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ANALYSIS OF PHASE SPACE STRUCTURE OF A 1-D DISCRETE SYSTEM USING GLOBAL AND LOCAL SYMBOLIC DYNAMICS*

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Dedicated to Professor Andrzej Fuliński on the occasion of his 70th birthday

Symbolic dynamics, in which the system trajectory is represented as a string of symbols, appears as a convenient method for the analysis of properties of chaotic attractors. In this paper, we show that, using a non-canonical coding scheme based on a moving partition point, we are able to access such properties of the phase space of a dynamical system as the localisation of unstable periodic orbits and of their stable invariant manifolds. Applying different coding schemes enables us to extract different information about the phase space structure from the chaotic trajectory. A judicial choice of the method of symbolic coding allows to obtain information which may be missing in the symbolic dynamics from the generating partition. We present results for the 1-D case taking the logistic map as a numerical example. The extension to higher dimension is also discussed. The theoretical background of the methods used is also given.

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1. Introduction

Poincaré noted that: *the apparent complexity of chaotic dynamics is such that it makes little sense to follow individual orbits. What is relevant is how regions of the state space are mapped between each other* (after [1]). This idea is expressed in a mathematical form as the symbolic dynamics theory. The theory states that the phase space should be divided into large partitions that are mapped one onto the other(s) under the action of the dynamical

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system equations. In one dimensional maps the partition is defined by setting the position of decision point (or points) that mark the boundaries between neighbouring partitions. All the partitions are labelled with symbols, the trajectory in phase space starting from an arbitrary initial condition may be represented as semi-infinite symbolic string, and the system equations are represented by the shift operator acting on this string [1, 4]. In the process of symbolic coding, however, one stringent condition must be met — the partition used must be so-called generating partition (GP).

The GP structure depends entirely on the topology of the central manifold of the studied dynamical system [1]. A defining property of the GP is such that it constitutes a one to one representation between each point in phase space and the unique symbolic string. This string is equivalent to the unique chaotic trajectory starting from this point. Finding the GP is a nontrivial task (apart from simple one-dimensional maps) and must be done for each studied system separately — in fact experimentally no general theoretical method for determination of GP from the equations of the system exists[1].

Using an arbitrary partition (*i.e.* non-GP) leads to severe diminishing of such complexity measures as metric entropy. The mechanism responsible for that effect is such that due to misplacement of the partition some admissible symbolic words appearing in a studied trajectory are falsely labelled as forbidden [2, 3].

There exists, however, a completely different approach to symbolic dynamics which is expressed also in this paper — regardless from the actual method of symbolic coding used, the complexity of the resulting symbolic "time series" may be evaluated using complexity measures of symbolic dynamics. This approach is sometimes called "a threshold-crossing technique" [3] — to stress the distinction from the "real" symbolic dynamics — and has been successfully applied in number of experimental conditions [5,6]. In fact, as we show in this paper, different methods of symbolic coding give different context to the complexity measures of symbolic dynamics.

In current paper we would like to show how, using different coding schemes, we may analyse the phase space structure of the generic 1-D discrete system. In this context a favourite example, the logistic map, was used [7]. Part of the material was already published as master thesis of one of the authors [16]. We also show how using syntactic analysis the two-dimensional system (Henon map) may be analysed.

[1] For all one-dimensional maps the decision points defining the GP must coincide with critical points of the map. In this way all different pre-images of an arbitrary point are labelled uniquely.

2. Chaotic attractor as structure in phase space

The chaotic attractor is typically a highly nonuniform structure. It consists of a set of unstable periodic orbits (UPO) together with their stable manifolds (SM) and unstable manifolds (UM). One of the formal definitions states that the attractor is the closure of the complete set of UPOs [8]. In chaotic states the UPO are attracting the chaotic trajectory along their stable directions (directions with negative Lyapunov exponents — all together spanning the stable manifold) and repelling toward unstable directions (with positive Lyapunov exponents) [8]. Neither the orbits nor the manifolds may be visited by the chaotic trajectory, as this would mean that the periodic orbit either is or was or will be visited. Different parts of the chaotic trajectory carry information about localisation of different UPO that may be extracted using many different methods [9–11].

