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BOOTSTRAP PERCOLATION IN CELLULAR AUTOMATA ON SMALL-WORLD DIRECTED NETWORK*

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Dedicated to Professor Andrzej Fuliński on the occasion of his 70th birthday

Effects of network topology are studied in a system of cellular automata driven by a totalistic rule. In particular, propagation of a signal is considered in the directed network obtained from a flat (square) lattice by adding directed connections. The model is motivated by features found in human neural system. Cooperation between local dynamics and network organization results in fast stabilization of the system. Simple model of neural pyramidal cell is proposed to stabilize the automata in the oscillating firing patterns form.

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1. Introduction

The motivation for cellular automata concept was biology [1]. During last 50 years the idea of the many element system which is discrete in time, space and states, and which evolves accordingly to the locally given rule has been successfully applied in many fields of scientific interest and not only of biological origin, [2]. Many aspects of self-organizations can be explained with the help of cellular automata. Therefore, one can say that the easiest way for understanding complex system is through cellular automata [3]. Unfortunately, the validity of this approach is usually qualitatively. It seems that something is missed in the concept of the cellular automata system organization.

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Coupled systems can be modeled as networks or graphs, where the vertices represent the elements of a system, and the edges represent interactions between them. The network topology may be random (each node is randomly wired to any other node) or regular (each vertex is connected in a fixed pattern to identical number of its neighboring nodes). Watts and Strogatz [4] showed that between these two extremes there is a regime, called small-world network, which presents properties of both networks. Small-world network can be highly clustered like regular networks and, at the same time, have small path lengths like random ones.

The topology of the network influences the dynamics. According to the universality hypothesis the details of the network are qualified as mean field approximation or other. In this sense the critical properties such as, *e.g.*, critical exponents, are sensitive to the details of topology. However, from examinations of human and technological systems and their organizations the importance of network effect has been emerged [5]. The lesson is that not only design of interactions matter greatly in reconstructing nature phenomena but the structure of connections: asymmetry and heterogeneity, matter significantly.

Many basic features of the network topology can be read from the vertex degree distribution, [5]. A vertex degree is the number corresponding to number of vertices to which the vertex is directly connected. The regular lattice has a delta distribution, the stochastic graph provides the Poisson sharply peaked distribution. The peak value corresponds to the average vertex degree and because of its high probability to occur this vertex degree is typical for a network. Small-world network gives also the exponential dependence on vertex degree. Networks, where the distribution of vertex degree is of power-law type are called a scale-free networks [6].

An important problem in neuroscience is to understand how the structure and functions at one level of organization is manifest in the structure and function at higher levels of organization [7–9]. For example, at the molecular level, the voltage and current across the cell membrane is controlled by ion channel proteins. At a higher level of organization, information of relevance is represented in the rate or timing of action potential. Experimental evidence suggests that synchronous activity of large assembles of neurons provide the basis of the remarkable computational performance of the brain [10]. Motivated by this, contrary to the presently used digital computer memories where information is encoded in the form of a given string of binary digits, there is an approach in which the encoding is embodied in oscillations of the activity of the memory nodes.

Information processing in the proposed dynamical memory is closely related to the percolation phenomena [11]. Much work has been done on the family of cellular automata known as bootstrap percolation [12]. The bootstrap percolation denotes the Z^d lattice where each site can be active or inactive and where initialy ($t = 0$) sites are active with probability ρ, and each time step a site becomes active if at least its l neighbors are active. The question is to find the critical probability ρ_{crit} above which all sites become active for large t and given l. In case of a square infinite lattice and $l = 2$ then $\rho_{\text{crit}} = \pi^2/18$ is known exactly [13]. For finite lattice systems ρ_{crit} is smaller about two times than the thermodynamic value.

Cellular automata of interacting locally elements with topology different from a regular lattice or a random network could be a conception for the absent ideas in the cellular automata system organization. This conception should at least provide hints how the underlying geometry cooperates with a dynamical rule in fixing properties of the stable system. The mentioned earlier achievements in neuroscience as well as discoveries about the organization in different natural networks [14] motivate us for such reconsidering basic features of cellular automata.

In the following we basically ask how allowing for long-range connections (hence, non-local interactions) influence the bootstrap percolation. We present computer simulation results obtained for cellular automata with topology modified by the presences of additional directed edges. The considered changes are made following the Barabasi and Albert concept [6] of preferential attachment. Thus, the network grows. However, the growth is limited to the number of edges, since the number of nodes is unchanged. Motivation for such a net construction comes from the fact that during the life time, the number of neurons in the human neuron system does not increase (precisely, the total number of cell diminishes with age) but the number of synapses (the communication links) on each neuron does [8]. These new edges introduced are directed to better imitate an axon to synapses connection. Instead of conclusion we propose a model of a neural network where the synchronous state means a precisely timed periodic firing pattern.

