A NEW GAMMA TYPE APPROXIMATION OF THE RUIN PROBABILITY*

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Dedicated to Professor Andrzej Fuliński on the occasion of his 70th birthday

In this paper we introduce a generalization of the De Vylder approximation of the ruin probability. Here the risk process is described in the language of a continuous time random walk. Our idea of approximation is to replace the risk process with the one with gamma claims, matching first four moments. We compare the two approximations studying mixture of exponentials and lognormal claims. We show that the proposed 4-moment gamma approximation works better than the original one.

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1. Introduction

Empirical evidence has been mounting that supports the possibility that a number of systems arising in disciplines as diverse as physics, biology, ecology and economics may have certain quantitative features that are intriguingly similar. For example, the continuous time random walk (CTRW) model, formerly introduced in statistical physics by Montroll and Weiss [1], see also for the recent development [2, 3], can provide a phenomenological description of tick-by-tick dynamics in financial markets [4]. Consequently,

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the survival probability for certain bond futures traded at LIFFE, London can be analysed via the Mittag–Leffler function [5]. The Merton option pricing formula [6], which is an extension of the Black–Scholes one, is based on the jump-diffusion model for the price process S_t and the jump part of S_t is just given by the CTRW. Here we will discuss a practical application of the CTRW model in the context of the ruin probability for the risk process describing the capital of an insurance company.

The recent increasing interplay between actuarial and financial mathematics has led to a surge of risk theoretic modeling. Especially actuarial ruin models under fairly general conditions on the underlying risk process have become focus of attention [6, 7]. Ruin theory is concerned with the excess of the income (with respect to a portfolio of business) over the outgo, or claims paid. This quantity, referred to as insurer's surplus, varies in time. Specifically, ruin is said to occur if the insurer's surplus reaches a specified lower bound, *e.g.* minus of the initial capital. One measure of risk is the probability of such an event, clearly reflecting the volatility inherent in the business. In addition, it can serve as a useful tool in long range planning for the use of insurer's funds.

Unfortunately, the ruin probabilities in infinite and finite time can only be calculated for a few special cases of the claim amount distribution. Thus, finding a reliable approximation, especially in the ultimate case when the straightforward Monte Carlo approach can not be utilized, is really important from a practical point of view.

Grandell [8] demonstrates that between possible simple approximations of ruin probabilities in infinite time the most successful is the De Vylder approximation, which is based on the idea to replace the risk process with the one with exponentially distributed claims and ensuring that the first three moments coincide.

We introduce a modification to the De Vylder approximation by changing the exponential distribution to the gamma and making the first four moments match. This modification is promising and works in many cases better than the original approximation. Observe that for empirical data there are no serious problems since no estimation of higher empirical moments is involved. We only use analytical form of the distribution which is fitted by the non-parametric procedure [9]. In order to compare De Vylder and 4-moment gamma (4MG) approximations we consider mixture of two exponentials and lognormal claims. We compute relative errors of the methods with respect to the exact values of the ruin probability. The ruin probability in the lognormal case is calculated via the Pollaczek–Khinchin formula using Monte Carlo simulations [10].

Let us now recall a standard model (called the risk process) for the capital of an insurance company. The initial capital is u, the Poisson process

N_t with intensity λ describes the number of claims in $(0,t]$ interval and claim severities are given by sequence of independent positive identically distributed random variables $\{X_k\}_{k=1}^{\infty}$ with mean μ and (if existing) raw moments $\mu^{(2)}$, $\mu^{(3)}, \ldots$. Furthermore, we assume that $\{X_k\}$ and $\{N_t\}$ are independent. To cover its liability, the company receives premium at a constant rate c, per unit time. Thus, the risk process $\{R_t\}_{t\geq 0}$ is given by

$$R_t = u + ct - \sum_{i=1}^{N_t} X_i\,. \tag{1}$$

Observe that the last term in Eq. (1) describing the insurance company's aggregate losses is modelled by the CTRW. If we add to the right side the diffusion term σW_t, where W_t stands for the Brownian motion, then the risk process R_t has a full jump-diffusion form. For the insurance company we have $\sigma = 0$.