If the studied system is irreversible, which is quite often among dynamical systems, each single point belonging to UPO is, in general, an image of multiple points. Only one of its preimages may belong to the UPO and all other belong to its SM[2]. This SM point may be back-iterated generating the cascade of other points that also belong to SM. The rate of growth of the number of these points depends on the system — for systems with known GP it is related with the number of unique symbols with which the different back-iteration points must be labelled[3]. Thus for a given system the set of the SM of all the UPO, together with the UPO themselves, forms a structure in the phase space of the system that is considered to be "a special part of the attractor" [8]. The invariant measure on this set is zero in chaotic states, so in fact they do not belong to the chaotic attractor as this would lead to a periodic evolution preceded by a chaotic transient[4]. As for hyperbolic systems UPO (and consequently the SM) are dense in the phase space so in ε-neighbourhood of any initial condition there are infinitely many points belonging to UPO and SM. For discrete time systems both SM and UM are in general non-compact objects of a fractional dimension, as the strange attractor itself.

3. Shifted partition method

In this paper a shifted partition method is used [13]. The principle of this method is such that a given chaotic trajectory is encoded many times using different positions of the decision point and some chosen complexity

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[2] Formally the UPO is neither part of the stable nor the unstable manifold as for flows the Lyapunov exponent in the direction of UPO is always zero.

[3] As the system is noninvertible, the inverse map is a set of functions — one for each monotonous branch of the original map. As in the GP the branches are labelled uniquely, the inverse functions may use the same labels.

[4] This discussion shows that the definition of the attractor as the closure of the set of UPO is questionable as the UPO themselves are not a part of the chaotic attractor.

measure is calculated as function of the spatial position of the decision point. In practice a finite grid of decision points is used. For each of resultant symbolic strings metric or block entropy is then calculated, which for symbolic dynamics may be done easily using the following formulae. The equation

$$H(n) = -\sum^{(n)} p\left(s_i\right) \ln p\left(s_i\right), \tag{1}$$

is a definition of the block entropy. Here summing is over all unique words s_i of length n and the probability of finding a word s_i in the symbolic trajectory is denoted $p\left(s_i\right)$. Metric entropy h may be calculated as the limit of the local slopes $h(n)$ of the block entropies $H(n)$ [12]:

$$h = \lim_{n \to \infty} h(n) = \lim_{n \to \infty} \left[H(n) - H(n-1) \right] . \tag{2}$$

For purpose of this work block entropies for a given value of n are used as the complexity measure. All the entropy plots are in arbitrary units as we were interested only in relative values. The logarithm used in calculation had base 10 instead of the commonly used cardinality of the alphabet (*i.e.* 2) or e, which for the same reason does not significantly affect the results.

The shifted partition method shows how the symbolic dynamics depend on the location of the decision point. Thus it is sensitive to local properties of the phase space of the system in the vicinity of the decision point.

Entropy plots of the type described above were first mentioned in [12] in a different context. In [2, 3] non-monotonicity of entropy plots was noticed, explained formally and reproduced analytically on a set of dyadic points [3]. In [13], however, they were used for the first time as a measurement method, with special interest in localisation and interpretation of the minima (see next chapter).

4. Global entropy

The impact of UPO and their SM on chaotic trajectory may be visualised using a shifted partition method with global coding scheme. The scheme is such that a single decision point d is selected and the chaotic trajectory is coded using two symbols:

$$s_i = \begin{cases} 0 & \text{for } \; x_i < d\,, \\ 1 & \text{for } \; x_i > d\,. \end{cases} \tag{3}$$

Using coded trajectory the block entropy for word length $n = 10$ may be calculated as function of the position of the decision point d. The entropy calculated using the global coding scheme will be further denoted as global entropy. The result of calculations is shown in Fig. 1.

