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STATIONARY DISTRIBUTION DENSITIES OF ACTIVE BROWNIAN PARTICLES*

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Dedicated to Professor Andrzej Fuliński on the occasion of his 70th birthday

We study the motion of active Brownian particles in 2d-external potentials. We give the stationary probability distribution in the four-dimensional phase space in several representations and show that it is maximized above the deterministic integrals of motion.

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1. Introduction

In this paper we study Brownian particles including an energy input from the surrounding modeled as negative friction [1–4]. This self moving objects were called *Active Brownian particles* [5,6] and stand for a simplified model of active biological motion [7–11]. It goes beyond Brownian or easy diffusional motion due to the nonlinear friction and due to being in nonequilibrium. We study them in symmetric external potentials. In particular, we will look up the motion in parabolic and Coulomb-like potentials.

Despite a wide diversity of possible applications with multifaceted features [12] our aim will be the investigation of the stationary probability density. From the view point of dynamical systems active Brownian motion on a plane represent a nonlinear oscillator with two degrees of freedoms and moving in the four dimensional phase space. Hence, the introduction of amplitude and phase variables appears to be favorable as well as the consideration of integrals of motion like energy and angular momentum. As will be seen attractors of these integrals will be provided with maximal probability in the noise driven case if in addition to the nonlinear dissipative and external forces random excitations act on the particles.

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The plan of the paper is in brief as follows: At first we will develop the model of active Brownian particles and reconsider the deterministic noise free case. Later on, we include the noisy forces and formulate Langevin equations in the Stratonovich calculus [13]. From the corresponding Fokker–Planck equations we find expressions for the stationary distributions in several limits. In contrast to former studies where identical expressions were presented our approach delivers the first systematic derivation of the stationary distribution densities.

2. Equations of motions

The motion of Brownian particles with general velocity- and space-dependent friction in a space-dependent potential $U(\vec{r})$ can be described by the Langevin equation:

$$\frac{d\vec{r}}{dt} = \vec{v}\,, \qquad m\frac{d\vec{v}}{dt} = \vec{F}_{\rm diss} - \nabla U(\vec{r}) + \vec{\mathcal{F}}(t)\,. \tag{1}$$

We assume that $\vec{r} = \{x_1, x_2\}$ and $\vec{v} = \{v_1, v_2\}$ are two-dimensional vectors, respectively, for the position and the velocity of the considered particle. $\vec{F}_{\rm diss}$ is a dissipative force which is in the simplest case given by a friction law

$$\vec{F}_{\rm diss} = -m\gamma(\vec{r},\vec{v})\,\vec{v}\,. \tag{2}$$

Therein $\gamma(\vec{r},\vec{v})$ is the friction function of the particle with mass m being at position $\vec{r}$ and moving with velocity $\vec{v}$. The friction $\gamma(\vec{r},\vec{v})$ may depend on space and velocity. $\vec{\mathcal{F}}(t)$ is a Gaussian stochastic force with strength D_p, independent components and a δ-correlated time dependence

$$\left\langle \vec{\mathcal{F}}(t) \right\rangle = 0\,; \quad \left\langle \mathcal{F}_i(t)\mathcal{F}_j(t')\right\rangle = 2D_p\,\delta_{i,j}\,\delta(t-t')\,. \tag{3}$$

In thermal equilibrium systems and in case of Stokes friction $\gamma(\vec{r},\vec{v}) = \gamma_0 =$ const. the noise strength of the momentum D_p is connected with the friction coefficient γ_0 due to the fluctuation-dissipation theorem: $D_p = mk_{\rm B}T\gamma_0$ where T is the temperature and $k_{\rm B}$ is the Boltzmann constant.