2. Cellular automata on directed network

2.1. Directed network construction

All basic connections of two dimensional square lattice are conserved. The regular links are considered as two directed edges. New directed edges are added with intensity governed by a parameter p, see Fig. 1. The value of the parameter p is chosen in such a way that it describes the average vertex degree after the network modification. By this procedure we would imitate all types of folding, bending, twisting the flat surface of neurons.

Depending on how the new edges are added: at random or intentionally (with linear preference to link to highly connected nodes), a different network is constructed, see Fig. 2 for the vertex degree distributions. The estimates

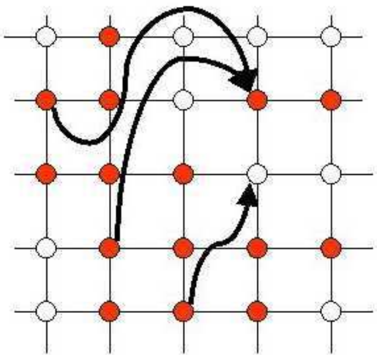

Fig. 1. New directed edges are added to a square lattice connections. Each directed edges increases by 1 the degree of the pointed vertex.

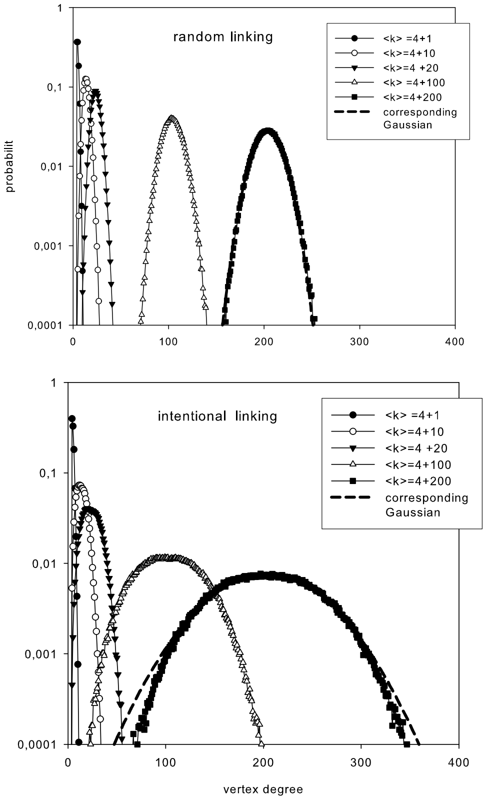

Fig. 2. Distribution of vertex degree in case of random adding (top panel) or with preference to link to highly connected nodes (bottom panel). $\langle k \rangle$ describes the average vertex *in*-degree. The dashed lines plot approximations by the Gaussian distributions.

of distributions in both cases (random and intentional edge adding) lead to the distributions which can be approximated by the Gaussian distribution centered at $\langle k\rangle = 4+p$ (see dashed lines in Fig. 2), $\langle k\rangle$ denotes the average vertex degree. The spread of the distributions, measured by standard deviations, is different in both cases. The difference grows with the value of parameter p. In result when p is large enough, namely $p > 100$, then the degree of a typical vertex means the wide interval in case of intentional linking. Therefore, we can expect that effects of heterogeneity of the network could appear.

2.2. Totalistic rule on directed network

The automata cell assigned to each vertex is considered as binary, *i.e.*, a cell can be *active* or *inactive* what denotes $\sigma_i \in \{1,0\}$, suitable. The state of automata cell is changed according to the following totalistic rule:

$$\sigma_i(t+1) = \begin{cases} 1 & \text{if } \sum\limits_{j\in N_i(t)} \sigma_j(t) > l \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where N_i denotes the set of in-neighbors of i-th vertex, *i.e.*, neighbors connected by the edge directed to the i-th vertex. Such a dynamics is often used to mimic spreading diseases and then the l parameter describes the number of sick neighbors which results that a central node becomes sick [15]. Notice, that a sick node becomes healthy only if it has a sufficient number of neighbors which are healthy. If one assumes that a sick node cannot recover from its disease, so the node which gets disease is sick for ever, the problem is known as the bootstrap percolation. On a square lattice if initial density of active cells ρ is low then the system quickly converges to a steady state of rectangular islands of active cells surrounded by a sea of inactive cells. However, as ρ crosses some threshold then every cell becomes active.