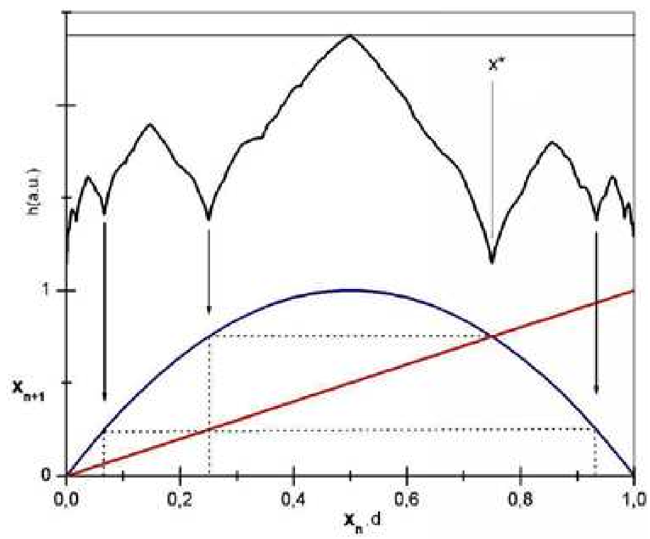

Fig. 1. Global entropy as the function of d for word length $n = 10$. The grid consists of 10000 equally spaced points. The minima of entropy marked with arrows are preimages of x^*. After few iterations they land on x^*, as shown by the dotted lines. The horizontal line shows the value of entropy when the partition is the GP (*i.e.* $\log 2$).

It may be seen that all the minima visible in this scale of the plot are located at preimages of the unstable fixed point x^*. This fixed point has relatively small positive Lyapunov exponent (as compared to other orbits). Therefore, it strongly influences the dynamics of the chaotic trajectory for nearly all values of control parameter[5]. The preimages of the UPO are located all over the attractor, constituting a fractal stable invariant manifold of x^*. For the calculation parameters used here (the density of the grid and the length of the trajectory), the eminent features of the plot seem to be all related with x^* due to its lowest instability. Increasing these parameters and/or taking every m-th point of the trajectory reveals the other minima related with UPO of higher order m and their SM [16].

It is possible to modify the above method by placing one decision point in one of the points belonging to some specific orbit and scan the phase space using second decision point. This enables us to expose the influence of this orbit on the chaotic trajectory [16].

[5] The only exceptions to this rule are the regions soon after the periodic windows where the large number of orbits created in a period-doubling cascade has instability comparable to that of x^* (in terms of Lyapunov exponent). Being of period higher than 1 they, however, loose stability much faster than x^*, which regains its dominating role. See also [13].

The series of minima of global entropy belonging to the SM are generated through an imaging mechanism. The principle of it will be described as following. Lets assume that d_1 is a decision point $d_1 \in [0,1]$ and d_2 is any of its preimages: $d_2 = f^{-1}(d_1)$. As the Lyapunov exponent is positive, the ε-neighbourhood of d_2 will be mapped onto the $\varepsilon \exp(\lambda)$-neighbourhood of d_1. Therefore, the shape of the entropy curve in the vicinity of d_1 resembles the rescaled curve from the vicinity of d_2. The imaging mechanism causes the structure of minima of global entropy to be self-similar. Understanding the reason for existence of the minimum of entropy at UPO requires introduction of the masking functions.

5. Masking functions

The values of entropy calculated using symbolic dynamics approach depend on a set of words that are forbidden or admissible. This set, in turn, depends on the position of the decision point and on the control parameter, as will be shown below.

The probabilities of words are equivalent to the integral over invariant measure taken in limits of the n-cylinder of the word. The n-cylinder of the word Σ_n [4] is a set of such points x that, taken as initial condition for the map, produce such trajectory that, after symbolic encoding, begins with Σ_n (Fig. 2). This equivalence will be denoted as $x \Rightarrow \Sigma_n$.

