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QUANTUM DYNAMICS OF PARTICLES IN A DISCRETE TWO-BRANES WORLD MODEL: CAN MATTER PARTICLES EXCHANGE OCCUR BETWEEN BRANES?

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In a recent paper, a model for describing the quantum dynamics of massive particles in a non-commutative two-sheeted spacetime was proposed. This model considers a universe made with two spacetime sheets embedded in a 5D bulk where the fifth dimension is restricted to only two points. It was shown that this construction has several important consequences for the quantum dynamics of massive particles. Most notably, it was demonstrated that a coupling arises between the two sheets allowing matter exchange in presence of intense magnetic vector potentials. In this paper, we show that non-commutative geometry is not absolutely necessary to obtain such a result since a more traditional approach allows one to reach a similar conclusion. The fact that two different approaches provide similar results suggests that standard matter exchange between branes might finally occur contrary to conventional belief.

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1. Introduction

The idea that our four dimensional spacetime is only a part of an extended multidimensional universe is a recurrent topic in literature. The idea traces back to the 1920's, to the works of Kaluza [1] and Klein [2] who tried to unify electromagnetism and gravitation by assuming that the photon originates from the fifth component of the metric. However, despite the interest of unification, the physical predictions of the model revealed unsustainable and the overall approach was abandoned for a while. In recent few years however, there has been a renewed interest for such multidimensional scenario in different contexts and using different mathematics (see references [3] to [13] for instance). An important breakthrough was undoubtedly made possible thanks to the emergence of non-commutative geometry (NCG) [14]. For the sake of simplicity, NCG is generally used to describe spacetimes composed of a continuous part (a four dimensional hypersurface) times a discrete part where non-commutativity acts [13–17]. Several extensions of general relativity have thus been proposed over the years. It has been shown that those models which can be seen as minimal extensions of present theories could give nice explanations to several puzzling phenomena, the most important one being the so-called hierarchy problem. In most approaches, spacetime is assumed to be two-sheeted. The two sheets where left and right fermions live are embedded in a discrete five dimensional space simply reduced to two points. The superiority of non-commutativity arises precisely by the way it elegantly allows to make mathematics in the discrete space. The idea of a doubled spacetime is also in agreement with some extensions of the standard model postulating the existence of a mirror sector (or hidden sector, depending on the approach) to explain parity violation problems [15, 16]. Yet, in spite of its powerfulness, the use of non-commutativity is still very limited in physics. The reason arises from the mathematical formalism which makes construction of new theories a hard task. A possible alternative is to keep the idea of discrete dimensional space while eliminating non-commutativity. Several papers have demonstrated that it is indeed possible to get similar results to NC approach without recourse to its formalism [9–12]. For instance, Kokado and coworkers have shown that it is possible to derive pure Einstein action on $M4 \times Z2$ geometry (leading to Brans–Dicke theory in four dimensional spacetime) simply by redefining the notions of parallel transport and Riemann curvature tensors in a way appropriate to those spacetimes [9]. More recently, Arkani-Hamed *et al.* [10] have developed the idea of discrete gravitational dimensions and developed an effective field theory for massive gravitons. Those spaces are defined by four dimensional sites in a compact discretized space taking the form of either a circle or an interval (in five dimensions). Each site is then endowed with its own four dimensional metric and the 5D Einstein–Hilbert action is simply discretized using a finite difference method. It is also the path followed by Defayet *et al.* in their approach to multigravity [11, 12]. Those authors have shown that different discretization schemes are even possible, some of them leading to theories free of ghosts and usual complications due to the recourse of a lattice extra space (at least at the linear level of approximation). So, it appears that a simple discretization of extra dimensions although simple could be useful to study.

In this paper we are following a similar approach applied to the fundamental equations of the quantum domain instead of general relativity. Hence, we are considering a five dimensional spacetime where the fifth dimension, a segment, is restricted to only two points. Relevant extensions of the classical Dirac's and Pauli's equations are then derived and the effects of the existence of a second spacetime sheet on the dynamics of massive particles are studied. In a previous paper [18], a similar work was proposed but a subsequent use of NCG was necessary to build the model. Besides, the approach we follow here is mathematically simplified, it will be shown that, except for secondary aspects, the basic results of the model are similar to those obtained with a non-commutative (NC) formalism. The results of the present paper appear also to be somewhat clearer. Nevertheless, it is shown that both approaches predict an electromagnetic coupling between the two sheets that might have dramatic experimental consequences. Indeed, since the particles wave functions are now five dimensional, particles are simultaneously present on both sheets although with different probabilities of presence. It is demonstrated that a high electromagnetic vector potential can modify those probabilities such that a particle initially localized in the first sheet can be transferred into the second sheet. The differences between NC and classical approach are also reviewed. The most noticeable ones concerns the confinement of the particles in their spacetime sheets. NC approach predicts that without electromagnetic vector potential, the particles are perfectly stable and remain in their original sheets. Contrarily, the "finite difference" approach suggests that particles oscillate between the two sheets with a time periodicity depending on the distance between the sheets. Hence, the observation of particle disappearance in laboratory conditions could provide a way of determining whether the universe is doubled and non-commutative or not.