The premium c is often written as $c = (1+\theta)\lambda\mu$ and $\theta > 0$ is called the relative safety loading. The loading has to be positive, otherwise c would be less than $\lambda\mu$ and thus with probability one the risk business would become negative in infinite time.

It is sometimes more convenient to work with the aggregate surplus process $\{S_t\}_{t\geq 0}$, namely $S_t = u - R_t = \sum_{i=1}^{N_t} X_i - ct$. Now, we are going to recall the definition of ruin probability, *i.e.* the probability that the capital drops below zero. The time to ruin is defined as

$$\tau(u) = \inf\{t \geq 0 : R_t < 0\} = \inf\{t \geq 0 : S_t > u\}\,.$$

Let $M = \sup_{0\leq t<\infty}\{S_t\}$. The ruin probability in infinite time is defined as

$$\psi(u) = \mathbb{P}(\tau(u) < \infty) = \mathbb{P}(M > u)\,. \tag{2}$$

2. Light- and heavy-tailed distributions

We distinguish here between light- and heavy-tailed distributions [11]. Distribution $F_X(x)$ is said to be light-tailed, if there exist constants $a > 0$, $b > 0$ such that $\bar{F}_X(x) = 1 - F_X(x) \leq ae^{-bx}$ or, equivalently, if there exist $z > 0$, such that $M_X(z) < \infty$, where $M_X(z)$ is the moment generating function. Distribution $F_X(x)$ is said to be heavy-tailed, if for all $a > 0$, $b > 0$ $\bar{F}_X(x) > ae^{-bx}$, or, equivalently, if $\forall z > 0\;\; M_X(z) = \infty$.

The most important distributions, often describing light- and heavy-tailed losses are presented in Table I. In this paper we focus on the mixture of exponentials and lognormal cases.

TABLE I

Densities of typical claim size distributions. In the loggamma case $x \geq 1$.

Light-tailed distributions		
Name	Parameters	pdf
Exponential	$\beta > 0$	$f_X(x) = \beta \exp(-\beta x)$
Gamma	$\alpha > 0,\ \beta > 0$	$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x)$
Weibull	$\beta > 0,\ \tau \geq 1$	$f_X(x) = \beta\tau x^{\tau-1} \exp(-\beta x^\tau)$
Mix. exp's	$\beta_i > 0,\ \sum_{i=1}^{n} a_i = 1$	$f_X(x) = \sum_{i=1}^{n} \{a_i \beta_i \exp(-\beta_i x)\}$
Heavy-tailed distributions		
Name	Parameters	pdf
Weibull	$\beta > 0,\ \ 0 < \tau < 1$	$f_X(x) = \beta\tau x^{\tau-1} \exp(-\beta x^\tau)$
Lognormal	$\mu \in \mathbb{R},\ \sigma > 0$	$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left\{-\frac{(\ln x-\mu)^2}{2\sigma^2}\right\}$
Loggamma	$\alpha > 0,\ \beta > 0$	$f_X(x) = \frac{\beta^\alpha (\ln x)^{\alpha-1}}{x^{\beta+1}\Gamma(\alpha)}$
Pareto	$\alpha > 0,\ \lambda > 0$	$f_X(x) = \frac{\alpha}{\lambda+x}\left(\frac{\lambda}{\lambda+x}\right)^\alpha$
Burr	$\alpha > 0,\ \lambda > 0,\ \tau > 0$	$f_X(x) = \frac{\alpha\tau\lambda^\alpha x^{\tau-1}}{(\lambda+x^\tau)^{\alpha+1}}$

In the case of light-tailed claims the adjustment coefficient (called also the Lundberg exponent) plays a key role in calculating the ruin probability. Let $\gamma = \sup_z \{M_X(z)\} < \infty$ and let R be a positive solution of the equation:

$$1 + (1+\theta)\mu R = M_X(R)\,, \qquad R < \gamma\,. \tag{3}$$

If there exists a non-zero solution R to the above equation, we call it an adjustment coefficient. Clearly, $R = 0$ satisfies Eq. (3), but there may exist a positive solution as well (this requires that X has a moment generating function, thus excluding distributions such as Pareto and the lognormal).