We consider velocity-dependent friction as a mechanism accelerating the Brownian motion. Velocity-dependent friction plays an important role *e.g.* in certain models of the theory of sound [14, 15]. In the simplest case we may assume the following friction force of the individual Brownian particle:

$$\gamma(\vec{r},\vec{v}) = -\gamma_1 + \gamma_2\vec{v}^{\,2} = \gamma_1\left(\frac{\vec{v}^{\,2}}{v_0^2} - 1\right) = \gamma_2(\vec{v}^{\,2} - v_0^2)\,. \tag{4}$$

This Rayleigh–Helmholtz model is a standard model studied in many papers on Brownian dynamics [1, 12, 16, 17]. We note that $v_0^2 = \gamma_1/\gamma_2$ defines a special value of the velocities where the friction is zero.

Due to the nonlinear pumping slow particles are accelerated and fast particles are damped. At definite conditions our active friction functions have a zero corresponding to stationary velocities v_0, where the friction function and the friction force disappear. Consideration of energy balance with $H = m\vec{v}^2/2 + U(\vec{r})$ results in ($\vec{\mathcal{F}}(t) = 0$)

$$\frac{d}{dt} H(t) = -m\,\gamma(\,\vec{v}\,)\,\vec{v}^{\,2}\,, \tag{5}$$

hence for small velocities energy is supplied to the particle whereas for large velocities if $\gamma(\vec{v})$ is positive $H(t)$ decreases. In both cases, the deterministic trajectory of our system moving on a plane is attracted by a cylinder in the four-dimensional phase space given by

$$v_1^2 + v_2^2 = v_0^2\,. \tag{6}$$

We are interested mainly in the statistical descriptions, *i.e.* in the probability $P(\vec{r}, \vec{v}, t)$ to find the particle at location $\vec{r}$ with velocity $\vec{v}$ at time t. As it is well known, this distribution function $P(\vec{r}, \vec{v}, t)$, which corresponds to the Langevin equations (1), obeys a Fokker–Planck equation:

$$\frac{\partial P}{\partial t} + \vec{v}\,\frac{\partial P}{\partial \vec{r}} - \frac{1}{m}\,\nabla U(\vec{r})\,\frac{\partial P}{\partial \vec{v}} = \frac{\partial}{\partial \vec{v}}\left[\gamma(\vec{r}, \vec{v})\,\vec{v}\,P + D_v\,\frac{\partial P}{\partial \vec{v}}\right], \tag{7}$$

where we have introduced $D_v = \frac{D_p}{m^2}$.

3. Deterministic motion in external potentials with rotational symmetry

In the following, we specify the potential $U(\vec{r})$ as a symmetric parabolic potential:

$$U(x_1, x_2) = \frac{1}{2} m\omega_0^2\,(x_1^2 + x_2^2)\,. \tag{8}$$

First, we restrict the discussion to a deterministic motion, which then is described by four coupled first-order differential equations:

$$\begin{aligned} \dot{x}_1 &= v_1\,, \qquad & \dot{v}_1 &= -\gamma\,(v_1, v_2)\,v_1 - \omega_0^2 x_1\,, \\ \dot{x}_2 &= v_2, \qquad & \dot{v}_2 &= -\gamma\,(v_1, v_2)\,v_2 - \omega_0^2 x_2\,. \end{aligned} \tag{9}$$

For the one-dimensional Rayleigh model it is well known that this system processes a limit cycle corresponding to sustained oscillations with the energy $H_0 = m\frac{\gamma_1}{\gamma_2}$. For the two-dimensional case we can show by simulation and theoretical considerations that two limit cycles in the four-dimensional phase space are developed [16]. The projections of both these periodic motions to the $\{v_1, v_2\}$ plane is the circle

$$v_1^2 + v_2^2 = v_0^2 = \text{const.} \tag{10}$$

The projection to the $\{x_1, x_2\}$ plane also corresponds to a circle

$$x_1^2 + x_2^2 = r_0^2 = \text{const.} \tag{11}$$

Due to the condition of equilibrium between centripetal and centrifugal forces on the limit cycle we have

$$\frac{m v_0^2}{r_0} = m r_0 \omega_0^2 \,. \tag{12}$$

Therefore the radius of the limit cycle is given by

$$r_0 = \frac{v_0}{\omega_0} \,. \tag{13}$$

From equation (12) follows

$$\frac{m}{2} v_0^2 = \frac{m \omega_0^2}{2} r_0^2 \,. \tag{14}$$

This means we have equal distribution of potential and kinetic energy on the limit cycle [5]. As for the harmonic oscillator in one-dimensional case, both parts of energy contribute the same amount to the full energy. Therefore the energy of motions on the limit cycle, which is asymptotically reached, is double the kinetic energy

$$H \longrightarrow H_0 = m v_0^2 \,. \tag{15}$$

The energy is a slow (adiabatic) variable which allows a phase average with respect to the phases of the rotation [16].

