

REFINEMENT OF THE FORMULAE FOR THE CASIMIR OPERATORS OF $wu(N)$ AND SOME OF ITS CONTRACTIONS

RUTWIG CAMPOAMOR-STURSBURG

Dpto. Geometría y Topología, Fac. CC. Matemáticas
Universidad Complutense de Madrid
Ciudad Universitaria s/n, E-28040 Madrid, Spain
e-mail: rutwig@mat.ucm.es

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The Casimir operators of the semidirect product Lie algebras $wu(N)$ and some contractions are obtained by purely determinantal methods, without involving the standard insertion method of Quesne.

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1. Introduction

Among the Lie algebras used in physics, unitary algebras $u(N)$ occupy a special place, since they appear naturally from the properties of the harmonic oscillator, and also constitute the appropriate approximation in the study of strong interactions and the quark model [1–3]. Certain semidirect products obtained from the unitary algebras, usually as a subgroup of the complete symmetry group that preserves certain properties. Whenever a Lie algebra is relevant to a physical model, their representations and Casimir invariants are an essential tool to analyse the states of the system (*e.g.* quantum numbers), and it is therefore important that they can be easily obtained.

In this work we obtain a matrix formula for the Casimir operators of the Lie algebras $wu(N)$ and some of its contractions. The advantage of this method compared with the so called insertion method of Quesne¹ [4] is that the invariants can be obtained directly using a determinant, when the classical boson operator realisation is used.

Unitary Lie algebras $u(N)$ can be realized by various methods [1], but for the analysis of Casimir operators, the boson realisation [5, 6] is very

¹ The procedure is based on the analysis of copies of $u(N)$ in the enveloping algebra of the semidirect product.

convenient. Consider the linear operators $a_i, a_j^\dagger$ ($i, j = 1 \dots N$) satisfying the commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad (1)$$

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0. \quad (2)$$

These are the usual creation and annihilation operators considered when studying the harmonic oscillators [5]. The operators $a_i^\dagger a_j$ ($1 \leq i, j \leq N$) generate the unitary Lie algebra $\mathfrak{u}(N)$, and the operators $a_i, a_i^\dagger$ transform as follows by the generators $a_i^\dagger a_j$ of $\mathfrak{u}(N)$:

$$[a_i^\dagger a_j, a_k^\dagger] = \delta_{jk} a_i^\dagger, \quad (3)$$

$$[a_i^\dagger a_j, a_k] = -\delta_{ik} a_j. \quad (4)$$

Therefore the operators $a_i^\dagger$ and a_j define a representation Γ of $\mathfrak{u}(N)$. Using the labelling

$$X_{i,j} = a_i^\dagger a_j, \quad 1 \leq i, j \leq N, \quad (5)$$

the brackets of $\mathfrak{u}(N)$ are determined by the formula:

$$[X_{i,j}, X_{k,l}] = \delta_{jk} X_{i,l} - \delta_{il} X_{k,j}, \quad 1 \leq i, j, k, l \leq N. \quad (6)$$

By (3)–(4), the operators $X_{i,j}, R_i = a_i^\dagger, S_i = a_i$ and Z for the identity operator $\mathbb{I}$ span a Lie algebra with nontrivial Levi decomposition isomorphic to the semidirect product $w\mathfrak{u}(N) := \mathfrak{u}(N) \overrightarrow{\oplus}_{\Gamma \oplus \Gamma_0} \mathfrak{h}_N$, where $\mathfrak{h}_N$ is the usual $(2N+1)$ -dimensional Heisenberg Lie algebra and Γ_0 the trivial representation. This induces a natural gradation $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ of $w\mathfrak{u}(N)$, where $\mathfrak{g}_0 = \mathfrak{u}(N) \oplus \langle Z \rangle$ and $\mathfrak{g}_1 = \sum_i \langle R_i \rangle \oplus \langle S_i \rangle$. Observe that by this decomposition $\mathfrak{g}_0$ is a subalgebra, while the other block is merely a linear subspace. This suggests to consider the contractions that preserve the subalgebra² $\mathfrak{g}_0$, which gives rise to a Lie algebra of inhomogeneous type³. To this extent,

² Actually this situation motivated the definition of Inönü–Wigner contractions. See *e.g.* [7].