An analytical solution to Eq. (3) exists only for few claim distributions. However, it is possible to obtain a numerical solution. The coefficient R satisfies the inequality:

$$R < \frac{2\theta\mu}{\mu^{(2)}}\,, \tag{4}$$

where $\mu^{(2)} = \mathrm{E}(X_i^2)$ [10]. Let $D(z) = 1 + (1+\theta)\mu z - M_X(z)$. Thus, the adjustment coefficient $R > 0$ satisfies the equation $D(R) = 0$. In order to get the solution one may use the Newton–Raphson formula:

$$R_{j+1} = R_j - \frac{D(R_j)}{D'(R_j)}\,, \tag{5}$$

with the initial condition $R_0 = 2\theta\mu/\mu^{(2)}$, where $D'(z) = (1+\theta)\mu - M_X'(z)$.

Moreover, if it is possible to calculate the third raw moment $\mu^{(3)}$, we can obtain a sharper bound than (4), [12]:

$$R < \frac{12\mu\theta}{3\mu^{(2)} + \sqrt{9(\mu^{(2)})^2 + 24\mu\mu^{(3)}\theta}}\,,$$

and use it as the initial condition in (5).

3. De Vylder and 4MG approximations

The idea of the De Vylder approximation is to replace the risk process with the one with $\theta = \bar{\theta}$, $\lambda = \bar{\lambda}$ and exponential claims with parameter $\bar{\beta}$, fitting first three moments [13]. Let

$$\bar{\beta} = \frac{3\mu^{(2)}}{\mu^{(3)}}\,, \qquad \bar{\lambda} = \frac{9\lambda\mu^{(2)^3}}{2\mu^{(3)^2}}\,, \qquad \text{and} \qquad \bar{\theta} = \frac{2\mu\mu^{(3)}}{3\mu^{(2)^2}}\theta\,.$$

Then De Vylder's approximation is given by

$$\psi_{\mathrm{DV}}(u) = \frac{1}{1+\bar{\theta}}\, e^{-\bar{\theta}\bar{\beta}u/(1+\bar{\theta})}\,. \tag{6}$$

Obviously, in the exponential case the method gives the exact result. For other claim distributions, in order to apply the approximation, the first three moments have to exist.

We now introduce a new 4-moment gamma approximation based on the De Vylder's idea to replace the risk process with another one for which the expression for $\psi(u)$ is explicit. We fit the four moments in order to calculate the parameters of the new process with gamma distributed claims and apply the exact formula for the ruin probability in this case which is given in Ref. [14]. The risk process with gamma claims is determined by the four parameters $(\bar{\lambda}, \bar{\theta}, \bar{\mu}, \bar{\mu}^{(2)})$. Since

$$\begin{aligned}
\mathrm{E}(S_t) &= -\theta\lambda\mu t\,,\\
\mathrm{E}(S_t^2) &= \lambda\mu^{(2)}t + (\theta\lambda\mu t)^2\,,\\
\mathrm{E}(S_t^3) &= \lambda\mu^{(3)}t - 3(\lambda\mu^{(2)}t)(\theta\lambda\mu t) - (\theta\lambda\mu t)^2\,,\\
\mathrm{E}(S_t^4) &= \lambda\mu^{(4)}t - 4(\lambda\mu^{(3)}t)(\theta\lambda\mu t) + 3(\lambda\mu^{(2)}t)^2\\
&\quad +6(\lambda\mu^{(2)}t)(\theta\lambda\mu t)^2 + (\theta\lambda\mu t)^4
\end{aligned}$$

and for the gamma distribution

$$\bar{\mu}^{(3)} = \frac{\bar{\mu}^{(2)}}{\bar{\mu}}(2\bar{\mu}^{(2)} - \bar{\mu}^2)\,, \quad \bar{\mu}^{(4)} = \frac{\bar{\mu}^{(2)}}{\bar{\mu}^2}(2\bar{\mu}^{(2)} - \bar{\mu}^2)(3\bar{\mu}^{(2)} - 2\bar{\mu}^2)\,,$$

the parameters $(\bar{\lambda}, \bar{\theta}, \bar{\mu}, \bar{\mu}^{(2)})$ must satisfy the equations