The explicite form we discuss on behalf of polar coordinates in the four-dimensional phase space. Introducing $v(t)$ and $\varphi(t)$ according to equations

$$v_1 = v(t)\, \cos(\varphi(t)) \,, \quad v_2 = v(t)\, \sin(\varphi(t)) \tag{16}$$

and in the coordinate space $r(t)$ and $\psi(t)$

$$x_1 = r(t) \cos(\psi(t)) \,, \quad x_2 = r(t) \sin(\psi(t)) \tag{17}$$

one obtains the dynamics

$$
\begin{aligned}
\dot{r} &= v\cos\theta\,, \\
\dot{v} &= \left(\gamma_1 - \gamma_2 v^2\right) v - \omega_0^2 r\cos\theta\,, \\
\dot{\theta} &= \left(\frac{\omega_0^2 r}{v} - \frac{v}{r}\right)\sin\theta\,, \\
\dot{\psi} &= \frac{v}{r}\,\sin\theta
\end{aligned} \tag{18}
$$

with $\theta(t) = \varphi - \psi$.

The stationary solutions can be readily found. The difference of the two angles θ approaches two values, $\theta = \pm\pi/2$. These two solutions resemble the two limit cycles with $v_0 = \gamma_1/\gamma_2$, $r_0 = v_0\omega_0$ and two stationary rotations (clockwise and counter clockwise) with stationary angular velocity $\dot{\psi} = \dot{\varphi} = \pm\omega_0$

Representing one cycle in the four-dimensional phase space reads with arbitrary initial phase Φ:

$$
\begin{aligned}
x_1 &= r_0\cos(\omega_0 t + \Phi)\,, \quad v_1 = -r_0\omega\sin(\omega_0 t + \Phi)\,, \\
x_2 &= r_0\sin(\omega_0 t + \Phi)\,, \quad v_2 = r_0\omega\cos(\omega_0 t + \Phi)\,.
\end{aligned} \tag{19}
$$

This means, the particle rotates even at strong pumping with the frequency given by the linear oscillator frequency ω_0. The trajectory defined by the above four equations looks like a hoop in the four-dimensional phase space. Most projections to the two-dimensional subspaces are circles or ellipses however there are to subspaces namely $\{x_1, v_2\}$ and $\{x_2, v_1\}$ where the projection is like a rod.

The second limit cycle is obtained by use of different initial conditions and replacing $\omega_0 \to -\omega_0$ which yields

$$
\begin{aligned}
x_1 &= r_0\cos(\omega_0 t - \Phi)\,, \qquad v_1 = -r_0\omega\sin(\omega_0 t - \Phi)\,, \\
x_2 &= -r_0\sin(\omega_0 t - \Phi)\,, \quad v_2 = -r_0\omega\cos(\omega_0 t - \Phi)\,.
\end{aligned} \tag{20}
$$

This second cycle forms also a hula hoop which is different from the first one, however both limit cycles have the same projections to the $\{x_1, x_2\}$ and to the $\{v_1, v_2\}$ plane. The projection to the $\{x_1, x_2\}$ plane has the opposite direction of rotation in comparison with the first limit cycle. The projections of the two hula hoops on the $\{x_1, x_2\}$ plane or on the $\{v_1, v_2\}$ plane are two-dimensional rings (figure 1). The hula hoop distributions intersect perpendicular the $\{x_1, v_2\}$ plane and the $\{x_2, v_1\}$ plane (see figure 1). The projections to these planes are rod-like and the intersection manifold with these planes consists of two ellipses located in the diagonals of the planes (see figure 1).