³ By inhomogeneous type we mean a semidirect product of an algebra $\mathfrak{s}$ with an Abelian algebra $\mathfrak{a}$. This enlarges naturally the definition of inhomogeneous algebra, which is the corresponding product determined by the standard representation of $\mathfrak{s}$.

consider the automorphism of $wu(N)$ defined by

$$\Phi(X_{i,j}) = X_{i,j}, \quad 1 \leq i, j \leq N, \quad (7)$$

$$\Phi(R_k) = \frac{1}{\sqrt{(t)}} R_k, \quad 1 \leq k \leq N, \quad (8)$$

$$\Phi(S_k) = \frac{1}{\sqrt{(t)}} S_k, \quad 1 \leq k \leq N, \quad (9)$$

$$\Phi(Z) = Z, \quad (10)$$

where t is a parameter. Then the nontrivial brackets of $wu(N)$ transform according to the following rules:

$$\Phi^{-1}[\Phi(X_{i,j}), \Phi(X_{k,l})] = \delta_{jk} X_{i,l} - \delta_{il} X_{k,j}, \quad (11)$$

$$\Phi^{-1}[\Phi(X_{i,j}), \Phi(R_k)] = \delta_{jk} R_i, \quad (12)$$

$$\Phi^{-1}[\Phi(X_{i,j}), \Phi(S_k)] = -\delta_{ik} S_j, \quad (13)$$

$$\Phi^{-1}[\Phi(R_i), \Phi(S_j)] = \frac{\delta_{ij}}{t} Z. \quad (14)$$

This shows that the brackets of the unitary part and the representation remain unchanged, while the brackets of the operators R_i and S_j are rescaled. Clearly the limit

$$[X, Y]_{\Phi} := \lim_{t \rightarrow \infty} \Phi^{-1}[\Phi(X), \Phi(Y)] \quad (15)$$

exists for any $X, Y \in wu(N)$, thus Φ defines an Inönü–Wigner contraction, and the contracted algebra is isomorphic to the decomposable Lie algebra $(\mathfrak{u}(N) \overrightarrow{\oplus}_\Gamma (2N)L_1) \oplus \langle Z \rangle$, where $(2N)L_1$ denotes the Abelian Lie algebra of dimension $2N$. Thus, over a basis $\{X_{i,j}, P_k, Q_k, Z\}$, the brackets of the contraction are given by

$$[X_{i,j}, X_{k,l}] = \delta_{jk} X_{i,l} - \delta_{il} X_{k,j}, \quad (16)$$

$$[X_{i,j}, P_k] = \delta_{jk} P_i, \quad (17)$$

$$[X_{i,j}, Q_k] = -\delta_{ik} Q_j, \quad (18)$$

$$[P_i, Q_j] = 0. \quad (19)$$

In the following we will consider the Lie algebras $\mathfrak{u}(N) \overrightarrow{\oplus}_\Gamma (2N)L_1$ as defined over the preceding basis, with the brackets (16)–(19).

In order to derive a matrix formula for the Casimir invariants of the algebras $\mathfrak{u}(N) \overrightarrow{\oplus}_\Gamma (2N)L_1$, we will use some properties of the Lie algebra $wu(N)$, which will be developed later in the context of invariants.