$$\begin{aligned}
\theta\lambda\mu &= \bar{\theta}\bar{\lambda}\bar{\mu}\,,\\
\lambda\mu^{(2)} &= \bar{\lambda}\bar{\mu}^{(2)}\,,\\
\lambda\mu^{(3)} &= \bar{\lambda}\frac{\bar{\mu}^{(2)}}{\bar{\mu}^2}(2\bar{\mu}^{(2)}-\bar{\mu}^2)\,,\\
\lambda\mu^{(4)} &= \bar{\lambda}\frac{\bar{\mu}^{(2)}}{\bar{\mu}^2}(2\bar{\mu}^{(2)}-\bar{\mu}^2)(3\bar{\mu}^{(2)}-2\bar{\mu}^2)\,.
\end{aligned}$$

Hence

$$\begin{aligned}
\bar{\lambda} &= \frac{\lambda(\mu^{(3)})^2(\mu^{(2)})^3}{(\mu^{(2)}\mu^{(4)}-2(\mu^{(3)})^2)(2\mu^{(2)}\mu^{(4)}-3(\mu^{(3)})^2)}\,,\\
\bar{\theta} &= \frac{\theta\mu(2(\mu^{(3)})^2-\mu^{(2)}\mu^{(4)})}{(\mu^{(2)})^2\mu^{(3)}}\,,\\
\bar{\mu} &= \frac{3(\mu^{(3)})^2-2\mu^{(2)}\mu^{(4)}}{\mu^{(2)}\mu^{(3)}}\,,\\
\bar{\mu}^{(2)} &= \frac{(\mu^{(2)}\mu^{(4)}-2(\mu^{(3)})^2)(2\mu^{(2)}\mu^{(4)}-3(\mu^{(3)})^2)}{(\mu^{(2)}\mu^{(3)})^2}\,.
\end{aligned}$$

We also need to assume that $\mu^{(2)}\mu^{(4)} < \frac{3}{2}(\mu^3)^2$ and to ensure that $\bar{\mu}, \bar{\mu}^{(2)} > 0$ and $\bar{\mu}^{(2)} > \bar{\mu}^2$. In case this assumption can not be fulfilled, we simply set $\bar{\mu} = \mu$ and do not calculate the fourth moment. This case leads to

$$\begin{aligned}
\bar{\lambda} &= \frac{2\lambda(\mu^{(2)})^2}{\mu(\mu^{(3)}+\mu^{(2)}\mu)}\,,\\
\bar{\theta} &= \frac{\theta\mu(\mu^{(3)}+\mu^{(2)}\mu)}{2(\mu^{(2)})^2}\,,\\
\bar{\mu} &= \mu\,, \quad \bar{\mu}^{(2)} = \frac{\mu(\mu^{(3)}+\mu^{(2)}\mu)}{2\mu^{(2)}}\,.
\end{aligned} \tag{7}$$

All in all, we get the approximation

$$\psi_{4\mathrm{MG}}(u) = \frac{\bar{\theta}(1-\frac{R}{\bar{\alpha}})e^{-(\bar{\beta}R/\bar{\alpha})u}}{1+(1+\bar{\theta})R-(1+\bar{\theta})(1-\frac{R}{\bar{\alpha}})} + \frac{\bar{\alpha}\bar{\theta}\sin(\bar{\alpha}\pi)}{\pi}\,I\,, \tag{8}$$

where

$$I = \int_0^\infty \frac{x^{\bar{\alpha}}e^{-(x+1)\bar{\beta}u}\,dx}{\left[x^{\bar{\alpha}}\left(1+\bar{\alpha}(1+\bar{\theta})(x+1)\right)-\cos(\bar{\alpha}\pi)\right]^2+\sin^2(\bar{\alpha}\pi)}\,,$$

R is the adjustment coefficient for the gamma distribution and $(\bar{\alpha}, \bar{\beta})$ are given by $\bar{\alpha} = \frac{\bar{\mu}^2}{\bar{\mu}^{(2)} - \bar{\mu}^2}$, $\bar{\beta} = \frac{\bar{\mu}}{\bar{\mu}^{(2)} - \bar{\mu}^2}$.