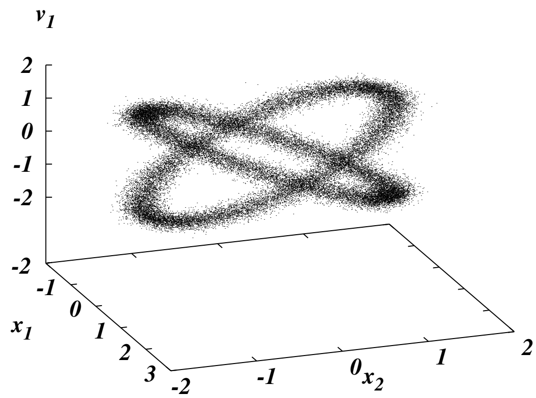


Fig. 1. Stroboscopic plot of the 2 limit cycles for driven Brownian motion. We show projections of solutions for $v_0 = 1$ to the subspace $\{x_1, x_2, v_1\}$. Parameters: $\gamma_1 = 2$, $D_v = 0.01$ and $\omega_0 = 1$.

In order to construct later solutions for stochastic motions we need beside $H = mv_0^2$ other appropriate invariants of motion. Looking at the first solution (19) we see, that the following relation is valid

$$v_1 + \omega_0 x_2 = 0\,; \qquad v_2 - \omega_0 x_1 = 0\,. \tag{21}$$

In order to characterize the first limit cycle we introduce the invariant

$$J_+ = H - \omega_0 L = \frac{m}{2}(v_1 + \omega_0 x_2)^2 + \frac{m}{2}(v_2 - \omega_0 x_1)^2\,, \tag{22}$$

where we have introduced the angular momentum $L = m(x_1 v_2 - x_2 v_1)$. We see immediately that $J_+ = 0$ holds on the first limit cycle which corresponds to positive angular momentum. In order to characterize the second limit cycle from equation (20) we use the invariant

$$J_- = H + \omega_0 L = \frac{m}{2}(v_1 - \omega_0 x_2)^2 + \frac{m}{2}(v_2 + \omega_0 x_1)^2. \tag{23}$$

We see that on the second limit cycle, which corresponds to negative angular momentum, holds $J_- = 0$.

4. Dynamics in inharmonic potentials

In the present section we will discuss briefly several extensions of the theory developed in the previous section. At first we will discuss the case of inharmonic potentials. For the general case of radially symmetric but inharmonic potentials $U(r)$ the equal distribution between potential and kinetic energy $mv_0^2 = m\omega_0^2 r_0^2$ which leads to $\omega_0 = v_0/r_0 = \omega$ is no more valid. It has to be replaced by the more general condition that on the limit cycle the attracting radial forces are in equilibrium with the centrifugal forces. This condition leads to

$$\frac{mv_0^2}{r_0} = |U'(r_0)| \,. \tag{24}$$

If v_0 is given, the equilibrium radius may be found from the implicit relation

$$v_0^2 = \frac{r_0}{m}|U'(r_0)| \,. \tag{25}$$

Then the frequency of the limit cycle oscillations is given by

$$\omega_0^2 = \frac{v_0^2}{r_0^2} = \frac{|U'(r_0)|}{mr_0} \,. \tag{26}$$

For the case of quartic oscillators

$$U(r) = \frac{k}{4}r^4 \tag{27}$$

we get the limit cycle frequency

$$\omega_0 = \frac{k^{1/4}}{v_0^{1/2}} \,. \tag{28}$$

Alternatively for attracting Coulomb forces (two charges on a plane)

$$U(r) \;=\; -\frac{Ze^2}{r} \tag{29}$$

we find the stable radius

$$r_0 \;=\; \frac{Ze^2}{mv_0^2} \,, \tag{30}$$

and the limit cycle frequency

$$\omega_0 = \frac{mv_0^3}{Ze^2} \tag{31}$$

and

$$H_0 = -\frac{1}{2}\,mv_0^2\,; \qquad L_0 = \pm\frac{Ze^2}{v_0} \,. \tag{32}$$

We note that this expression diverges for $v_0 \to 0$ (similarly as in quantum theory the Bohr radius diverges for $h \to 0$).