The admissibility of words for a given position of the decision point may be determined analytically using masking functions. The position of the decision point will be denoted as p to maintain consistence with the original theory [3]. For the same reason, the symbols 0 and 1 will be denoted A and B. For the map on interval $\mathcal{I} = [0,1]$, with parameter $p \in \mathcal{P} = [0,1]$ we define a masking function $s_j(x,p,i) : \mathcal{I} \times \mathcal{P} \times \mathcal{N} \to \{0,1\}$ which has value 1 for such $\{x,p\}$ pairs for which the symbol s_j is admissible on the position i in a word. Such masking function may be defined for each symbol s_j, thus $j = 1 \ldots C$, where C is the cardinality of the alphabet. The set of $\{x,p\}$ for which $s_j(x,p,i) = 1$ (for some constant i) is called admissibility region $\Gamma(s_j)$. For $i = 0$ the masking functions describe the admissibility regions for symbols A and B, that are shown in Fig. 3(a). The space $\mathcal{I} \times \mathcal{P}$ may be expressed as a sum of $\Gamma(A)$ and $\Gamma(B)$[6]. The admissibility regions are separated by a boundary line $x = p$. This is a simple consequence of a coding scheme that assigns B to $x > p$ and A to $x < p$. For $i = 1$ the boundary line separates the admissibility region of words $^*A$ and $^*B$ (having, respectively, A or B as a second symbol (* denotes any symbol). The shape

[6] Not that the line $p = x$ is excluded according to a silent assumption typical to the symbolic dynamics that the trajectory will never visit the decision point. The set of $p = x$ has measure zero.

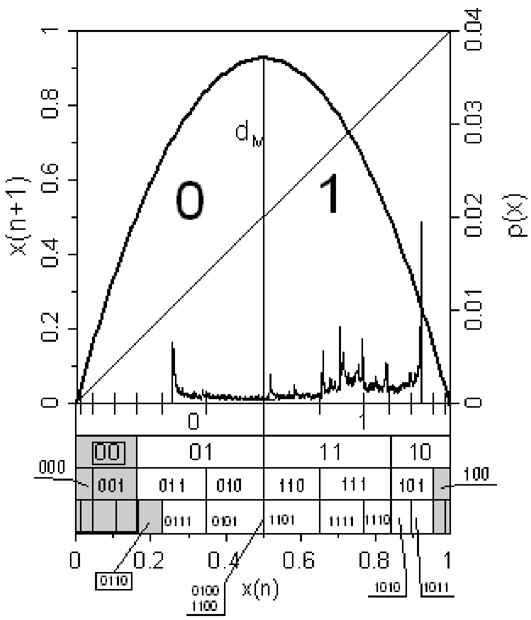

Fig. 2. The n-cylinder structure for a logistic map at $r = 3.7$ and the invariant measure. The n-cylinders that are outside the attractor are marked gray — they are related with the forbidden words for $r = 3.7$. As the attractor grows with rising r they will disappear one by one. The words in rectangles are minimal (irreducible) forbidden words (explanation in text).

of the boundary line $x = p$ for $i = 1$ may be obtained by substituting x by $f(x)$. This gives $f(x) = p$ which is equivalent to the shape of the map itself (Fig. 3(b)). Accordingly for the higher iterates the boundary line is given

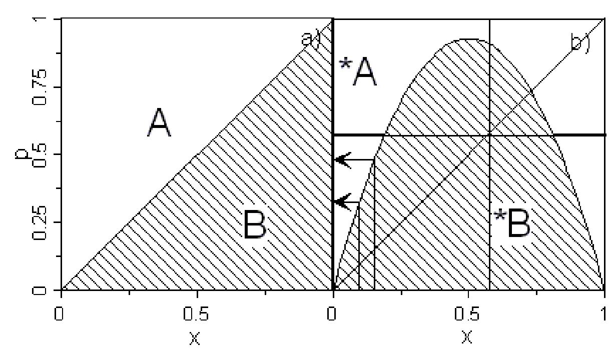

Fig. 3. Masking functions for $i = 0$ (a) and $i = 1$ (b). The lines with arrows in (b) show typical iterates that begin from $^*A$ and end in A for some value of p (horizontal and vertical line).

