are used to transform the so-obtained information into the “effective” axis-ratio of an “associated ellipsoid”.

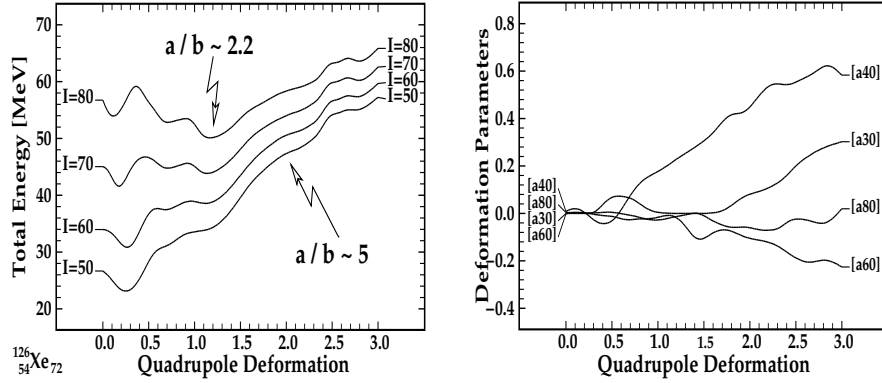


Fig. 3. Microscopic calculation results obtained using the cranking Strutinsky method and deformed Woods–Saxon Hamiltonian with the universal parameters. Left: Total energies have been minimised over α_{30} , α_{40} and α_{60} at each quadrupole deformation. Four curves correspond to four spins, *viz.* $I = 50, 60, 70$ and 80 . Right: Typical evolution of the high-multipole deformations in function of the quadrupole deformation for spin $I = 70$. Similar relations hold for other spin values.

^{126}Xe nucleus for which new experimental results have been recently announced [12]. Fig. 3 presents the results of the multidimensional Strutinsky type calculations obtained with the “universal” parameters and the automatic minimisation routine.

The results presented in Fig. 3 can be considered typical for a number of nuclei in the mass range under discussion; they indicate first of all the presence of two large-deformation competing-minima, both at the very high spins only. We will focus on the more pronounced minimum at $\alpha_{20} \sim 1.1$ with the calculated axis ratio $a : b \approx 2.2$. From the behaviour of the energy curves it follows that the hyperdeformed bands associated with such minima will contain ~ 10 transitions at the most.

The shapes corresponding to the oscillations around the predicted hyperdeformed minima are illustrated in Fig. 4. The presence of a pronounced necking deserves noticing. Such a necking has an important impact on the spatial behavior of the nuclear mean-field potential implying that many orbitals will have a dominating presence in either the left- or in the right-fragment, a mechanism that is not present according to theory in the superdeformed minima. One of the consequences of this situation will be a necessity of applying a more “sophisticated” pairing formalism that takes into account a well-defined two-center character of the mean-field Hamiltonian in such a case.

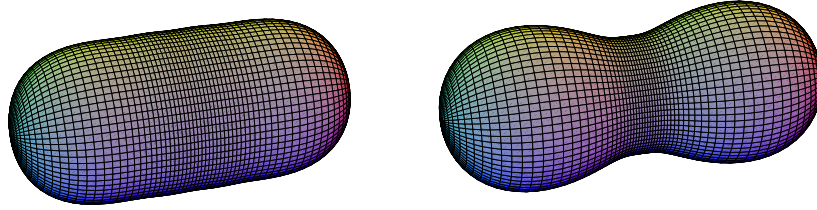


Fig. 4. Nuclear shapes corresponding to $\alpha_{20} = 0.9$ (left) and $\alpha_{20} = 1.2$ (right). These two numbers have been chosen because the hyper-deformed energy minimum of Fig. 4 lies in a plateau extending approximately between those values.