If the equation (25) has several solutions, the dynamics might be much more complicated, *e.g.* we could find Kepler-like orbits oscillating between the solutions for r_0. In other words we may find then beside driven rotations also driven oscillations between the multiple solutions of equation (25).

An interesting application of the theoretical results given above, is the following: Let us imagine a system of Brownian particles which are pairwise bound by a Lennard–Jones-like potential $U(r_1 - r_2)$ to dumb-bell-like configurations. Then the motion consists of two independent parts: The free motion of the center of mass, and the relative motion under the influence of the potential. The motion of the center of mass is described by the equations given in the previous section and relative motion is described by the equations given in this section. As a consequence, the center of mass of the dumb-bell will make a driven Brownian motion but in addition the dumb-bells are driven to rotate around there center of mass. What we observe then is a system of pumped Brownian molecules which show driven translations with respect to their center of mass. On the other side the internal degrees of freedom are also excited and we observe driven rotations and in general (if equation (25) has several solutions) also driven oscillations. In this way we have shown that the mechanisms described here may be used also to excite the internal degrees of freedom of Brownian molecules.

5. Stochastic motion in symmetric external potentials

Since the main effect of noise is the spreading of the deterministic attractors we may expect that the two hoop-like limit cycles are converted into a distribution looking like two embracing hoops with finite size, which for strong noise converts into two embracing tires in the four-dimensional phase space. In order to get the explicite form of the distribution we may introduce different variables, like the amplitude and phase description as used in the previous sections. Here we introduce the energy and angular momentum as variables and derive reduced densities. We point out that throughoutly the Stratonovich calculus is used [13].

On the basis of the amplitude and phase representation (16) and (17) we get for the Hamiltonian

$$H = \frac{m}{2} v(t)^2 + \frac{m}{2} \omega_0^2 r(t)^2 \,. \tag{33}$$

The angular momentum is given as

$$L = m(x_1 v_2 - x_2 v_1) = m v(t) r(t) \cos(\theta) \,. \tag{34}$$

Values corresponding to the two limit cycles are

$$L = +L_0\,; \qquad L = -L_0\,; \qquad L_0 = \frac{mv_0^2}{\omega_0} \tag{35}$$

with $v_0^2 = \gamma_1/\gamma_2$. Both limit cycles are located on the sphere with $H = mv_0^2$.

Considering harmonic oscillators and using equipartition of potential and kinetic energy (see equation (15)) we find for motions on the limit cycle $v^2 = H/m$. Assuming that $v^2 \simeq H/m$ holds also near to the limit cycle, the dynamic system is converted to a canonical dissipative system with

$$\gamma(v^2) \simeq \gamma\left(\frac{H}{m}\right) = \gamma_H(H)\,. \tag{36}$$

Outgoing from the equations (9) we come for the Rayleigh-model to the energy balance

$$\frac{d}{dt}H(t) = -\gamma_H(H)\,H + \sqrt{2D_H H}\xi_H(t)\,, \tag{37}$$

where $\xi_H(t) = \xi_1\cos(\varphi) + \xi_2\sin(\varphi)$ is again Gaussian white noise and $D_H = D_v m$. This corresponds to the Fokker–Planck equation in energy representation

$$\frac{\partial}{\partial t}P(H,t) = \frac{\partial}{\partial H}\left[(\gamma_H(H)\,H - D_H)\,P + D_H\,\frac{\partial}{\partial H}\,H\,P\right] \tag{38}$$

which stationary solution reads

$$P_0(H) = \mathcal{N}\exp\left[-\frac{1}{D_H}\int \gamma_H(H)dH\right]. \tag{39}$$

The most probable value of the energy is the energy on the limit cycle. In case of the Rayleigh model it is

$$\tilde{H} = H_0 = \frac{\gamma_1}{\gamma_2} = mv_0^2\,. \tag{40}$$

The stationary distribution can be shaped in compact shape ($H \geq 0$)