Lemma 1 *The operators $X'_{i,j} := X_{i,j} - R_i S_j$ ($1 \leq i, j \leq N$) generate a copy of $\mathfrak{u}(N)$ in the enveloping algebra $\mathfrak{U}$ of $w\mathfrak{u}(N)$ that commutes with R_k and S_k for all k .*

Proof.

$$\begin{aligned} [X'_{i,j}, X'_{k,l}] &= [X_{i,j}, X_{k,l}] - [R_i S_j, X_{k,l}] - [X_{i,j}, R_k S_l] + [R_i S_j, R_k S_l] \\ &= \{X_{k,l} R_i S_j - R_i (X_{k,l} S_j + \delta_{kj} S_l)\} - R_k (\delta_{il} S_j + X_{i,j} S_l) + (\delta_{jk} X_{i,l} - \delta_{li} X_{k,j}) \\ &= -X_{i,j} R_k S_l \{\delta_{jk} R_i S_l + R_i R_k S_j S_l - R_k S_l R_i S_j\} + \delta_{jk} (X_{i,l} - R_i S_l) \\ &\quad - \delta_{li} (X_{k,j} - R_k S_j) = \delta_{jk} X'_{i,l} - \delta_{li} X'_{k,j}. \end{aligned} \quad (20)$$

$$[X'_{i,j}, R_k] = \delta_{jk} R_i - \{R_i (R_k S_j + \delta_{jk}) - R_k R_i S_j\} = 0. \quad (21)$$

$$[X'_{i,j}, S_k] = -\delta_{ik} S_j - \{R_i S_j S_k - (R_i S_k + \delta_{ik}) S_j\} = 0. \quad (22)$$

■

As a consequence of this result, any operator commuting with the variables $X'_{i,j}$ will also commute with the operators R_i and S_i , thus will provide invariants of the algebra $w\mathfrak{u}(N)$. This method is in essence the insertion method, and was first presented in [4].

The most extended procedure to determine the (generalised) Casimir invariants of a Lie algebra $\mathfrak{g}$ is the analytical method, which turns out to be more practical than the traditional method of analysing the centre of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. This is particularly convenient in the study of completely integrable Hamiltonian systems, where Casimir operators in the classical sense do not have to exist, and where the transcendental invariant functions are not interpretable in terms of $\mathcal{U}(\mathfrak{g})$.

We recall briefly the analytical method. Given a basis $\{X_1, \dots, X_n\}$ of the Lie algebra $\mathfrak{g}$ and the structure tensor $\{C_{ij}^k\}$, $\mathfrak{g}$ can be realized in the space $C^\infty(\mathfrak{g}^*)$ by means of differential operators:

$$\hat{X}_i = -C_{ij}^k x_k \frac{\partial}{\partial x_j}, \quad (23)$$

where $[X_i, X_j] = C_{ij}^k X_k$ ($1 \leq i < j \leq n$) and $\{x_1, \dots, x_n\}$ is a dual basis of $\{X_1, \dots, X_n\}$. In this context, an analytic function $F \in C^\infty(\mathfrak{g}^*)$ is called an invariant of $\mathfrak{g}$ if and only if it is a solution of the system of PDEs:

$$\left\{ \hat{X}_i F = 0, \ 1 \leq i \leq n \right\}. \quad (24)$$

Polynomial solutions F correspond, after symmetrisation, to the classical Casimir operators, while nonpolynomial solutions of system (24) are usually called “generalised Casimir invariants”. The cardinal $\mathcal{N}(\mathfrak{g})$ of a maximal set

of functionally independent solutions (in terms of the brackets of the algebra $\mathfrak{g}$ over a given basis) is easily obtained from the classical criteria for PDEs:

$$\mathcal{N}(\mathfrak{g}) := \dim \mathfrak{g} - \text{rank} \left(C_{ij}^k x_k \right)_{1 \leq i < j \leq \dim \mathfrak{g}}, \quad (25)$$

where $A(\mathfrak{g}) := \left(C_{ij}^k x_k \right)$ is the matrix which represents the commutator table of $\mathfrak{g}$ over the basis $\{X_1, \dots, X_n\}$. Evidently this quantity constitutes an invariant of the algebra. We remark that $\mathcal{N}(\mathfrak{g})$ can also be obtained from the Maurer–Cartan equations of the Lie group [8, 9], which turns out to be more practical for certain types of Lie algebras.