3.2. Hyper-deformed nuclei: How to get them?

It has been the discovery of only some recent years that the population (or not) of the highly-deformed nuclei at high spins is a matter of the existence (or not) of the Jacobi transition. In order to introduce the discussion of this particular aspect for the hyper-deformed nuclei let us have a close look at this problem in the case of the ^{152}Dy nucleus where, we believe, the situation is relatively well known. The calculated results in the form of the total energy cross-sections in this nucleus obtained by using a five-dimensional deformation space are given in Fig. 5. There are clearly three types of minima seen in the figure: the SD bands associated with

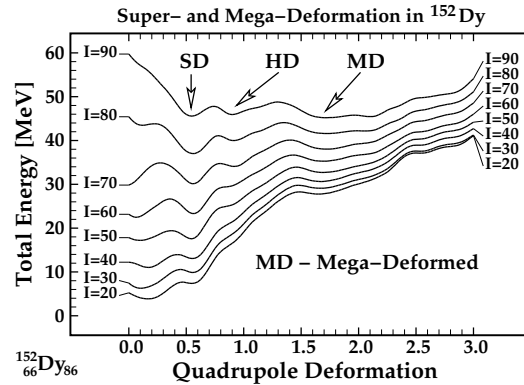


Fig. 5. Total energy calculation results of the ^{152}Dy nucleus using minimisation over γ (triaxial quadrupole deformation) as well as α_{40} , α_{60} and α_{80} axial deformations, for spins between 20 and 90. In addition to the super-deformed minimum, the often discussed hyper-deformed one (but only in a very narrow spin window) and a mega-deformed one at a much larger spin window deserve noticing.

the first one (marked SD) have been seen in numerous experiments. The second minimum marked HD, was the one predicted long ago⁸ but according to the calculations it should “survive” a few E2 transitions only what makes the corresponding experiments very difficult at the present time. The third one, called mega-deformed (MD) represents probably the most interesting hypothesis at this time: its “left” barriers are significantly broader than the ones in the case of HD, and the barrier-heights remain relatively high (≥ 1.5 MeV) down to the spins of $I \sim 50$ or so. The above conditions are probably not much more different as compared to the ^{124}Xe case discussed below, but the spin-stretch seems much larger in the Dysprosium case. The experimental results for the SD yrast band in ^{152}Dy correspond to spins $I = 26 \rightarrow 66$ from where we must conclude that the expected transitions from the SD to the normal deformed configurations are too strong (the super-deformed nuclei stop existing) when the barrier heights between the SD and the ND minimum decrease down to about 1.5 MeV. At the top of the band, we believe, there are about 4 E2-transitions feeding the last, $I = 66$ state, what brings us to the conclusion that the excited compound nuclei from which the γ -cascades take their origin correspond to the spins about $I = 74$. What are the potential energy surfaces describing ^{152}Dy nucleus? Assuming that the final nuclei after the neutron evaporation are populated at temperatures of the order of $T \sim 1$ MeV or more, we may expect that the shell effects are effectively washed out and approximate the total nuclear energy by the macroscopic energy only, here taken in the form of the LSD approximation of Ref. [16]. The results are presented in Fig. 6; they show that at spins $I \sim 74$ the equilibrium deformations correspond exactly to the superdeformation of $\alpha_{20} \sim 0.6$ (!) In other words, at spins very close to the central value of $I \approx 74$ and independently of the excitation (*i.e.* at possibly higher and higher temperatures) the whole population flux ends up in the super-deformed ^{152}Dy nuclei!!!

Another important aspect deserves noticing: The height of the fission barrier at $I \sim 74$ according to our LSD calculations amounts to about 5 MeV; this has been obtained in the calculations that involve the minimisation at each (β, γ) -point over the multipole deformations with $\lambda \leq 8$. By changing the calculation technique *i.e.* not producing the energy *maps* but rather calculating directly the LSD fission barriers we have convinced ourselves that the inclusion of the deformation multipolarities with $\lambda \leq 16$ lowers the

⁸ The first mention of this configuration goes back to the 1987 Gordon Conference in Nuclear Chemistry, where one of the authors (J.D.) presented it as the super-superdeformed configuration. The name “hyper-deformation” was proposed at that time by F. Stephens. In the mean-time several attempts to observe this configuration in experiments have been undertaken but no conclusive positive result has been obtained so far.