2. The model

Let us consider a quantum model for a two-branes universe based on a non-trivial generalization of the Dirac equation. In this model, the fifth dimension is reduced to two points with coordinates $\pm\delta/2$. The branes are assumed to be located at those points and δ is the distance between the two branes. From our point of view, this distance should be considered as a phenomenological one. More precisely, in the context of a two-branes universe, the $M4 \times Z2$ manifold representation is a convenient approach to formalize the two-branes world problem. The $Z2$ dimension has not necessarily a physical existence and can be considered just as an abstract dimension. As a consequence, we must take care about the fact that the distance mentioned in the present work does not directly correspond to the concept of distance between branes often mentioned in previous works. Hence, the fifth dimensional generalization of the covariant Dirac equation can be written as

$$i\gamma^{\mu}\partial_{\mu}\psi + i\gamma^{5}\partial_{5}\psi - m\psi = 0\,, \tag{1}$$

where the matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ anticommutes with the usual Dirac gamma matrices γ^{μ} such that $\mu = 0, 1, 2, 3$. Now, let us define ψ_+ (respectively ψ_-) the wave function at the point $+\delta/2$ (respectively $-\delta/2$). Then, the discrete derivative $\partial_5\psi$ can be simply written as a finite difference involving ψ_+ and ψ_- through

$$(\partial_5\psi)_{\pm} = \pm g(\psi_+ - \psi_-) \tag{2}$$

with $g = 1/\delta$. Note that the NCG [13–17] also uses a discrete derivative along the fifth dimension to generalize the classical Dirac operator in the case of a two-sheeted spacetime. Our approach is however quite different and in fact, no non-commutative mathematics will be explicitly used throughout this paper. Using the discrete derivative, it can be shown that the 5D Dirac equation breaks down into a set of two coupled differential equations

$$\left\{ \begin{array}{l} i\gamma^{\mu}\partial_{\mu}\psi_+ + ig\gamma^5\psi_+ - ig\gamma^5\psi_- - m\psi_+ = 0 \\ i\gamma^{\mu}\partial_{\mu}\psi_- + ig\gamma^5\psi_- - ig\gamma^5\psi_+ - m\psi_- = 0 \end{array} \right. \tag{3}$$

which can be rewritten in a more compact form using a matrix formalism, *i.e.*

$$\left\{ i\Gamma^{\mu}\partial_{\mu} + ig\Gamma^5 - m \right\} \Psi = 0\,, \tag{4}$$

where

$$\Psi = \left(\begin{array}{c} \psi_+ \\ \psi_- \end{array} \right) \tag{5}$$

with $\Gamma^{\mu} = \left[\begin{array}{cc} \gamma^{\mu} & 0 \\ 0 & \gamma^{\mu} \end{array} \right]$ and $\Gamma^5 = \left[\begin{array}{cc} \gamma^5 & -\gamma^5 \\ -\gamma^5 & \gamma^5 \end{array} \right]$.

We get

$$\left[\Gamma^{\mu}, \Gamma^5\right] = 0 \tag{6}$$

and

$$\Gamma^{5\ 2} = 2\Gamma = 2 \left[\begin{array}{cc} \mathbf{1}_{4\times4} & -\mathbf{1}_{4\times4} \\ -\mathbf{1}_{4\times4} & \mathbf{1}_{4\times4} \end{array} \right]. \tag{7}$$

Then, it is straightforward to show that the Lagrangian associated with Eq. (4) takes the form

$$\mathcal{L} = \overline{\Psi} \left\{ i\Gamma^{\mu}\partial_{\mu} + ig\Gamma^5 - m \right\} \Psi\,, \tag{8}$$

where

$$\overline{\Psi} = \left(\begin{array}{cc} \overline{\psi}_+ & \overline{\psi}_- \end{array} \right) . \tag{9}$$

We see that our model describes two interacting Dirac's fields, each one being related to a specific brane. The interaction arises from the $ig\Gamma^5$ term which equals zero in the case of infinitely separated sheets. We stress that $\mathcal{L}$ is the simplest non trivial Lagrangian relevant for describing quantum interactions between different branes. It can be verified that $\mathcal{L}$ is CPT, PT and C invariant but it is not P and T invariant. It would be interesting to study if the fixed time arrow of our spacetime could be linked to this broken symmetry. If this assumption holds, then it would be very tempting to assume an opposite time arrow in the other sheet to restore symmetry. At this stage, we just underline that if we suppress the diagonal terms γ^5 from Γ^5, we then retrieve the equations obtained for a two-sheeted spacetime using the NC formalism [18]. We are now going to study the consequences of this subtle difference between both approaches.

3. Free field eigenmodes

Let us determine the eigenmodes of the two-sheeted Dirac equation in the free field case. The solution can be easily derived by introducing a potential Φ such that

$$\Psi = \left\{ i\Gamma^\mu \partial_\mu + ig\Gamma^5 + m \right\} \Phi \, . \tag{10}$$

Using Eq. (4) and Eq. (10), it can then be demonstrated that Φ satisfies the equation

$$\left\{ \Box + 2g^2\Gamma + m^2 \right\} \Phi = 0 \tag{11}$$

such that the solutions Ψ of Eq. (4) can be deduced from the solutions Φ of Eq. (11). Let us look for solutions of the form

$$\Phi = \Phi_0 e^{-i\varepsilon p\cdot x} = \Phi_0 e^{-i\varepsilon(E_p t - \boldsymbol{p}\cdot\boldsymbol{x})} \, , \tag{12}$$

where $\varepsilon = +1$ for positive energies and $\varepsilon = -1$ for negative ones. By replacing the expression of Φ in Eq. (11), one gets

$$\left[\begin{array}{cc} -E_p^2 + p^2 + m^2 + 2g^2 & -2g^2 \\ -2g^2 & -E_p^2 + p^2 + m^2 + 2g^2 \end{array} \right] \Phi_0 = 0 \tag{13}$$

which yields two solutions.