As example of the method, consider the unitary Lie algebra $\mathfrak{u}(3)$. The system of PDEs corresponding to it with respect to the basis above is

$$\begin{aligned} & x_{1,2} \frac{\partial F}{\partial x_{1,2}} + x_{1,3} \frac{\partial F}{\partial x_{1,3}} - x_{2,1} \frac{\partial F}{\partial x_{2,1}} - x_{3,1} \frac{\partial F}{\partial x_{3,1}} = 0, \\ & -x_{1,2} \frac{\partial F}{\partial x_{1,1}} + (x_{1,1} - x_{2,2}) \frac{\partial F}{\partial x_{2,1}} + x_{1,2} \frac{\partial F}{\partial x_{2,2}} + x_{1,3} \frac{\partial F}{\partial x_{2,3}} - x_{3,2} \frac{\partial F}{\partial x_{3,1}} = 0, \\ & -x_{1,3} \frac{\partial F}{\partial x_{1,1}} - x_{2,3} \frac{\partial F}{\partial x_{2,1}} + (x_{1,1} - x_{3,3}) \frac{\partial F}{\partial x_{3,1}} + x_{1,2} \frac{\partial F}{\partial x_{3,2}} + x_{1,3} \frac{\partial F}{\partial x_{3,3}} = 0, \\ & x_{2,1} \frac{\partial F}{\partial x_{1,1}} + (x_{2,2} - x_{1,1}) \frac{\partial F}{\partial x_{1,2}} + x_{2,3} \frac{\partial F}{\partial x_{1,3}} - x_{2,1} \frac{\partial F}{\partial x_{2,2}} - x_{3,1} \frac{\partial F}{\partial x_{3,2}} = 0, \\ & -x_{1,2} \frac{\partial F}{\partial x_{1,2}} + x_{2,1} \frac{\partial F}{\partial x_{2,1}} + x_{2,3} \frac{\partial F}{\partial x_{2,3}} - x_{3,2} \frac{\partial F}{\partial x_{3,2}} = 0, \\ & -x_{1,3} \frac{\partial F}{\partial x_{1,2}} - x_{2,3} \frac{\partial F}{\partial x_{2,2}} + x_{2,1} \frac{\partial F}{\partial x_{3,1}} + (x_{2,2} - x_{3,3}) \frac{\partial F}{\partial x_{3,2}} + x_{2,3} \frac{\partial F}{\partial x_{3,3}} = 0, \\ & x_{3,1} \frac{\partial F}{\partial x_{1,1}} + x_{3,2} \frac{\partial F}{\partial x_{1,2}} + (x_{3,3} - x_{1,1}) \frac{\partial F}{\partial x_{1,3}} - x_{2,1} \frac{\partial F}{\partial x_{2,3}} - x_{3,1} \frac{\partial F}{\partial x_{3,3}} = 0, \\ & -x_{1,2} \frac{\partial F}{\partial x_{1,3}} + x_{3,1} \frac{\partial F}{\partial x_{2,1}} + x_{3,2} \frac{\partial F}{\partial x_{2,2}} + (x_{3,3} - x_{2,2}) \frac{\partial F}{\partial x_{2,3}} - x_{3,2} \frac{\partial F}{\partial x_{3,3}} = 0, \\ & -x_{1,3} \frac{\partial F}{\partial x_{1,3}} - x_{2,3} \frac{\partial F}{\partial x_{2,3}} + x_{3,1} \frac{\partial F}{\partial x_{3,1}} + x_{3,2} \frac{\partial F}{\partial x_{3,2}} = 0. \end{aligned}$$

This system can be still integrated directly, but a maximal independent set of solutions can be obtained from the characteristic polynomial P of the matrix

$$A_3 := \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix}. \quad (26)$$

We have

$$\begin{aligned} P = & (-x_{1,1} - x_{2,2} - x_{3,3})T^2 + (-x_{2,3}x_{3,2} - x_{1,3}x_{3,1} - x_{1,2}x_{2,1} + x_{1,1}x_{2,2} \\ & + x_{2,2}x_{3,3} + x_{1,1}x_{3,3})T + (-x_{1,2}x_{2,3}x_{3,1} + x_{1,3}x_{2,2}x_{3,1} + x_{1,2}x_{2,1}x_{3,3} \\ & - x_{1,3}x_{2,3}x_{3,2} + x_{1,1}x_{2,3}x_{3,2} - x_{1,1}x_{2,2}x_{3,3}). \end{aligned} \quad (27)$$

It can be easily verified that the three polynomial coefficients of P are solutions of the system, thus they define invariants of the unitary algebra $\mathfrak{u}(N)$ after symmetrisation. In fact, the same matrix procedure holds for any N .