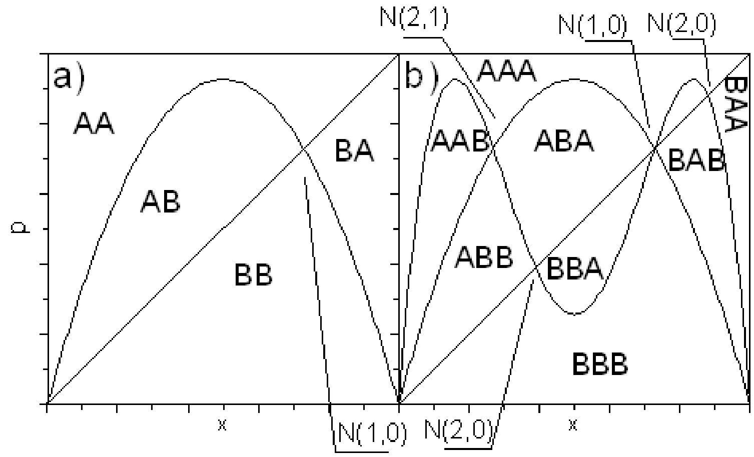

Fig. 4. The space $\mathcal{I} \times \mathcal{P}$ partitioned into admissibility regions for $i = 2$ (a) and $i = 3$ (b). Nodal points are marked with lines.

by higher orders of the map $f^n(x) = \underbrace{f \circ \ldots \circ f}_{n} = p$. It may be seen that for a given p (horizontal line in Fig. 3(b)) all the x belonging to $^*A$ iterated once fall into A (below the horizontal line in Fig. 3(b) — few such iterates are shown with arrows).

The $\Gamma(BB)$ may be expressed using masking functions: it consists of such $\{x, p\}$ pairs for which $B(x, p, 0) \cdot B(x, p, 1) = 1$. All the $\Gamma(\Sigma_n)$ may be expressed accordingly. For any length of the word n the space $\mathcal{I} \times \mathcal{P}$ may be partitioned into disjoint $\Gamma(\Sigma_n)$ of all possible words Σ_n, as Fig. 4 for $i = 2$ and 3.

For any value of p we may obtain all the admissible words and their admissibility regions. It may be seen from Fig. 4 that the limits of $\Gamma(\Sigma_n)$ are given by: $x = p, x = f(p), \ldots, x = f^l(p)$. For some values of p the limits of $\Gamma(\Sigma_n)$ coincide with the intersection points of such lines. Positions of such points are defined by:

$$f^l(x) = p = f^m(x)\,, \tag{4}$$

(denoting x as $f^0(x)$). They may be called nodal points of order l, m: $N(l, m)$.

Now we will show that the nodal points are either the UPO or their preimages. The condition (4) may be rewritten as:

$$x = f^{-l}(f^m(x))\,. \tag{5}$$

Then, due to the fact that the map is non-invertible — $f^{-1}(x)$ is not unique — each point has two preimages. Thus the condition 5 defines a set of 2^l

points. Only m of them belong to an UPO of order m, the other must belong to its stable invariant manifold as after l iterations they fulfil the condition $x = f^m(x)$. The nodal points being the UPO are the nodal points $N(*, 0)$

Therefore, we see in Fig. 4 that the limits of $\Gamma(\Sigma_n)$ are related with the values of p related to either UPO or their SM. The entropy $H(n)$ (for the word length n) depends on probabilities of words (see Eq. (1)). When $H(n)$ is calculated as a function of p it has minima for $p^* = N(l, m)$ as for $p > p^*$ one word is forbidden (see *e.g.* Fig. 4(a) at nodal point $N(1, 0)$ — the word BB). At $p = p^*$ this word becomes forbidden which significantly diminishes the entropy. When a new word appears (as a function of p) its admissibility region increases from width 0, therefore, the shape of the entropy is smooth around p^*. Note from the construction of $\Gamma(\Sigma_n)$ that $H(n)$ is sensitive to all UPO of orders $m < n$. If we calculate the borders of admissible regions analytically, the probabilities of words and the entropy as a function of p may be calculated using the natural measure. The results obtained are in concordance with those obtained from the numerical experiment [17].