The first solution is obtained in the case $E_p = \sqrt{p^2 + m^2}$ (with $\varepsilon = \pm 1$) for which we get

$$\Phi_0 \to \left[\begin{array}{c} \phi_{\varepsilon,\lambda} \\ \phi_{\varepsilon,\lambda} \end{array} \right] . \tag{14}$$

The second solution corresponds to $\widetilde{E}_p = \sqrt{p^2 + m^2 + 4g^2}$ (with $\varepsilon = \pm 1$) for which we get

$$\Phi_0 \to \left[\begin{array}{c} \varphi_{\varepsilon,\lambda} \\ -\varphi_{\varepsilon,\lambda} \end{array} \right], \tag{15}$$

where $\phi_{\varepsilon,\lambda}$ and $\varphi_{\varepsilon,\lambda}$ are 4-spinors. λ is set to $\pm 1/2$ and refers to the two possible helicity states. From Eq. (10) the solutions of Ψ can then be easily derived.

•$E = E_p$ and $\varepsilon = +1$

$$u_\lambda(\boldsymbol{p}) = \frac{1}{2\sqrt{E_p(E_p+m)}} \left(\begin{array}{c} (E_p+m)\, R\chi_\lambda \\ 2\lambda p R\chi_\lambda \\ (E_p+m)\, R\chi_\lambda \\ 2\lambda p R\chi_\lambda \end{array} \right) \tag{16}$$

•$E = E_p$ and $\varepsilon = -1$

$$v_\lambda(\boldsymbol{p}) = \frac{1}{2\sqrt{E_p(E_p+m)}} \left(\begin{array}{c} -2\lambda p R i\sigma_2\chi_\lambda \\ (E_p+m)\, R i\sigma_2\chi_\lambda \\ -2\lambda p R i\sigma_2\chi_\lambda \\ (E_p+m)\, R i\sigma_2\chi_\lambda \end{array} \right) \tag{17}$$

•$E = \widetilde{E}_p$ and $\varepsilon = +1$

$$\widetilde{u}_\lambda(\boldsymbol{p}) = \frac{1}{2\sqrt{\widetilde{E}_p(\widetilde{E}_p+m)}} \left(\begin{array}{c} (\widetilde{E}_p+m) R\chi_\lambda \\ (2\lambda p + i2g) R\chi_\lambda \\ -(\widetilde{E}_p+m) R\chi_\lambda \\ -(2\lambda p + i2g) R\chi_\lambda \end{array} \right) \tag{18}$$

•$E = \widetilde{E}_p$ and $\varepsilon = -1$

$$\widetilde{v}_\lambda(\boldsymbol{p}) = \frac{1}{2\sqrt{\widetilde{E}_p(\widetilde{E}_p+m)}} \left(\begin{array}{c} (-2\lambda p + i2g) R i\sigma_2\chi_\lambda \\ (\widetilde{E}_p+m) R i\sigma_2\chi_\lambda \\ -(-2\lambda p + i2g) R i\sigma_2\chi_\lambda \\ -(\widetilde{E}_p+m) R i\sigma_2\chi_\lambda \end{array} \right) \tag{19}$$

where χ_λ is such that $\chi_{1/2} = (1,0)$, and $\chi_{-1/2} = (0,1)$. We have also

$$R = \exp\left[-\frac{i}{2}\sigma_2\theta \right] \tag{20}$$

and

$$\boldsymbol{p} = (p\sin\theta, 0, p\cos\theta)\ . \tag{21}$$

Note that the solutions given by Eq. (18) and (19) have been derived assuming that the charge conjugation operator is given by $C = i\Gamma^2\Gamma^0$ as a natural extension of its one sheeted counterpart. Note that $E = E_p$ refers to symmetric states, whereas $E = \widetilde{E}_p$ refers to antisymmetric states.

4. Free field and fermion oscillations

Using the 8-spinors solutions given by Eq. (16) and Eq. (18) we can now try to build states corresponding to particles localized in a specific sheet. For instance, a convenient state relative to a particle localized initially in the (+) sheet is given by

$$\begin{aligned}\Psi(x) \;=\; & \frac{1}{\sqrt{2}\sqrt{V}}\Big(N_{1/2}\left[u_{1/2}(\boldsymbol{p})e^{-ip\cdot x}+\widetilde{u}_{1/2}(\boldsymbol{p})e^{-i\widetilde{p}\cdot x}\right] \\ & +N_{-1/2}\left[u_{-1/2}(\boldsymbol{p})e^{-ip\cdot x}+\widetilde{u}_{-1/2}(\boldsymbol{p})e^{-i\widetilde{p}\cdot x}\right]\Big). \end{aligned} \tag{22}$$

The polarization P_e for such a particle is defined as

$$P_e = \frac{N_{1/2}^2 - N_{-1/2}^2}{N_{1/2}^2 + N_{-1/2}^2} = N_{1/2}^2 - N_{-1/2}^2 \tag{23}$$

such that $-1 < P_e < 1$ where $N_{1/2}^2 + N_{-1/2}^2 = 1$.