Proposition 1 *Let $N \geq 2$. Then the Casimir operators C_k of $\mathfrak{u}(N)$ are given by the coefficients of the characteristic polynomial*

$$|A_N - T.id_N| = T^N + \sum_{k=1}^N C_k T^{N-k}, \quad (28)$$

where

$$A_N = \begin{pmatrix} x_{1,1} & \dots & x_{1,N} \\ \vdots & & \vdots \\ x_{N,1} & \dots & x_{N,N} \end{pmatrix}. \quad (29)$$

Moreover $\deg C_k = k$ for $k=1 \dots N$.

Observe that the matrix A_N can be rewritten as

$$A_N = \sum_{i=1}^N x_{i,j} \Gamma(X_{i,j}), \quad (30)$$

where $\Gamma(X_{i,j})$ is the matrix corresponding to the generator $X_{i,j}$ by the representation Γ defined by the boson operators (3) and (4).

The objective of this work is to show that for the Lie algebras

$$\mathfrak{u}(N) \oplus_{\Gamma} (2N)L_1$$

the Casimir operators can also be obtained using characteristic polynomials. As intermediate result, we will also find a closed matrix expression for the invariants of the semidirect product $w\mathfrak{u}(N)$.

2. Illustrating example

Consider the Lie algebra $\mathfrak{g} := \mathfrak{u}(2) \overrightarrow{\oplus}_{\Gamma} 4L_1$ in dimension 8. Over the basis $\{X_{1,1}, X_{2,2}, X_{1,2}, X_{2,1}, P_1, P_2, Q_1, Q_2\}$ the commutator matrix is given by

$$A(\mathfrak{g}) = \begin{pmatrix} 0 & 0 & X_{1,2} & -X_{2,1} & P_1 & 0 & -Q_1 & 0 \\ 0 & 0 & -X_{1,2} & X_{2,1} & 0 & P_2 & 0 & -Q_2 \\ -X_{1,2} & X_{1,2} & 0 & X_{1,1} - X_{2,2} & 0 & P_1 & -Q_2 & 0 \\ X_{2,1} & -X_{2,1} & X_{2,2} - X_{1,1} & 0 & P_2 & 0 & 0 & -Q_1 \\ -P_1 & 0 & 0 & -P_2 & 0 & 0 & 0 & 0 \\ 0 & -P_2 & -P_1 & 0 & 0 & 0 & 0 & 0 \\ Q_1 & 0 & Q_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & Q_2 & 0 & Q_1 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (31)$$

The matrix has rank six, so the algebra has two invariants. As commented above, the Casimir operators of $\mathfrak{u}(2)$ can be computed from the characteristic polynomial of the matrix

$$A_2 = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}, \quad (32)$$

and we obtain

$$F_1 := |M_2 - \text{Id}_2| = T^2 - (x_{1,1} + x_{2,2})T + (x_{1,1}x_{2,2} - x_{1,2}x_{2,1}). \quad (33)$$

Now we consider the matrix

$$D_2 = \begin{pmatrix} x_{1,1} & x_{1,2} & p_1 \\ x_{2,1} & x_{2,2} & p_2 \\ -q_1 & -q_2 & 0 \end{pmatrix} \quad (34)$$

and take its characteristic polynomial

$$F_2 := T^3 - (x_{1,1} + x_{2,2})T^2 + (x_{1,1}x_{2,2} - p_1q_1 - q_2p_2 - x_{2,1}x_{1,2})T - q_1x_{1,2}p_2 + x_{1,1}q_2p_2 + p_1q_1x_{2,2} - x_{2,1}p_1q_2. \quad (35)$$