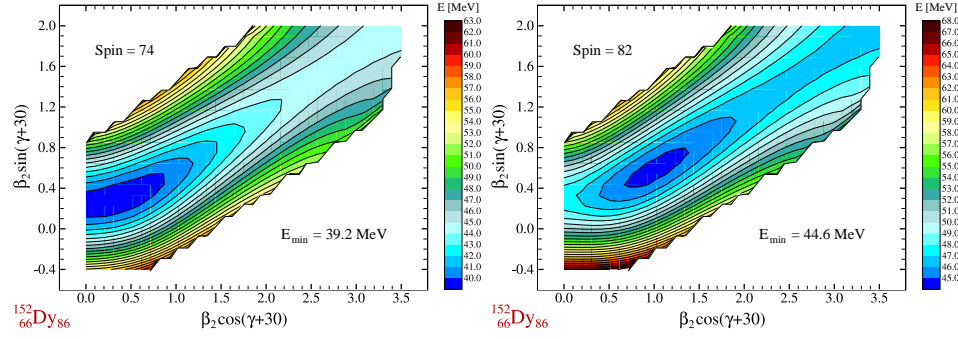


Fig. 6. Total energy surfaces at $I = 74$ and 82 illustrating Jacobi transition in ^{152}Dy . At $I \sim 74$, *i.e.* where the feeding of the super-deformed yrast band begins according to experiment, according to theory the population of the super-deformation is privileged (the minimum of the energy lies at $\alpha_{20} \sim 0.6$).

barriers typically by ~ 1.5 MeV, and consequently the estimates associated with the results in Fig. 7 are lower: ~ 3.5 MeV instead of ~ 5 MeV at $I \approx 74$ and nearly vanishing barrier at $I \approx 82$. This brings us to the conclusion that in no case the high spin states with $I \geq 76-78$ will be populated through particle evaporation reactions with the consequence (*cf.* Fig. 6) that there may only be 2–4 transitions associated with the hyper-deformation in this nucleus.

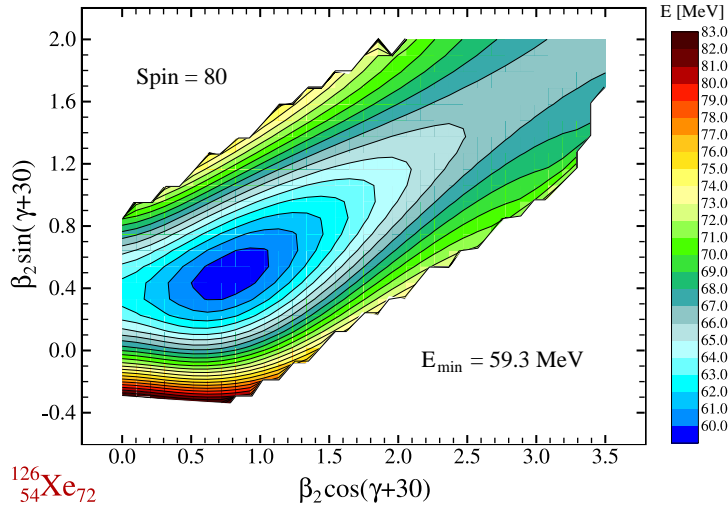


Fig. 7. Total energy calculation results of the ^{152}Dy nucleus using minimisation over γ (triaxial quadrupole deformation) as well as α_{40} , α_{60} and α_{80} axial deformations, for spins between 20 and 90. In addition to the super-deformed minimum, the often discussed hyper-deformed one (but only in a very narrow spin window) and a mega-deformed one at a much larger spin window deserve noticing.

Returning to the comparison between the *super-deformation* formation conditions in ^{152}Dy and those for the *hyper-deformation* in ^{126}Xe , it is worth noticing the similarities, *cf.* Fig. 6 and compare with Fig. 7. Clearly, at spins $I \sim 80\text{--}84$ the total energy minimum corresponds to $\alpha_{20} \sim 0.9\text{--}1.1$, *i.e.* the population of the hyper-deformed configurations of Fig. 3 (left) is privileged while at the same time the barrier heights amount to 7–5 MeV. We expect that the mechanisms of stability of the nuclei against fission do not depend dramatically on the mass range and thus conclude that *since the super-deformation in ^{152}Dy is populated at the barrier heights $\sim(3\text{--}5)$ MeV, we believe that the hyper-deformation in ^{126}Xe will be populated as well.*

At this stage the experimental analysis is still in progress as discussed in Ref. [12], but it may be interesting to compare the experimental results for the average dynamical moments, $\mathcal{J}^{(2)}$, coming from the $E_\gamma - E_\gamma$ correlation spectra even if they should be considered as preliminary. The hyper-