To illustrate the basic predictions of the model, let us consider the case of a fully polarized particle with a positive energy

$$\psi = \frac{1}{\sqrt{2}}\left(u_\lambda(\boldsymbol{p})+\widetilde{u}_\lambda(\boldsymbol{p})\right) . \tag{24}$$

It can be verified that in the limit of zero coupling, this state reduces to

$$\lim_{g\to 0}\psi = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{E_p(E_p+m)}}\begin{pmatrix}(E_p+m)R\chi_\lambda \\ 2\lambda pR\chi_\lambda \\ 0 \\ 0\end{pmatrix} \tag{25}$$

such that there is no field contribution in the $(-)$ sheet. In that case, the spinor takes the form of the usual solution of the Dirac equation. At the first order of approximation, however, one gets

$$\psi \sim \frac{1}{\sqrt{2}}\frac{1}{\sqrt{E_p(E_p+m)}}\begin{pmatrix}(E_p+m)R\chi_\lambda \\ (2\lambda p+ig)R\chi_\lambda \\ 0 \\ -igR\chi_\lambda\end{pmatrix}. \tag{26}$$

Now, one can see that the field component in the $(-)$ sheet is no more strictly equal to zero. Such a result has several interesting consequences for the particle. As an example, let us consider the probability $P(t)$ for a particle of positive energy to be localized in the $(+)$ sheet. We need to consider the first four components of the spinor Ψ, *i.e.* the spinor ψ_+ and integrate $|\psi_+|^2$ through the whole space of the $(+)$ sheet. Then, one gets from Eq. (22)

$$P(t) = \frac{1}{2}\left[1 + A\cos\left[(\widetilde{E}_p - E_p)t\right] + B\sin\left[(\widetilde{E}_p - E_p)t\right]\right], \qquad (27)$$

where

$$A = \frac{(E_p + m)(\widetilde{E}_p + m) + p^2}{2\sqrt{E_p(E_p + m)}\sqrt{\widetilde{E}_p(\widetilde{E}_p + m)}} \qquad (28)$$

and

$$B = \frac{gpP_e}{\sqrt{E_p(E_p + m)}\sqrt{\widetilde{E}_p(\widetilde{E}_p + m)}}\,. \qquad (29)$$

The form of $P(t)$ indicates that the particle oscillates between the two sheets. Assuming $g \ll E_p$, the period of oscillations T_0 can be expressed as

$$T_0 = \frac{\pi}{g^2}E_p\left(1 + g\frac{pP_e}{2\pi E_p(E_p + m)} + \mathcal{O}[g]^3\right). \qquad (30)$$

It is worth noticing that particles of high energy undergo oscillations of larger period than particles of low energy. This is a very interesting result suggesting that contrarily to what happens usually in branes models, the oscillations are strongly suppressed for highly massive or energetic particles. Obviously such oscillations would be observed from the perspective of a brane observer as a violation of conservations law. But, in fact, no violation occurs from a 5D point of view since the sum of the energy on both sheets remain constant. The fact that such a process has not been observed yet suggests a very weak coupling constant g. In the limit where $g \to 0$, the two sheets are completely decoupled and no oscillation occurs. We would like to stress that $P(t)$ is not invariant through P or T transformations besides it is PT invariant. Notice that the results and considerations reported in this paragraph hold for a negative energy particle as well.

For illustrative purpose, let us consider an electron such that $p = 0$. We are going to assume a half period of oscillations of the order of the estimated proton lifetime, *i.e.*: $T_0/2 \sim 10^{34}$ years. Then one gets $g \sim 2 \times 10^{-19}$ m^{-1}. This value corresponds to a separation distance between both sheets of about 510 l.y. This is a particularly huge value in comparison with the usual distances considered in branes theories. Alternatively, assume that $\delta \sim$

10^{-3} m and let us consider an electron with a kinetic energy of 1 keV. Then, the half period of oscillation becomes $T_0/2 \sim 10^{-2}$ s which corresponds to a travel distance of about 187 km. If the kinetic energy of the electron is subsequently decreased, *i.e.* $E \sim 25$ meV, the covered distance is still 1 km. So, in both cases, the disappearance of the particle into the other sheet is a phenomenon that cannot be easily observed. One may suggest to use a beam of particles instead of individual particles to reveal the oscillations. Nevertheless, interactions between particles in a beam cannot be neglected anymore and strong suppression of the oscillations will likely occur in that case. We will return to this important problem later on in the paper. At last, these oscillations for a free particle are typically a consequence of the present "finite difference" approach. In the paper cited before [18] where non-commutativity was used, these oscillations do not appear at all.

5. Incorporation of an electromagnetic field into the model

In this section, the electromagnetic field will be introduced into the two-sheeted Dirac equation. The choice of the electromagnetic force, by contrast to electroweak or strong force, rests on the choice of a simplified case of gauge group and just serves as an illustration. One can easily convince oneself that the results presented here for electromagnetism can be extended to the case of other interactions like electroweak interaction or chromodynamics for instance.

Logically, each sheet possesses its own current and charge density distribution. It is also considered that photons cannot travel from one sheet to the other. That means that each electromagnetic field is confined within its own brane from which it is native. In each brane, the electromagnetic field is then described by the corresponding 4-vector potential A_μ^+ and A_μ^-, each one being independent of the other.