Now we compute the difference $F_2 - TF_1$ and obtain the polynomial

$$F := -(p_1q_1 + p_2q_2)T - q_1p_2x_{1,2} + x_{1,1}p_2q_2 + p_1q_1x_{2,2} - x_{2,1}p_1q_2. \quad (36)$$

It is straightforward to verify that the functions $C_1 = p_1q_1 + p_2q_2$ and $C_2 = -q_1p_2x_{1,2} + x_{1,1}p_2q_2 + p_1q_1x_{2,2} - x_{2,1}p_1q_2$ are solutions of the system (24) corresponding to $\mathfrak{u}(2) \overrightarrow{\oplus}_{\Gamma} 4L_1$, so that the symmetrisation of C_1 and C_2 provide the Casimir operators of the algebra. The finality of this article is to show that for any N a fundamental system of invariants of $\mathfrak{u}(N) \overrightarrow{\oplus}_{\Gamma} (2N)L_1$ can be obtained by this method.

3. The matrix formula

In order to prove the preceding assertion for any N , we will first obtain some formulae concerning the Casimir invariants of the Lie algebras $wu(N)$, and then use the contraction onto the direct sum $\mathfrak{u}(N) \xrightarrow{\Gamma} (2N)L_1 \oplus \langle Z \rangle$.

Proposition 2 *Let $N \geq 2$. Then the noncentral Casimir operators C_{k+1} of $wu(N)$ are given by the coefficients of the polynomial*

$$|B - T.id_{N+1}| = T^{N+1} + \sum_{k=1}^N z^{k-1} C_{k+1} T^{N+1-k}, \quad (37)$$

where

$$B = \begin{pmatrix} zx_{1,1} & \dots & zx_{1,N} & p_1 T \\ \vdots & & \vdots & \vdots \\ zx_{N,1} & \dots & zx_{N,N} & p_N T \\ -q_1 & \dots & -q_N & 0 \end{pmatrix}. \quad (38)$$

Moreover $\deg C_{k+1} = k + 1$ for $k=1 \dots N$.

Proof. The idea is to use lemma 1 combined with the matrix formula (28). We have that the operators $X'_{i,j}$ span a copy of $\mathfrak{u}(N)$ that commutes with R_i and S_i . Thus the insertion of the corresponding variables $zx'_{i,j}$ ⁴ into (28). The invariants are given by the following determinant:

$$\Delta = \begin{vmatrix} zx_{1,1} - p_1 q_1 - T & \dots & zx_{1,N} - p_1 q_N \\ \vdots & & \vdots \\ zx_{N,1} - p_N q_1 & \dots & zx_{N,N} - p_N q_N - T \end{vmatrix}. \quad (39)$$

Now this can be simplified using the elementary rules for determinants. We observe that the second summand in each column of (39) is a multiple of $(p_1, \dots, p_N, q_1, \dots, q_N)^t$. Therefore Δ reduces to:

$$\begin{aligned} \Delta &= \begin{vmatrix} zx_{1,1} - T & \dots & zx_{1,N} \\ \vdots & & \vdots \\ zx_{N,1} & \dots & zx_{N,N} - T \end{vmatrix} \\ &+ \sum_{j=1}^N \begin{vmatrix} zx_{1,1} - T & \dots & zx_{1,j-1} & p_1 q_j & zx_{1,j+1} & \dots & zx_{1,N} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ zx_{N,1} & \dots & zx_{N,j-1} & p_N q_j & zx_{N,j+1} & \dots & zx_{N,N} - T \end{vmatrix}. \end{aligned} \quad (40)$$

⁴ Recall that Z is the generator of the centre of $wu(N)$, corresponding to the unit operator, thus it commutes with any element. The reason to include it in the variables is to obtain a homogeneous polynomial in the coefficients.