6. Syntactic analysis

Besides entropy, there exist also other complexity measures that may be used to characterise the dynamics of the system, such as the Riemann ζ function [4]. One of such measures is the number of *irreducible forbidden words* IFW [15]. If we analyse the structure of forbidden words, we will see that they are built upon the skeleton of the IFW: *i.e.* such forbidden words that do not contain any other forbidden words. Such words may appear when the chaotic attractor does not cover the whole structure of all possible n-cylinders, *i.e.* does not fill all the phase space (see *e.g.* Fig. 2). The analysis of the IFW may be called syntactic analysis as, in fact, the IFW determine the syntax of the symbolic trajectory. The structure of IFW may be analysed using the shifted partition method. The regions in $\mathcal{P}$ space, in which certain words are irreducible forbidden, are related to the structure of characteristic points of the chaotic attractor (*e.g.* the critical point, the UPO, the boundaries of chaotic bands or similar). This is also a clear conclusion from the previous chapter: *i.e.* the word BB is a IFW for $p > x^*$ (Fig. 4).

7. Local entropy

Main drawback of the global entropy method is such that we are unable to distinguish between the minima related to the UPO and those related to the SM. The minima related to SM are "preimages" of those related to UPO, created by the imaging mechanism described in Section 4. Therefore, it would be valuable to have a method sensitive only to the minima related with the UPO. This may be done by a suitable redefinition of the symbolic string: applying a different coding scheme enables us to obtain different results from the same method of entropy calculation. Note here that the different coding scheme puts the same data vector $\{x\}$ in a different context. The global entropy is sensitive to such chaotic trajectories which start from the vicinity of SM and fall into the vicinity of UPO: they cannot be easily distinguished from those related only with UPO as their symbolic future is the same — they differ only by the first symbol. Such orbits may be excluded by applying a local coding scheme: to limit the domain of the coding function to a small $2d$-neighbourhood of the decision point. Therefore, the coding function would be the following:

$$s_i = \begin{cases} 0 & \text{for } x_i \in (p-d, p)\,, \\ 1 & \text{for } x_i \in (p, p+d)\,, \\ \emptyset & \text{otherwise.} \end{cases} \tag{6}$$

The empty symbol means that no symbol will be put into the symbolic string. The half-width of the window d is a parameter of the method.

Due to the presence of the empty symbol a symbolic string does not give the complete symbolic representation of the system dynamics. The neighbouring symbols are not necessarily related with the neighbouring (in time) points of the trajectory. They, however, remain neighbours in space as the domain of the coding function is narrow. Therefore, this symbolic string describes the properties of the dynamics locally in phase space which legitimises the name "local entropy".

The results obtained using this method are shown in Fig. 5.

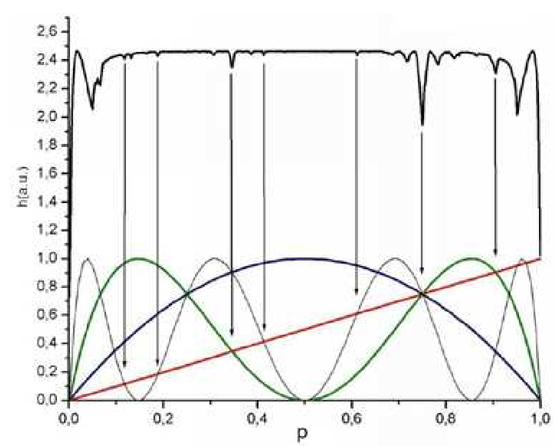

Fig. 5. Local entropy as the function of p. The window half-width $d = 0.1$. The minima of entropy marked with arrows are related only with the UPO. Other nodal points are not detected.