Classically, the electromagnetic field satisfies the gauge condition

$$A'_\mu = A_\mu + \partial_\mu \Lambda \tag{31}$$

in order for the electromagnetic field tensor

$$F^{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu \tag{32}$$

to be invariant through a gauge transformation. These conditions can be easily extended to a five dimensional problem, even in the case of a discretized extra-dimension. Usually, Λ depends on x_μ coordinates. What is going on then if Λ also depends on the extra-dimension? It will imply the existence of a fifth potential vector components A_5. To keep safe actual electromagnetic laws implies that $A_5^\pm = 0$, *i.e.* we must impose that Λ is independent of the fifth space dimension. Indeed, from Eq. (31) one must verify that

$$A_5^{\prime\pm} = A_5^{\pm} + (\partial_5 \Lambda)^{\pm} = 0 \tag{33}$$

and using the definition of the discrete derivative along the extra dimension, this last equation corresponds to

$$\pm\frac{1}{\delta}\left(\Lambda_+ - \Lambda_-\right) = 0 \tag{34}$$

i.e. $\Lambda_+ = \Lambda_- = \Lambda$. Considering the two-branes wave function, the previous result means that we have a gauge transformation $\exp(iq\Lambda)$ which must be applied to both spacetime sheets simultaneously. This result could have seemed to be in conflict with locality of the gauge transformation. In fact, the gauge is global from the point of view of both branes locations only. Of course, the locality is conserved according to the x_μ coordinates and we should not be surprised about this result which keeps unchanged the electromagnetic potential properties. It allows to introduce two copies of the electromagnetic field, each confined to a single brane, without introducing a new electromagnetic potential component which would be difficult to easily reconcile with actual observations. A similar hypothesis was assumed in the NC approach of the two-sheeted spacetime [18].

Now, the incorporation of the gauge field in Eq. (4) leads to

$$\left\{ i\Gamma^\mu(\partial_\mu + iq\boldsymbol{A}_\mu) + ig\Gamma^5 - m \right\} \Psi = 0\,, \tag{35}$$

where

$$\boldsymbol{A}_\mu = \left[\begin{array}{cc} A_\mu^+ & 0 \\ 0 & A_\mu^- \end{array} \right] \tag{36}$$

corresponding to a Lagrangian density given by

$$\mathcal{L} = \overline{\Psi} \left\{ i\Gamma^\mu \partial_\mu + ig\Gamma^5 - m \right\} \Psi - q\overline{\Psi}\Gamma^\mu \Psi \boldsymbol{A}_\mu\,. \tag{37}$$

In the following, we will restrict ourselves to the case where the kinetic energy is much smaller than the rest energy m. In that case, we will show that it is possible to derive a Pauli like equation valid for the two-sheeted spacetime.

6. Extended Pauli equation

To clarify the effect of the electromagnetic fields, let us determine the non-relativistic limit of Eq. (35). Inspired from the classical treatment, we are now looking for an equation satisfying

$$i\hbar\partial_t \chi = \boldsymbol{H}\chi \tag{38}$$

with

$$\chi = \begin{bmatrix} \chi_+ \\ \chi_- \end{bmatrix}, \tag{39}$$

where χ_+ and χ_- are two component spinors related to the wave functions of the two sheets.

Eq. (35) can be written as

$$i\partial_0 \Psi = -i\Gamma^0 \Gamma^\eta (\partial_\eta + iq\boldsymbol{A}_\eta) \Psi - ig\Gamma^0 \Gamma^5 \Psi + m\Gamma^0 \Psi + q\boldsymbol{A}_0 \Psi\,. \tag{40}$$

When m is large compared with the kinetic energy, the most rapid time dependence arises from a factor $\exp(\pm imt)$. For a free positive energy particle, the general solution is then given by the product of $\exp(-imt)$ by one of the solutions (16) and (18). For small kinetic and electromagnetic energies, therefore, we look for solutions of the form

$$\Psi = \begin{bmatrix} \chi_+ \\ \theta_+ \\ \chi_- \\ \theta_- \end{bmatrix} e^{-imt}, \tag{41}$$

where χ_+, θ_+, χ_-, θ_- are two-component spinors.

Then, the two-sheeted Dirac equation leads to the following system of coupled differential equations

$$\begin{aligned}
i\partial_0\chi_+ &= -i\sigma_\eta \left(\partial_\eta + iqA_\eta^+\right)\theta_+ + qA_0^+\chi_+ - ig\left(\theta_+ - \theta_-\right)\,, &(42)\\
i\partial_0\chi_- &= -i\sigma_\eta \left(\partial_\eta + iqA_\eta^-\right)\theta_- + qA_0^-\chi_- + ig\left(\theta_+ - \theta_-\right)\,, &(43)\\
i\partial_0\theta_+ &= -i\sigma_\eta \left(\partial_\eta + iqA_\eta^+\right)\chi_+ + qA_0^+\theta_+ + ig\left(\chi_+ - \chi_-\right) - 2m\theta_+\,, &(44)\\
i\partial_0\theta_- &= -i\sigma_\eta \left(\partial_\eta + iqA_\eta^-\right)\chi_- + qA_0^-\theta_- - ig\left(\chi_+ - \chi_-\right) - 2m\theta_-\,. &(45)
\end{aligned}$$