Now consider the matrix B . The polynomial $\Delta' = |B - T.id_{n+1}|$ is given by the determinant

$$\Delta' = \begin{vmatrix} zx_{1,1} - T & \dots & zx_{1,N} & p_1 T \\ \vdots & & \vdots & \vdots \\ zx_{N,1} & \dots & zx_{N,N} - T & p_N T \\ q_1 & \dots & q_N & -T \end{vmatrix}. \quad (41)$$

Solving it by the elements of the last row, we decompose the determinant into:

$$\Delta' = T \left(\sum_{j=1}^N (-1)^{N+j} q_j |(B - T.id_{N+1})_{N+1,j}| - |(B - T.id_{N+1})_{N+1,N+1}| \right), \quad (42)$$

where $(B - T.id_{N+1})_{i,j}$ is the minor of $B - T.id_{N+1}$ obtained deleting the i^{th} row and j^{th} column. Inserting the variable q_j in the minor $(B - T.id_{N+1})_{N+1,j}$, we recover the summands of (40). Comparing both determinants, we see that they obey the relation

$$\Delta T + \Delta' = 0. \quad (43)$$

■

4. The Casimir operators of the contraction

By the automorphism Φ of Section 1, the preceding algebra $wu(N)$ contracts onto the direct sum of inhomogeneous algebra and the base field, *i.e.*

$$wu(N) \rightsquigarrow \mathfrak{u}(N) \overrightarrow{\oplus}_{\Gamma(2N)} L_1 \oplus \langle Z \rangle. \quad (44)$$

Therefore the matrix formula (39) could be used to determine the Casimir operators of the contraction (it can easily be verified that $\mathcal{N}(\mathfrak{u}(N) \overrightarrow{\oplus}_{\Gamma(2N)} L_1) = \mathcal{N}(wu(N)) - 1 = N$ (see *e.g.* [4, 8]), so that the contraction procedure allows to obtain a maximal set of independent invariants of algebra of inhomogeneous type. However, a variation of the preceding method will enable us to propose a matrix formula for the invariants of these algebras without having to consider the contraction explicitly. This formula can be easily computed for any N and provides a direct method.

Theorem 1 *Let $N \geq 2$. Then the Casimir operators C_k of $\mathfrak{u}(N) \overrightarrow{\oplus}_{\Gamma(2N)} L_1$ are given by the coefficients of the polynomial*

$$F(x_{i,j}, p_k, q_l) := |C - T.id_{N+1}| + |A_N - T.id_N| T = \sum_{k=1}^N C_{k+1} T^{N-k}, \quad (45)$$

where

$$C = \begin{pmatrix} x_{1,1} & \cdots & x_{1,N} & p_1 \\ \vdots & & \vdots & \vdots \\ x_{N,1} & \cdots & x_{N,N} & p_N \\ -q_1 & \cdots & -q_N & 0 \end{pmatrix}. \quad (46)$$

Moreover $\deg C_{k+1} = k + 1$.

Proof. We will use the matrix formula obtained for $wu(N)$ to obtain the formula for $\mathfrak{u}(N) \vec{\oplus}_\Gamma(2N)L_1$. As a consequence of the contraction, the Casimir operators can be obtained by contracting the Casimir operators of $wu(N)$. In particular, using the determinant (39), they follow from the limit:

$$\lim_{t \rightarrow \infty} \frac{1}{t} \begin{vmatrix} zx_{1,1} - tp_1q_1 - T & \cdots & zx_{1,N} - tp_1q_N \\ \vdots & & \vdots \\ zx_{N,1} - tp_Nq_1 & \cdots & zx_{N,N} - tp_Nq_N - T \end{vmatrix}. \quad (47)$$

This determinant can be reduced in analogous manner as (39), and taking the limit we obtain the sum of determinants:

$$\sum_{j=1}^N \begin{vmatrix} zx_{1,1} - T & \cdots & zx_{1,j-1} & p_1q_j & zx_{1,j+1} & \cdots & zx_{1,N} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ zx_{N,1} & \cdots & zx_{N,j-1} & p_Nq_j & zx_{N,j+1} & \cdots & zx_{N,N} - T \end{vmatrix}. \quad (48)$$