Apart from two deep artificial minima near d and $1-d$ (for which the window boundary exceeds the edge of the attractor) all the other minima are related with the UPO of smallest periods. Free parameters of the method are the number of points required to fill the window and the window width. By increasing the number of points and decreasing the width we are able to improve the resolution of these calculations, this however, requires a long chaotic trajectory. For smaller window half-width the minima related with UPO become significantly steeper and thus easy to identify.

For the results presented in Fig. 5 the 10^5 points falling within the coding window were used. The length of the whole trajectory cannot be estimated as the invariant measure is non uniform and the probability to find the trajectory point in the coding window may differ from point to point. Thus it is easier to find UPO located in the regions of large invariant measure. This, however, is a property common to all the methods for UPO detection that use the trajectory of the system. For a constant length of the trajectory the resolution of the measurement depends on the fraction of the invariant measure contained within the coding window.

8. Syntactic analysis of a two-dimensional system

The issue whether the method of symbolic dynamics may be applied successfully to a multi-dimensional system is an open question. To partially address this problem we used the Henon map as an example of a two-dimensional, discrete-time system

$$\begin{aligned} x_{i+1} &= 1 - a x_i^2 + b y_i \,, \\ y_{x+1} &= x_i \,. \end{aligned}$$

Next, the sequence of iterations of the x variable was analysed using the shifted partition method. As the shape of the entropy did not show any evident minima which could be easily interpreted, we decided to use a more subtle technique, that is, the analysis of irreducible forbidden words (IFW). The results obtained are shown in Fig. 6.

From the IFW, that appear for the Henon system as function of the position of the decision point, two families may be discerned. It is typical that when some irreducible forbidden word Σ_{IFW} becomes admissible, some other forbidden word(s) of the structure $\Sigma_{\mathrm{IFW}}\Sigma$ become an IFW. The words 00 and 000 may be taken as example. At some point (the beginning of a chaotic branch of the attractor) the word 00 is no longer forbidden. Therefore, at the same point, 000 becomes an IFW. Thus the whole family of words may be discerned, as it is shown in the left of the two bars in Fig. 6. The majority of points, where the IFW change are related with the chaotic branches of the attractor, are denoted with the dashed lines. The asymptotic limit of

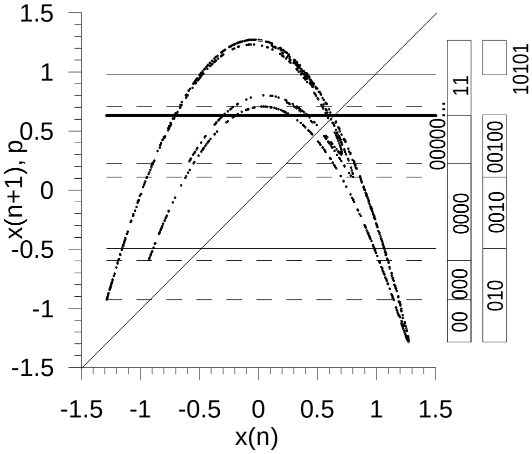

Fig. 6. The attractor of the Henon map in the (x_n, x_{n+1}) projection together with the regions of appearance of two families of IFW: (00, 000, 0000, 00000 ..., 11) and (010, 0010, 00100, ... 10101). The address of each word is shown within its region of appearance. The dashed lines show the positions of the "skeleton" points of the attractor: the beginnings and the ends of some of the branches. The thick solid line marks the position of the period-1 UPO. Two thin solid lines mark the points belonging to a period-2 UPO.

regions of appearance of this family of IFW is located at the point where the word 11 becomes an IFW *i.e.* at the UPO of order 1 (thick solid line). The second family is related with the other chaotic branches of the attractor and with the two points belonging to period-2 orbits (thin solid lines).

If the symbolic dynamics of the system is not known *a priori* it seems impossible to discern between the points marking the limits of chaotic branches and those related with UPO. On the other hand, nearly all the points where the IFW appear or disappear have a special meaning in the structure of the chaotic attractor.

Note that the set of UPOs of the Henon map consists of a single period-1, a single period-2 and two period-6 orbits [18], thus the lack of orbits of a higher period (*e.g.* 3, 4, 5 ...) is not a limitation of the method but the property of the system studied here.