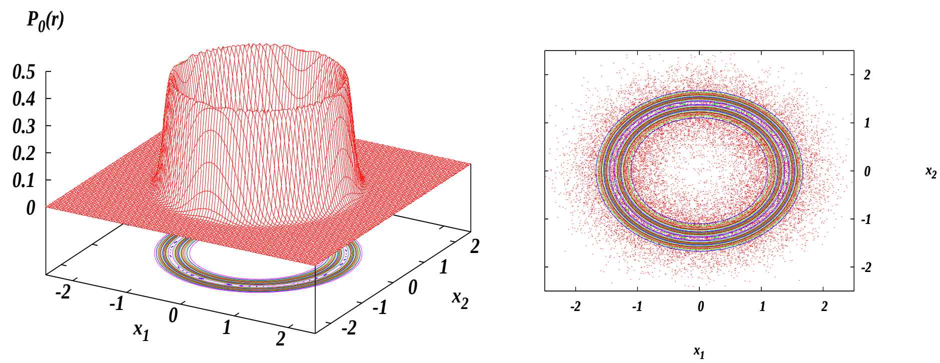
$$P_0(H) = \mathcal{N}\exp\left[-\frac{\gamma_2}{2m^2 D_v}\,(H\, - H_0)^2\right] \tag{41}$$

which is a Gaussian at positive energies.

This probability is in fact distributed on the surface of the four-dimensional sphere. By using equation (33) we get for the Rayleigh model of pumping in the approximation of equipartition of energy the following distribution of the coordinate with $r^2 = x_1^2 + x_2^2$

$$P_0(x_1, x_2) \simeq \exp\left[\frac{\gamma_1 \omega_0^2}{D_v} r^2 \left(1 - \frac{r^2}{2r_0^2}\right)\right]. \tag{42}$$

We see in figure 2 that the probability crater is located above the trajectory obtained from simulations of an Active Brownian particle. This way the maximal probability corresponds indeed to the deterministic limit cycle.



(a) (b)

Fig. 2. Probability density for the Rayleigh-model represented over the $\{x_1, x_2\}$ plane. (a) The probability density (42). (b) Contour plot of $P_0(r)$ superimposed with data points out of simulations of the Active Brownian dynamics. Parameters: $\gamma_1 = 2$, $D_v = 0.1$ and $\omega_0 = 1$

So far we represented only a projection on the $\{x_1, x_2\}$ plane. The full probability distribution in the four-dimensional phase space is not constant on the four-dimensional sphere $H = mv_0^2$ as suggested by equation (39) but should be concentrated around the limit cycles which are closed curves on the four-dimensional sphere $H = mv_0^2$. This means, only a subspace of this sphere is filled with probability. The correct stationary probability has the form of two noisy distributions in the four-dimensional phase space, which look like hula hoops. This characteristic form of the distributions was confirmed also by simulations (see figure 2 and [3, 16]). The projections of the distribution to the $\{x_1, x_2\}$ plane and to the $\{v_1, v_2\}$ plane are noisy tori in the four-dimensional phase space. The hula hoop distribution intersects perpendicular the $\{x_1, v_2\}$ plane and the $\{x_2, v_1\}$ plane. The projections to these planes are rod-like and the intersection manifold with these planes consists of two ellipses located in the diagonals of the planes.

In order to refine the description we find the distribution of the angular momenta. We derive from the Langevin equations (9)

$$\frac{dL}{dt} = -\gamma\left(v^2\right) L + \sqrt{2D_v} r\, m\, \xi_{\mathrm{L}}(t) \tag{43}$$

with $\xi_{\mathrm{L}} = \xi_y \cos(\varphi) - \xi_x \sin(\varphi)$ being Gaussian white noise. On the limit cycles it holds

$$L(t) = \pm mr(t)v(t), \quad v(t) = \omega_0\, r(t), \tag{44}$$

respectively, the different signs for the different cycles and $r(t) = r_0$ and $v(t) = v_0$. To find a closed description we assume that the equations (44) hold and replace