In the exponential and gamma case this method gives the exact results. For other claim distributions in order to apply the approximation, the first four (or three) moments have to exist. In Section 5 will show that it gives a slight correction to the De Vylder approximation, which is said in Ref. [8] to be the best among simple approximations.

4. Pollaczek–Khinchin formula

This time we use the representation (2) of the ruin probability and the decomposition of the maximum M as a sum of ladder heights. Let L_1 be the value that process $\{S_t\}$ reaches for the first time above the zero level. Next, let L_2 be the value which is obtained for the first time above the level L_1; $L_3, L_4, \ldots$ are defined in the same way. The values L_k are called ladder heights. Since the process $\{S_t\}$ has stationary and independent increments, $\{L_k\}_{k=1}^{\infty}$ is the sequence of independent and identically distributed variables. One may show that the number of ladder heights K to the moment of ruin is given by a geometric distribution with the parameter $q = \frac{\theta}{1+\theta}$. Thus, random variable M may be expressed by

$$M = \sum_{i=1}^{K} L_i \,. \tag{9}$$

This implies that random variable M has a compound geometric distribution given by the distribution function

$$F_M(x) = \frac{\theta}{1+\theta} \sum_{n=0}^{\infty} G^{*n}(x) \,, \tag{10}$$

where G^{*n} is the nth convolution of the distribution with the defective density

$$g(x) = \frac{1}{\mu(1+\theta)} \bar{F}_X(x) = \frac{1}{1+\theta} b_0(x) \,, \tag{11}$$

and the density

$$b_0(x) = \frac{\bar{F}_X(x)}{\mu} \,. \tag{12}$$

The above fact together with the representation (2) leads to the Pollaczek–Khinchin formula for the ruin probability:

$$\psi(u) = \mathbb{P}(M > u) = \frac{\theta}{1+\theta} \sum_{n=0}^{\infty} \left(\frac{1}{1+\theta} \right)^n \bar{B}_0^{*n}(u)\,, \tag{13}$$

where $\bar{B}_0$ is the tail of the distribution corresponding to the density b_0 [10].

One can use it to derive explicit solutions for a variety of claim amount distributions, particularly those whose Laplace transform is a rational function [12]. Unfortunately, the lognormal case is not included. However, in order to calculate the ruin probability the formula can be also applied directly. Using Eqs. (9) and (13), the ruin probability $\psi(u) = \mathrm{E}(Z)$, where $Z = 1(M > u)$, may be calculated via Monte Carlo simulations.

SIMULATION ALGORITHM

1. Generate a random variable K from the geometric distribution with the parameter $q = \frac{\theta}{1+\theta}$.
2. Generate random variables $X_1, X_2, \cdots, X_K$ from the density $b_0(x)$.
3. Calculate $M = X_1 + X_2 + \cdots + X_K$.
4. If $M > u$, let $Z = 1$, otherwise let $Z = 0$.

The main problem seems to be simulating random variables from the density $b_0(x)$. In the lognormal case the density does not have a closed form. Consequently, in order to generate random variables X_k we use formula (12) and controlled numerical integration.

It was shown in Ref. [15] that the computer approximation via the Pollaczek–Khinchin formula can be chosen as the reference method for calculating the ruin probability in infinite time. Hence we will call ruin probability values obtained by virtue of the above procedure exact.

5. De Vylder versus 4MG approximation

We now aim to compare De Vylder and 4-moment gamma approximations. To this end we consider the ruin probability as a function of the initial capital u, with two claim amount distributions, namely mixture of two exponentials representing the light-tailed case and lognormal being a prominent example of the heavy-tailed case. In order to show the relative errors of the methods we compare results of the approximations with the exact values.

In the case of mixture of two exponentials distribution, exact values of the ruin probability can be computed using inversion of Laplace transform technics [12]. Fig. 1(a) depicts the exact ruin probability values and results of the De Vylder and 4-moment gamma approximations. Fig. 1(b) demonstrates that the relative error of the latter is less than 8% and proves that it gives much better results than the original method which reaches the 50% error.