2.3. Results

In computer simulation experiments we test what is the effect of growing number of connections and especially we are interested in what kind of the stable state is reached if we start from a random picture or from a picture which is invariant with respect to the (1) dynamics set on a square lattice. So that we are able we observe properties of final configurations when the evolution starts at either a random state with a given ρ or with a square filled with cells in active state. The size of the square corresponds to ρ.

The experiment goes as follows:

- For a given initial density of 1's ρ, $\rho \in [0,1]$, the initial state of automata cells is prepared either at random or as the square of 1's that corresponds to ρ.

- For a given number of p, $p = 1, 2, \ldots$, the directed long-ranged connections are established either randomly choosing a destination or intentionally choosing a destination.

- For a given l, $l = 0, 1, 2, \ldots$ the totalistic rule (1) is applied synchronously many times to fix the stable state.

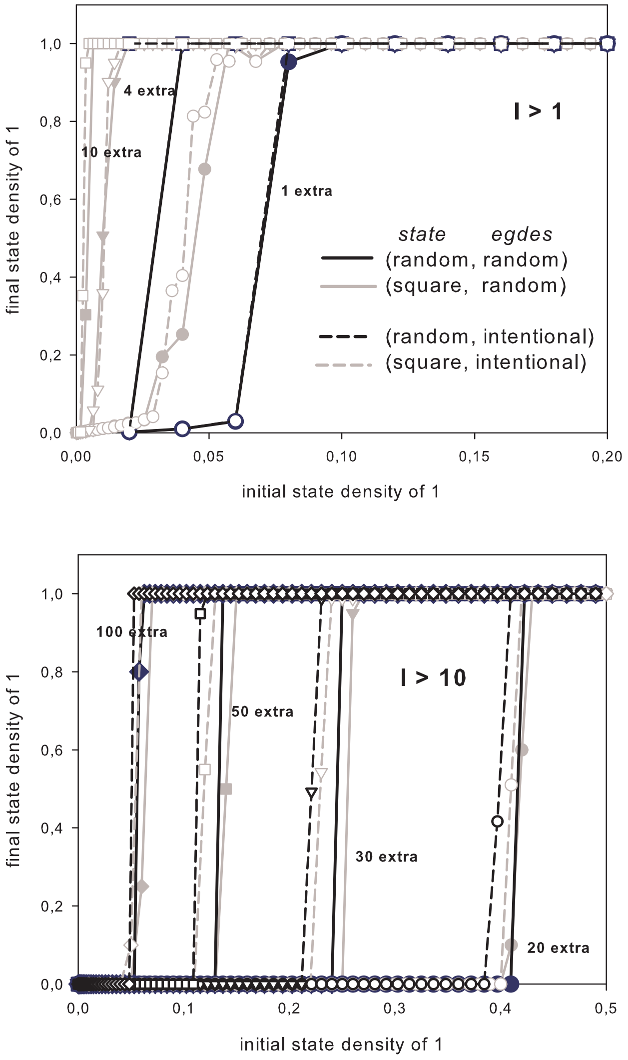

Fig. 3. Bootstrap percolation on cellular automata with directed edges added randomly or preferentially and if the initial state is at random or a rectangle of 1's. All four possible combinations are listed in the top panel.

Fig. 3 presents results for $l = 1$ and $l = 10$. If each vertex has the one extra edge at average then the stabilization with other than the homogeneous configuration of all cells 1's or all cell 0's appears only if $\rho \in (0.04, 0.10)$ in case of random initial state. But when the initial state is fixed by a rectangle then inhomogeneous final state happens only if $\rho \in (0.02, 0.07)$. This difference denotes that 1's arranges in the square shape work more efficiently on transferring state to *all cells active*. Both the size of uncertainty interval in ρ and the distance between uncertainty intervals when different initial states happens, decrease with the increase of the number of extra edges. This is a consequence of loosing geometry of a square lattice. When $p = 10$ then systems stops to feel the fixed initial configuration. When l is high enough, *e.g.*, $l > 10$, see bottom figure of Fig. 3, then the system is a two-state system. The uncertainty intervals squeeze to 0.01 what means here the precision of the performed simulations. However, the discrepancy between dashed lines (which represent intentionally prepared network) and solid lines (which represent randomly prepared network) is persistent. It seems that the heterogeneity of the network that is modified intentionally decreases the threshold for bootstrap percolation.

3. Modeling firing patterns

The hippocampus is perhaps the most studied structure in the brain [9]. The hippocampus is formed by two interlocking sheets of cortex and in cross-section has a very defined laminar structure with layers visible where rows of pyramidal cells are arranged. The connections within the hippocampus generally follow this laminar format and, as a rule, are uni-directional.