$$r = \sqrt{\frac{L}{m\omega_0}}, \quad v^2 = L\,\frac{\omega_0}{m}, \quad \gamma(v^2) = \gamma\left(L\frac{\omega_0}{m}\right) = \gamma_{\mathrm{L}}(L)\,, \tag{45}$$

where we have used the positive sign and hence $L > 0$. It follows

$$\frac{dL}{dt} = -\gamma_{\mathrm{L}}(L)\, L + \sqrt{2D_{\mathrm{L}} L}\, \xi_{\mathrm{L}}(t) \tag{46}$$

with

$$D_{\mathrm{L}} = \frac{D_v m}{\omega_0}\,. \tag{47}$$

The corresponding Fokker–Planck equation is similar to the energy representation

$$\frac{\partial}{\partial t} P(L,t) \;=\; \frac{\partial}{\partial L}\left[(\gamma_{\mathrm{L}}(L)\, L \;-\; D_{\mathrm{L}})\, P + D_{\mathrm{L}}\, \frac{\partial}{\partial L}\, L\, P\right]. \tag{48}$$

Obviously its stationary solution reads

$$P_0(L) = \mathcal{N} \exp\left[-\frac{1}{D_{\mathrm{L}}} \int \gamma_{\mathrm{L}}(L) dL\right] \tag{49}$$

and eventually after introducing the most probable angular momentum $L_0 = mr_0v_0 = H_0/\omega_0$ at the limit cycle the stationary solution becomes ($L > 0$)

$$P_0(L) = \mathcal{N} \exp\left[-\frac{\gamma_2 \omega_0^2}{2m^2 D_v} (L - L_0)^2\right]. \tag{50}$$

A corresponding solution can be found for the second cycle by replacing $L_0 \to -L_0$ for momenta with $L < 0$. Since due to symmetry both values are provided with same probability one may expect a linear superposition of the two solutions

$$P_0(L) = \mathcal{N}\left(\exp\left[-\frac{\gamma_2\omega_0^2}{2m^2D_v}(L-L_0)^2\right] + \exp\left[-\frac{\gamma_2\omega_0^2}{2m^2D_v}(L+L_0)^2\right]\right). \tag{51}$$

The given method does not provide a complete solution in the four-dimensional phase space, but gives us a good idea about the projections on different planes. In order to find a distribution in the four-dimensional phase space we combine the previously found distributions and introduce the invariants J_+, J_- which leads to the following ansatz:

$$\begin{aligned} P_0(x_1,x_2,v_1,v_2) &= \mathcal{N}\exp\left[-\frac{\gamma_2}{2D_p}(H-H_0)^2\right] \\ &\quad\times\left(\exp\left[-\frac{\gamma_2}{2D_p}J_+^2\right] + \exp\left[-\frac{\gamma_2}{2D_p}J_-^2\right]\right). \end{aligned} \tag{52}$$

Equation (3) in mind $D_p = m^2D_v$. We may convince ourselves that this ansatz agrees with all projections derived above. Furthermore, it is in agreement with the general ansatz derived in earlier work from information theory [18]. Since our new expression for the stationary distribution does not contain any parameter characterizing the concrete potential it may be applied to arbitrary radially symmetric potentials, in particular we may use it for describing the stationary distributions for Coulomb confinement.

6. Summary

In this article we have extended the theory of Brownian motion for systems which remain far from equilibrium due to permanent energy uptake out of the environment. We considered particles with negative friction at low velocities and (positive) dissipation at high velocities and which are in addition affected by random forces.

The corresponding evolution equations for the probability densities of Active Brownian particles, the Fokker–Planck equations for appropriate choices of the phase space coordinates are derived and the stationary solutions of it are calculated. Simple confinements, which can be formulated in potential form, have been regarded. These confinements like external fields could be a good approximation to explain stable rotational states as they can be observed in active biological motion. It was shown that the stationary probability densities possess maxima above the integrals of motions which characterize the limit cycles. These are the Hamiltonians corresponding to the constant energy according to the pumping and the constant positive and negative angular momenta corresponding to clockwise and counterclockwise rotations of particles in symmetric parabolic potentials.

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