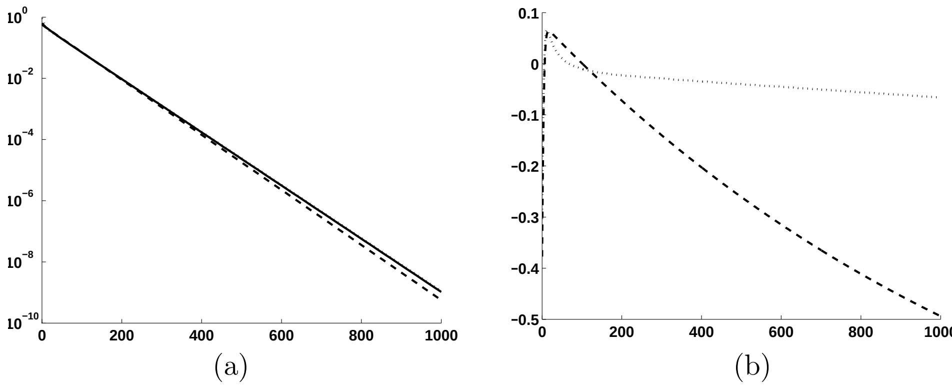

Fig. 1. Illustration of (a) the ruin probability $\psi(u)$ in the logarithmic scale and (b) the relative error of the approximations. Solid line represents exact values of the ruin probability. Dashed and dotted lines correspond to De Vylder and 4-moment gamma approximations, respectively. The mixture of two exponentials case with $\beta_1 = 0.04$, $\beta_2 = 2$, weight $a = 0.002$, $\theta = 0.1$ and $u \leq 1000$.

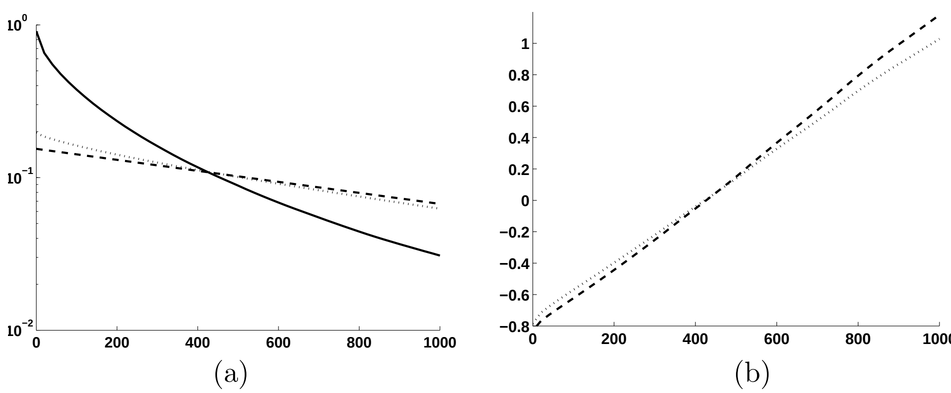

Fig. 2. Illustration of (a) the ruin probability $\psi(u)$ in the logarithmic scale and (b) the relative error of the approximations. Solid line represents ruin probability values obtained via the Pollaczek–Khinchin formula. Dashed and dotted lines correspond to De Vylder and 4-moment gamma approximations, respectively. The lognormal case with $\mu = -3$, $\sigma = 2.1$, $\theta = 0.1$ and $u \leq 1000$.

When the claim amount distribution is lognormal, the formula for the ruin probability does not have a closed form, therefore we employ the Pollaczek–Khinchin formula to obtain exact results. For the Monte Carlo method purposes we generated 100 blocks of 100000 simulations. The computations were realized in the Matlab package. We also note that the variance within the results derived from the blocks was always below 3×10^{-6}. Fig. 2(a) illustrates the exact ruin probability values and results of the De Vylder and 4-moment gamma approximations. Fig. 2(b) shows that the relative error of the 4-moment gamma approximation is always significantly less than the error of the original one.

Finally, let us note that we have conducted similar studies for other light- and heavy-tailed claim size distributions, *e.g.* Weibull, Pareto, Burr and loggamma, with different parameters. With the usage of XploRe package they are presented in Ref. [16] and justify the thesis the 4-moment gamma approximation works better than the De Vylder approximation.

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