Since the mass m is large in comparison with the kinetic energy and Coulomb terms, the components θ_+ and θ_- are small and can be approximated by

$$\theta_+ \approx -i\frac{1}{2m}\sigma_\eta\left(\partial_\eta + iqA_\eta^+\right)\chi_+ + i\frac{g}{2m}\left(\chi_+ - \chi_-\right)\,, \tag{46}$$

$$\theta_- \approx -i\frac{1}{2m}\sigma_\eta\left(\partial_\eta + iqA_\eta^-\right)\chi_- - i\frac{g}{2m}\left(\chi_+ - \chi_-\right)\,. \tag{47}$$

By substituting Eq. (46) and Eq. (47) into Eq. (42) and Eq. (43), one finds

$$\begin{aligned}
i\partial_0\chi_+ = &-\frac{1}{2m}\sigma_\eta\sigma_\nu\left(\boldsymbol{\nabla} - iq\boldsymbol{A}_+\right)^\eta\left(\boldsymbol{\nabla} - iq\boldsymbol{A}_+\right)^\nu\chi_+ + q\Phi_+\chi_+ \\
&+\frac{g^2}{m}\left(\chi_+ - \chi_-\right) + iq\frac{g}{2m}\sigma\cdot\left\{\boldsymbol{A}_+ - \boldsymbol{A}_-\right\}\chi_-\,,
\end{aligned} \tag{48}$$

$$\begin{aligned} i\partial_0 \chi_- &= -\frac{1}{2m}\sigma_\eta \sigma_\nu \left(\boldsymbol{\nabla} - iq\boldsymbol{A}_-\right)^\eta \left(\boldsymbol{\nabla} - iq\boldsymbol{A}_-\right)^\nu \chi_- + q\Phi_- \chi_- \\ &\quad -\frac{g^2}{m}\left(\chi_+ - \chi_-\right) - iq\frac{g}{2m}\sigma\cdot\left\{\boldsymbol{A}_+ - \boldsymbol{A}_-\right\}\chi_+ , \end{aligned} \tag{49}$$

where $\Phi_\pm = A^{0,\pm}$ and $\boldsymbol{A}_{\pm,\eta} = A^{\eta,\pm}$ are the usual electric potential and magnetic vector potential. The Pauli matrices satisfying the identity

$$\sigma_i \sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k , \tag{50}$$

where ε_{ijk} is the Levi–Civita symbol, one gets:

$$\sigma_\eta \sigma_\nu \left(\boldsymbol{\nabla} - iq\boldsymbol{A}_\pm\right)^\eta \left(\boldsymbol{\nabla} - iq\boldsymbol{A}_\pm\right)^\nu = \left(\boldsymbol{\nabla} - iq\boldsymbol{A}_\pm\right)^2 + q\boldsymbol{\sigma}\cdot B_\pm . \tag{51}$$

The Hamiltonian of Eq. (38) is then given by a sum of several different terms

$$\boldsymbol{H} = \boldsymbol{H}_k + \boldsymbol{H}_m + \boldsymbol{H}_p + \boldsymbol{H}_c + \boldsymbol{H}_{cm} \tag{52}$$

which are (in natural units):

$$\boldsymbol{H}_k = -\frac{\hbar^2}{2m}\begin{bmatrix} \left(\boldsymbol{\nabla} - i\frac{q}{\hbar}\boldsymbol{A}_+\right)^2 & 0 \\ 0 & \left(\boldsymbol{\nabla} - i\frac{q}{\hbar}\boldsymbol{A}_-\right)^2 \end{bmatrix}, \tag{53}$$

$$\boldsymbol{H}_m = -g_s\mu\frac{\hbar}{2}\begin{bmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{B}_+ & 0 \\ 0 & \boldsymbol{\sigma}\cdot\boldsymbol{B}_- \end{bmatrix}, \tag{54}$$

$$\boldsymbol{H}_p = \begin{bmatrix} q\Phi_+ + V_+ & 0 \\ 0 & q\Phi_- + V_- \end{bmatrix}, \tag{55}$$

$$\boldsymbol{H}_c = \frac{g^2\hbar^2}{m}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \tag{56}$$

$$\boldsymbol{H}_{cm} = igg_c\mu\frac{\hbar}{2}\begin{bmatrix} 0 & \boldsymbol{\sigma}\cdot\left\{\boldsymbol{A}_+ - \boldsymbol{A}_-\right\} \\ -\boldsymbol{\sigma}\cdot\left\{\boldsymbol{A}_+ - \boldsymbol{A}_-\right\} & 0 \end{bmatrix} \tag{57}$$

with $\mu = q/2m$ the Bohr magneton and $g_s = g_c = 2$.

The first three terms of the Hamiltonian (52), *i.e.* $\boldsymbol{H}_k$, $\boldsymbol{H}_m$ and $\boldsymbol{H}_p$, remind the usual terms of the classical Pauli equation in presence of an electromagnetic field: $\boldsymbol{H}_k$ relates to the kinetic part of the Hamiltonian taking into account the vector potential. $\boldsymbol{H}_m$ is the coupling term between the magnetic field and the magnetic moment of the particle $g_s\mu$ with g_s the gyromagnetic factor and $\boldsymbol{H}_p$ is the Coulomb term. In addition to these terms, the Hamiltonian contains a coupling term $\boldsymbol{H}_c$ linking the two sheets together and which was previously responsible for the oscillations of the free particle. Note that except this last term and some minor differences regarding the notations (as the use of natural units and the explicit introduction of the magneton) the extended Pauli equation derived in this paper is actually the same as the one obtained using a NC formalism [18]. Hence, we confirm that a classical treatment using finite differences in discrete space permits to reproduce quite easily the NC results.