9. Discussion

Once the symbolic string is obtained symbolic dynamics provides a number of measures and techniques to access various dynamical properties of the system represented by this string. What remains a general yet unsolved, problem is how to construct a reliable representation of the system dynamics. This paper, as well as some other cited herein [5, 6, 13] do not give a solution to this problem. Instead they attempt to show that applying a non-generating coding scheme, without knowledge of the generating partition, it is possible to use symbolic dynamics methods to study various aspects of the system dynamics. In this paper we show that application of the proper coding scheme lets us obtain the information we need. The coding scheme may be seen as a context in which the chaotic trajectory is put prior to application of tools of nonlinear dynamics. Depending on the question asked, the proper coding method may be helpful in giving an exact answer. We show this on example of localisation of phase space structures: such as UPO and their SM, but this list is by no means closed.

Such an approach extends the possible usage of symbolic dynamics. One must keep in mind, however, that the symbolic description is not unique in this case and *e.g.* the absolute values of entropy cannot be directly interpreted as a measure of complexity of the studied system but only taken as a crude approximation. This was studied in [3] and [13] from the points of view of pure *versus* applied symbolic dynamics, respectively. The other thing that must be taken into account is that the structure of generating partition changes with the control parameter, thus the values of entropy based on an arbitrary partition cannot be calculated as a function of the control parameter [13]. Note that the line of reasoning presented in Section 5 holds also for higher dimensions. *E.g.* for the 2-D systems the x–p space becomes 4-dimensional and two moving decision lines must be used instead of one point. The admissibility regions are then 4-D objects. The entropy plots for 2-D become the 2-D surfaces and the deepest minima appear when both decision lines are placed appropriately, therefore, we may predict that the extension of the method towards higher dimensions is in general possible. Alternatively a single, suitably chosen variable of the multi-dimensional systems may be analysed.

In Section 8 we showed that it is, in principle, possible to analyse a single variable of a multi-dimensional system using the shifted partition method. It is not surprising that a single variable may carry information about the topology of the attractor. In fact, for the Henon map, the curve defining the generating partition is nearly a straight line, that may be found at least numerically [1]. Thus, if we analyse a single variable perpendicular to this line, we obtain very good results. Problems appear for positions of the decision points that are close to the generating partition line. There may appear such point locations, at which a single variable is not enough to assign the proper symbol. We have observed the existence of the IFW whose inadmissibility region begins and ends at the points that are close to a UPO. Thus, probably, if the full symbolic dynamics was taken into account, they would be found to be related to this UPO.

A possible solution in many dimensions is to find either such affine transformation of variables for which the partition line would be moved parallel to the generating one, or to apply a nonlinear transformation to make the generating partition curve a straight line, but this has yet to be done.

10. Conclusions

Two methods for the examination of the phase space of one-dimensional systems were proposed. Both apply the symbolic dynamics approach but place the standard complexity measure (block entropy) in a different context by using two different coding schemes. The global entropy method enables one to find the UPO and their stable manifolds, whereas the local entropy method is sensitive only to the occurrence of the UPO. A theoretical background for the global entropy method was also given.

The tools discussed represent a new approach to the standard symbolics dynamics analysis of trajectories of chaotic systems. The choice of the method of symbolic coding has a strong effect on the information content of the symbolic dynamics obtained. It appears that, besides coding by means of the generating partition, the true trajectory of the system in phase space may be encoded in different ways. The different symbolic dynamics obtained in this way each carry different information about the system. A judicial choice of the method of symbolic coding (in 1-dimension, the choice of the location of the decision point) allows to obtain information which may be missing in the symbolic dynamics from the generating partition. In this paper, we discussed the detection of UPO in this way.

The methods were presented taking the logistic map as a numerical example but they may be applied to any 1-D map such as one given analytically or a return map of either a set of ordinary differential equations or any experimental signal. The possibility to extend the methods to higher dimensions was also discussed. A single variable from the Henon map was analysed and the UPO of lowest orders were found by the method developed here.

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