Since the contraction of $wu(N)$ is a direct sum of $\mathfrak{u}(N) \vec{\oplus}_\Gamma(2N)L_1$ and Z , the latter algebra being Abelian, we can replace in (48) the variable z by 1⁵. We thus obtain:

$$\begin{aligned} & F(x_{i,j}, p_k, q_l) \\ &:= \sum_{j=1}^N \begin{vmatrix} x_{1,1} - T & \cdots & x_{1,j-1} & p_1q_j & x_{1,j+1} & \cdots & x_{1,N} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_{N,1} & \cdots & x_{N,j-1} & p_Nq_j & x_{N,j+1} & \cdots & x_{N,N} - T \end{vmatrix}. \end{aligned} \quad (49)$$

Expanding the sum we obtain the polynomial

$$F(x_{i,j}, p_k, q_l) = \sum_{k=1}^N C_{k+1} T^{N-k}, \quad (50)$$

⁵ Since obviously the Casimir operators of $\mathfrak{u}(N) \vec{\oplus}_\Gamma(2N)L_1$ are independent of z .

where the C_{k+1} are homogeneous polynomials in $x_{i,j}, p_k, q_l$ of degree $k+1$. These functions can be taken as the invariants of $\mathfrak{u}(N) \oplus_{\Gamma} (2N)L_1$ (adding z as the only invariant of the direct summand $\langle Z \rangle$). We now consider the matrix (46) and expand the determinant:

$$|C - T.id_{n+1}| = \sum_{j=1}^N (-1)^{N+1+j} q_j |(C - T.id_{N+1})_{N+1,j}| + T |(C - T.id_{N+1})_{N+1,N+1}|. \quad (51)$$

Here $(C - T.id_{N+1})_{i,j}$ is the minor of $(C - T.id_{N+1})$ obtained deleting the i^{th} row and j^{th} column. For any $j \in \{1 \dots N\}$ it is straightforward to verify that

$$\begin{vmatrix} x_{1,1} - T & \dots & x_{1,j-1} & p_1 q_j & x_{1,j+1} & \dots & x_{1,N} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_{N,1} & \dots & x_{N,j-1} & p_N q_j & x_{N,j+1} & \dots & x_{N,N} - T \end{vmatrix} = (-1)^{N+1+j} q_j |(C - T.id_{N+1})_{N+1,j}|. \quad (52)$$

Therefore we conclude that

$$F(x_{i,j}, p_k, q_l) - |C - T.id_{n+1}| = -T |(C - T.id_{N+1})_{N+1,N+1}|, \quad (53)$$

but this is nothing but the determinant of (28) multiplied by T . Thus

$$F(x_{i,j}, p_k, q_l) = |C - T.id_{N+1}| + |A_N - T.id_N| T. \quad (54)$$

■

This result shows that the Casimir operators of $\mathfrak{u}(N) \oplus_{\Gamma} (2N)L_1$ can be obtained directly as the difference of two characteristic polynomials, and that no limit or contraction must be used for its computation.

5. Concluding remarks

We have shown that the Casimir operators of the semidirect product $w\mathfrak{u}(N)$ can be obtained by evaluation of a determinant, and that, in addition, this formula provides also closed matrix expressions for the invariants of certain contractions. These formulae both follow from the generalisation to non-classical algebras of the well known matrix methods developed in the 50's by Gel'fand [10], thus showing that using characteristic polynomials to compute Casimir invariants is not exclusive of classical Lie algebras. Actually similar determinantal and matrix methods have recently been developed for various types of semidirect products of Lie algebras [11, 12], basing on

different properties of its semisimple subalgebras. However, the method presented here generalises naturally the formula for $\mathfrak{u}(N)$ by using the gradation of the semidirect product $w\mathfrak{u}(N)$, which remains valid also for the contraction. Under certain circumstances, the method should also be applicable to other types of Lie algebras $\mathfrak{s} \oplus_{\overrightarrow{R}} \mathfrak{t}$ exhibiting this gradation, for which there exists a matrix formula to determine the Casimir operators of $\mathfrak{s}$.

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