So, $\boldsymbol{H}_{cm}$ introduces a pure electromagnetic coupling term involving the magnetic vector potential and something like a magnetic moment given by $g_c\mu$, where g_c is analogous to the gyromagnetic factor. Of course, we still have $g_s = g_c = 2$. At that point, it seems important to remind that g_s is not strictly equal to 2 due to vacuum effects accurately predicted by QED. So, there is no certainty that $g_c\mu$ closely corresponds to the magnetic moment $g_s\mu$. In doubt, we now refer to $g_c\mu$ as being the isomagnetic moment and g_c the isogyromagnetic factor by analogy with $g_s\mu$ the magnetic moment and g_s the gyromagnetic factor. For the proton and the neutron, one usually uses the nuclear magneton (which is defined by the mass and the charge of the proton instead of the values of the electron). Then, the g_s factors derived using the classical Dirac equation are 2 and 0, respectively. There is a large discrepancy between the predicted values and the experimental ones which are 5.58 and -3.82, respectively. It is well known that the difference arises from the fact that the electron is a fundamental particle whereas proton and neutron which are composed by quarks, are not. As a consequence, the Dirac equation which applies normally only to point like particles cannot be applied directly to the proton and the neutron. Practically, the use of the Dirac equation to describe these particles requires the use of the experimental values of the gyromagnetic factors g_s thus defining the anomalous magnetic moment. In the same spirit, one can expect that the g_c factors are not exactly 2 and 0 for the proton and the neutron and we may assume something like an anomalous isomagnetic moment as well. If this assumption holds, then it would mean that the coupling $\boldsymbol{H}_{cm}$ between the two sheets occurs also for the neutron despite its zero electrical charge.

It could be objected that neutron oscillations are suspicious since it neglects the internal structure of the particle made of quarks always interacting *via* gluons exchange. But, as for the electromagnetic force, we assumed that the strong force exists as two copies of the gluons fields each one confined in its own brane. In our model, only the quarks, which are fermions, would be able to oscillate from one brane to the other, contrary to the gluons. So, what about the neutron? The quarks form a strongly bounded and entangled system and they must oscillate together. As the neutron oscillates, at each time, it is delocalized on each brane at the same time. Of course, it is the same thing for the related quarks bag. A related assumption is that the cohesion of a delocalized bag in one brane is then ensured by the gluons field associate with this brane. In this way, as the neutron is transferred, the quarks are transferred, and the role of the gluons field of the first brane is progressively substituted by the gluons field of the second brane. Note that similar considerations can also be applied to the NC approach of the problem [18].

In the following we shall admit these assumptions as true and consider two cases of coupling with the aim to highlight some basic features of our model.

7. Neutron in a constant scalar potential

Let us consider the case of a single neutron embedded in a region of constant potential in the (+) sheet. In the rest frame of the particle, the Hamiltonian $\boldsymbol{H}$ reduces to:

$$\boldsymbol{H} = \begin{bmatrix} V_0 & 0 \\ 0 & 0 \end{bmatrix} + \frac{g^2\hbar^2}{m} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} . \tag{58}$$

If one assumes that the particle is originally located in the (+) sheet, a simple calculus based on Eq. (38) gives the probability to find the particle in the second sheet. One gets:

$$\mathcal{P}_{\mathrm{inv}} = \frac{1}{1+\gamma^2} \sin^2\left(\hbar^{-1}\beta\sqrt{1+\gamma^2}t\right) , \tag{59}$$

where $\beta = g^2\hbar^2/m$ and $\gamma = V_0/(2\beta)$.

When $\gamma = 0$, *i.e.* without any potential, the particle oscillates freely between the two sheets with a period $T = \pi m/(g^2\hbar)$ similar to the one found previously (*cf.* Eq. (30)). But, when the potential is switched on, the period of oscillations changes. In fact, the greater the potential is, the lower the period is. Moreover, the amplitude of the oscillations drops very quickly with the potential. Such a result shows that the field induces a confinement of the particle in its sheet. Perhaps, such a mechanism could be responsible for the matter stability even for a large coupling constant g. Let us assume for instance that $g = 10^3$, assuming a neutron with a kinetic energy of about 25 meV, the period is now of the order $T \sim 50$ s which is a too small value to be compatible with the matter stability. Nevertheless, the corresponding value of beta is $\beta \sim 4 \times 10^{-17}$ eV which is quite small. By contrast, the lowest temperatures obtained in laboratory are around $T_{\mathrm{emp}} \sim 50 \times 10^{-9}$ K and correspond to energy of the order $k_{\mathrm{B}}T_{\mathrm{emp}} \sim 4 \times 10^{-12}$ eV. Even a value of V_0 corresponding to such a low energy is still associated with a $\gamma \sim 5 \times 10^4$, *i.e.* a maximum probability amplitude of 4×10^{-10}. It is thus clear that even if the coupling constant is large, the particles are quickly and strongly confined in their own sheet. Obviously, the environmental effects inhibit dramatically the particle oscillations between the two sheets. So we see that even if the present approach predicts free oscillations by contrast to the NC approach, where particles are perfectly stable in their own sheet, the confinement discussed here leads to a similar physical result: in a usual physical environment, particles do not oscillate at all, they are glued in the brane.