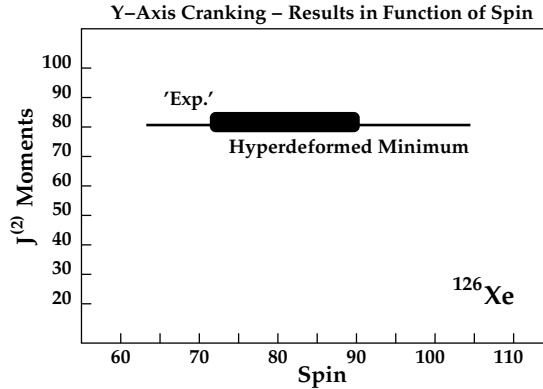


Fig. 8. Calculated dynamical moment, $\mathcal{J}^{(2)}$, for the hyper-deformed minimum of Fig. 4, left. The experiment is represented by a bar taking into account the experimental uncertainties. The length of the experimental bar is strictly speaking unknown; we assume here ten ($\Delta I = 2$) transitions. Also the lowest spin is experimentally unknown and the horizontal position of the bar corresponds to the spin range for which theoretical barriers of at least 2 MeV exist.

deformed minimum generates a constant $\mathcal{J}^{(2)}$ -moment as seen in Fig. 8, but there exist another high-elongation minimum as well corresponding to the super- rather than hyper-deformation. The related minimum is interesting as well because it corresponds to a rather significant octupole deformation, *cf.* Fig. 4 (right) and the dynamical moment that is not constant. Its presence remains so far a pure theoretical prediction.

4. Physics motivations behind the nuclear hyper-deformation

Despite numerous efforts the nuclear hyper-deformation has not been observed in discrete transitions while the continuum studies seem to provide the very first positive results (referring once again to Herbert Hübeler's presentation during this Conference). On the basis of theoretical results presented in the preceding Section we believe that there are valid analogies between the successful population of the super-deformation in ^{152}Dy and the hyper-deformation in ^{126}Xe and that, hopefully sooner than later, a valid final confirmation will be published. If true, this will open a new research field in the nuclear structure. What are the *real physics motivations behind*, other than merely another deformed nucleus?

In the following sections we would like to present our view of this problem.

4.1. Hyper-deformed nuclei, pseudo SU_3 symmetry and multiplets

According to a relatively extensive discussion in Sec. 2, the pseudo SU_3 multiplet structure seems to be by far the most profound aspect in this sub-field of nuclear physics because *its importance goes far beyond the pure structure of highly-deformed nuclei*. As it happens, by studying the super- and the hyper-deformed nuclei we will examine one of the most spectacular symmetry in nuclear structure physics, *cf.* Sec. 2.

Yet: is it sure that the multiplet structures will continue down to the larger and larger elongations on the way to fission?

The answer is: certainly not in general case. The most likely scenario will depend on whether the more and more elongated nuclei such as hyper-deformed or mega-deformed ones develop the necking relatively early on the way to fission, with the shapes somewhat similar to those in Fig. 4 (right), or whether their shapes will be rather like the one in Fig. 4 (left). In the case of the early neck building nuclei, the newly-forming fragments will have the structure that more and more resembles the object composed of two sub-systems, each of which having the mass of the order of $A/2$ rather than a single object with the original mass A . Consequently, there will be the low-degeneracy multiplets corresponding to the left and to the right fragment separating gradually rather than a single heavy nucleus with all its original N -shells occupied.

Concluding: Under the supposition that the hyper-deformation studies will enable us to observe the discrete states we will be able to identify and/or confirm the theoretical predictions of the multiplet structures like the ones in Fig. 2 in analogy with the present-day situation concerning the identification of the super-deformed gap structures through the information about numerous SD nuclei known today.

4.2. Hyper-deformed nuclei and the additivity of moments

In the preceding sections we have formulated the conditions for the rather unusual feature: the additivity of moments, both in terms of, first of all the quadrupole moments as well as in terms of the alignments and the implied kinematical and dynamical moments, $\mathcal{J}^{(1)}$ and $\mathcal{J}^{(2)}$.