Let us take a look at a pyramidal cell from the hippocampus, see Fig.4, to better understand the model [16]. It has a pyramid-shaped soma and two large dendritic trees, the apical dendrites at the top, and the smaller basal dendrites at the bottom. There is also a long thin axon. Obviously, all this extended branching can allow for a lot of information processing of its many inputs, by making use of both temporal and spatial summation. A cortical pyramidal cell has approximately 20–30000 synapses on it, and makes as many synapses on other neurons.

Pyramidal cells in the hippocampus show a range of interesting firing patterns, [16, 17]:

— with no external input, a cell fires bursts of action potentials every few seconds;

— with increasing amounts of stimulation, the frequency of bursts increases;

— with even larger amounts of input, the cell switches over to a pattern of regular firing instead of bursting.

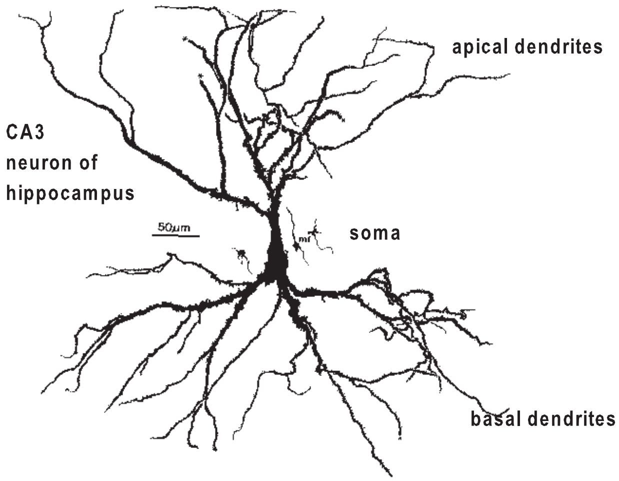

Fig. 4. Neuron parts and functions: *dendrites* (apical and basal) — receive information from other neurons in the form of electrical signals; *soma* (cell body) — processes information by adding together all the signals; *axon* — sends output to other neurons; *Action Potential* — electric impulse traveling from soma to axon terminals; *synapse* — transmits information to a next neuron. Axon terminals release chemicals (neurotransmitters) onto dendrites of another neuron that either excites or inhibits that an neuron. In the picture, a typical cortical pyramidal cell, from [16]. (This particular cell is of rat olfactory cortex.)

For example, during an epileptic seizure, the bursts in different cells become synchronized over large portions of the hippocampus. So, there are lots of good reasons for wanting to understand more about the behavior of these cells.

Basing on the above description we propose the following model.

(i) Let a neural cell state be described by the product $\sigma_i \times \Sigma_i$ where

- $\sigma_i \in \{1, 0\}$ corresponds to the axon state of the i-th neuron: *fire* — Action Potential is produced and *quiet* none Action Potential is produced;
- $\Sigma_i \in \{Active, Resting, OverReactive\}$ corresponds to a state of the soma of the i-th neuron. The soma states are governed by timing: *activity interval* — time in which each time step the axon alternatively changes its state, and *refractory interval* — number of time steps in which the axon is in 0 state independently of any external signal.

(ii) Let the isolated from external signals neuron periodically switch between the *Active* state of its soma with a burst of Action Potentials in the axon (of length of the activity interval) to the *Resting* state of soma in which no axon activity (of length of the refractory interval) is present.

(iii) Let the stimulus from external axons change the length of the refractory interval:

$$\mathrm{refractory\_interval}(t+1) = \mathrm{refractory\_interval}(t) - \left(\sum_{j\in N_i} \sigma_j(t)\right)^2 .$$

Thus, the stimulus from other axon decreases the refractory time. If the refractory time becomes zero then the soma of a neuron changes its state to *OverReactive*, what causes that its axon is in the permanent fire state for the whole time of the activity interval. Then the soma switches to the *Resting* state.

The parameters used by us in simulations are as follows: number of additional edges $p = 500$, the activity interval is set to 20, the refractory interval is set to 100. The initial state is prepared at random in all state space of axon and soma. It appears that the cellular automata system leads to the stabilization with precisely timed periodic firing patterns with the period equal to 22 independently of p and the refractory interval. The length of the period depends on the activity interval while the refractory interval determines the time to reach the periodic solution.

Our findings are only preliminary, and are included here to present new challenges which open to cellular automata concept if one admits manipulating with geometry of the underlying network.

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