One of the pre-conditions for this feature was the weak pairing condition allowing to approximate the pairing occupation factors by either 0's or 1's. This condition is clearly met so far in the Rare-Earth region where the relatively big SD shell-gaps and the presence of the strongly down-sloping intruder orbitals causes the near disappearance of pairing because of the blocking mechanism and because of the presence of large gaps.

In the case of the hyper-deformed nuclei the situation seems to be much less favorable in this respect. Firstly since the moments of inertia are much larger as compared to the SD configurations and at the same time all the orbitals much more “deformation aligned” the disappearance of the Kramers degeneracy in function of the rotational frequency is much slower and as the result the Coriolis (anti-pairing) mechanism slowed down. Secondly, in many nuclei with the necking being formed the clear-cut multiplet structures with the systematic sequences of large gaps as in Fig. 2 is not expected to take place. As a result the pairing correlations are likely to be much more pronounced, something like the pairing correlations in rotating large deformation nuclei in the Mercury and Cerium regions.

Summarising: One should not expect the spectacular additivity rules in the case of the hyper-deformed nuclei (unless an un-expected discovery of an unknown mechanism will change that expectation).

4.3. Hyper-deformed nuclei and the identical bands

It follows from the results discussed already in the case of the nuclear super-deformation that near-zero and or negative inertia polarisations may become possible when the nuclear elongation increases. Indeed, by writing

$$\delta\mathcal{J} = \frac{\mathcal{J}}{3\Sigma_{\perp}} \left\{ \frac{\Sigma_{\perp}}{\Omega} - \delta\Sigma_x + \frac{5 - \alpha^2}{1 + \alpha^2} \delta\Sigma_y + \frac{5\alpha^2 - 1}{\alpha(\alpha^2 + 1)} \delta\Sigma_z \right\}, \quad (22)$$

cf. Eq. (22), we can see that by combining appropriately n_x and n_y orbitals of the added “valence” particle at $n_z = 0$ we may arrive at the possibility that the inertia polarisation becomes negative, *i.e.* that *by increasing the nuclear mass we are going to effectively lower its moment of inertia.*

Summarising: It is not clear at present whether this anti-intuitive mechanism will be achieved since the nuclear elongations needed may be larger than the hyper-deformations of the realistic nuclei that are expected to reach the axis ratio $\alpha \sim 2$ rather than 3 as discussed earlier. However, it is interesting

to remark that the large- n_y orbitals are those that contribute to the nucleus becoming triaxial with the flattening towards the y -axis, the mechanism that is analogous to the one accompanying the Jacobi transition discussed earlier.

5. Summary and conclusions

In this article we have based the discussion of the physics of the hyper-deformed nuclei on the extrapolations from the solid experimental and theoretical information available today about the nuclear super-deformation. We believe that the mechanisms discovered on the occasion of the super-deformation studies were essential in bringing our understanding of the nuclear deformed mean-field theory to a much higher level as compared to the status about 20 years back, when the super-deformation studies have begun. More precisely:

Super-deformation: Achieved discoveries

- The superdeformation studies have confirmed indirectly but convincingly the presence of the pseudo SU_3 nuclear multiplet scheme rather than that of the SU_3 scheme of the harmonic oscillator. In particular we know today that the axis ratios of the super-deformed nuclei are not very close to the oscillator prediction $a : b = 1 : 2$, and that the deformations of the super-deformed states correspond very well with the predictions of the realistic models that are close to the pseudo SU_3 scheme rather than to the “fixed-deformation scheme” of the oscillator.
- The additivity of the quadrupole moments discovered in the SD nuclei of the Rare Earth region brought up the evidence of the weak pairing correlations, in addition to that of the indirect arguments based on the microscopic calculations of the moments of inertia with the models using the pairing interactions explicitly.
- The discovery of the identical band mechanism brought an evidence for the selfconsistency conditions between the nuclear mass density distributions and the forms of the nuclear potentials offering at the same time the possibilities of identifying (or helping to identify) the single-nucleonic orbitals.

The above physics mechanisms have been selected among several others since, in our opinion, they bring new light on the forms of nuclear behaviour that are more general and go much beyond the very properties of the super-deformed nuclei only. We believe, therefore, that it is important to discuss first of all the role that the hyper-deformation studies may have in possibly extending/modifying our knowledge coming from the super-deformation studies. By comparison, the following differences should be expected:

***Hyper-deformation: Expected discoveries
and physics motivations behind***

- The conclusions about the pseudo SU_3 symmetry multiplets and their characteristic regularity as obtained in the calculations for the super-deformed states and confirmed through large scale Strutinsky calculations and detailed comparison with experiment will most likely be different in the case of hyper-deformed nuclei. In particular, the nuclei developing the neck at relatively low quadrupole deformations will most likely produce a strongly perturbed multiplet scheme with possibly increased number of single-particle level crossings in function of the elongation. In contrast, the large elongation nuclei that fission from the initial states that have very poorly developed neck will most likely manifest the orbital regularities known from the pseudo SU_3 scheme.
- The additivity of the quadrupole moments, because of the expected presence of the relatively strong pairing correlations, is expected to be much less pronounced if at all.
- The identical band mechanism, expected to be strengthened by the absence of the pairing correlations, is therefore, less good a candidate. However, because of the large elongations it may happen that several orbitals close to the Fermi energies in those strongly elongated nuclei will simultaneously contribute to a very weak and/or possibly negative inertia polarisations. Consequently, the whole areas of paired but nonetheless nearly identical inertia nuclei may be expected.

In principle one can imagine that the hyper-deformed configurations can form chain structures as the super-deformed nuclei do: for instance, at the proton number fixed, *e.g.* $Z = 66$, there are several isotopes from $N = 87$ down to $N = 80$ that produce the SD effect with, on the average, decreasing quadrupole deformations. This is the result of the pseudo SU_3 structure visible from Fig. 2. The big gaps arise because of the presence of the high-degeneracy multiplets *that are necessarily up-sloping*. In reference to this figure: grouping of levels in a narrow energy stripe necessarily produces a “vacuum” in some other areas, the absence of levels creating a large spacing. These large spacing areas are cut by singlets or low degeneracy multiplets that are necessarily *down-sloping*. Out of these two mechanisms the pattern of the chains of large SD gaps of deformation increasing with the particle number are created. As a consequence, however, the hyper-deformed gaps are created among the same high-degeneracy pseudo SU_3 multiplets.

Finally, although both the super- and the hyper-deformed nuclear configurations belong to the high-elongation family, the pseudo SU_3 symmetry implies that the susceptibility to produce super- and hyperdeformed nucleus is in a way anti-correlated. This is because as discussed in the paper, the pseudo SU_3 symmetry implies that, except for an “accidentally big gap here and there” the systematic tendency is to produce the hyper-deformed minima in the nuclei in which the super-deformed minima exist as well. Consequently, the barriers separating the HD minima from the normal-deformed ones are either significantly lowered or totally cut at relatively high spins as presented in this paper. As a consequence the most often prevailing mechanism is that of a competition between the hyper- and the super-deformed minima. Since the latter are lower in energy, the stability of the hyper-deformed configurations is the looser in the competition. Therefore, in searching for the hyper-deformation the pre-requisites are totally different as compared to the search for the super-deformation:

Hyper- versus super-deformation: Expected differences in experimental approaches and suggested new strategies

- The discrete hyper-deformed bands are predicted to be much shorter than the super-deformed bands: 5–8 transitions typically, compared to 20 or more in the super-deformation case. Consequently, the experimental criteria based on the “picket-fence” like spectra must take this into account.
- The Jacobi transition seems a necessary condition: only nuclei that produce the Jacobi transitions, and therefore, the minima at high spins, high temperatures and at the same time at the hyper-deformed shapes can be populated through the fusion–evaporation reactions. Consequently, in contrast to the discussions existing so far in the literature, the first theoretical criterion should be “the nucleus of interest for the hyper-deformation studies must produce the Jacobi transition”. Only on top of that we must apply the shell-closure criteria, negative shell energies *etc.*
- As the result of the previous observation, one should seriously consider a drastic change in the experiment objectives: instead of hunting for the long discrete bands (that are anyway predicted to be absent) concentrate on the γ - γ - γ correlation analyses that give precious information at this time: the average length of the HD bands, the numbers of the excited bands in a given energy window *etc.*

We believe that the successes in the hyper-deformation studies have been delayed partly by the low statistics as compared to the sensitivity of the present day detection facilities, but also partly because of the relatively late discovery of the importance of the Jacobi transitions in the mechanism of populating the high-elongation nuclei.

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