8. Case of a neutron embedded in a region of constant magnetic vector potential

In our previous paper [18], it was shown that the particle oscillations could be enhanced in some appropriate situations involving intense magnetic potentials. Let us now demonstrate that this result can also be obtained with the present approach. As previously, we are considering the simplified case of a neutron, assuming that g_c is not strictly equal to zero as suggested previously. We make the complementary assumption $g_c \sim g_s$. Assume that there is a constant magnetic vector potential in the $(+)$ sheet in the region where the particle is initially located. Such a potential could be experimentally realized for instance, by considering a uniform current map along a hollow cylinder. If the current intensity is I then the magnetic vector potential $\boldsymbol{A}$ appearing inside the hollow part of the cylinder has a module of the order $A \sim \mu_0 I$.

In the Hamiltonian given by Eq. (52), the eigenvalues of $\boldsymbol{H}_c$ are $2g^2\hbar^2/m$ and 0, each one being doubly degenerated. The eigenvalues of $\boldsymbol{H}_{cm}$ are $\pm(1/2)gg_c\mu\hbar\,|\boldsymbol{A}|$ and are also doubly degenerated. The typical order of magnitude for the energies related to $\boldsymbol{H}_c$ and $\boldsymbol{H}_{cm}$ are E_c and E_{cm}, respectively, such that the ratio between both contributions is approximately $E_{cm}/E_c \sim |q|\,|\boldsymbol{A}|/(g\hbar)$. If we set $|\boldsymbol{A}| \gg g\hbar/|q|$ then we get $E_{cm} \gg E_c$ such that $\boldsymbol{H}_c$ can now be treated as a weak perturbation of $\boldsymbol{H}_{cm}$. For a small coupling constant g, this condition is not very restrictive and we may assume that it could be achieved practically. Thus to simplify further the calculations, we are neglecting $\boldsymbol{H}_c$. In such conditions, the Hamiltonian reduces to (in the rest frame of the neutron and with $\boldsymbol{A} = A\boldsymbol{e}$):

$$\boldsymbol{H} = \begin{bmatrix} V_0 & 0 \\ 0 & 0 \end{bmatrix} + i\hbar\Omega \begin{bmatrix} 0 & \boldsymbol{\sigma}\cdot\boldsymbol{e} \\ -\boldsymbol{\sigma}\cdot\boldsymbol{e} & 0 \end{bmatrix}, \tag{60}$$

where $\Omega = (1/2)gg_s\mu A$ with

$$\boldsymbol{e} = \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix}. \tag{61}$$

Provided that the neutron is originally located in the sheet $(+)$ and considering that (θ,φ) gives the relative direction between $\boldsymbol{A}$ and the spin, it is straightforward to derive the transfer probability, *i.e.*

$$\mathcal{P}_{\text{inv}} = \frac{1}{1+\widetilde{\gamma}^2} \sin^2 \left(\Omega \sqrt{1+\widetilde{\gamma}^2} t \right) \tag{62}$$

with $\widetilde{\gamma} = V_0/(2\Omega\hbar)$.

Eq. (62) exhibits similar properties to the one derived in Eq. (59). Nevertheless, the magnetic vector potential plays now the role of a coupling constant which supplements g. Thus, the probability of transfer is now directly related to the value of A which can be controlled experimentally. Note that Eq. (62) is consistent with the one derived using a NC formalism (see Eq. (51)) in the aforementioned paper). Hence, from the point of view of artificial oscillations, both approaches are in perfect agreement. It is useful to define a critical value for A given by $A_c = V_0/(\hbar g g_s \mu)$ insuring that the maximum probability amplitude would be equal to 1/2 at least. This value is defined in accordance with the estimated value of V_0 which can be seen, in this idealized case, as an indicator of the environmental effects. Let us assume for instance that $g = 10^3$. In a typical cooled ($T_{\text{emp}} \sim 1$ K) and insulated environment, one can expect for $V_0 \sim k_B T_{\text{emp}} \sim 86$ μeV leading to $A_c \sim 0.72$ T.m. The related current intensity required to satisfy these conditions is $I \sim 0.6 \times 10^6$ A. At room temperature, one can consider instead $V_0 \sim 25$ meV corresponding to a value of A_c around 207.5 T.m. The related current intensity to produce oscillations now becomes $I \sim 0.2 \times 10^9$ A. At first glance, one may think that the most simple way to produce artificial oscillations would be to use a cooled and insulated device with a neutron beam flowing inside. However, one must keep in mind that the coldest neutron beams still correspond to energies about 25 meV. In such circumstances, the interactions between neutrons in the beam could likely inhibit the oscillations and force the particles to stay in their own spacetime sheet. A possible solution could be to use a weakly intense source of cold neutrons in order to prevent their interactions. As an example, a typical experimental device should be constituted by a one-by-one ultra cold neutron source (25 meV) and a conducting cylinder with a current intensity $I \sim 0.6 \times 10^6$ A. Of course, the experimental device should be placed in a cooled environment at a temperature of about 1 K. In such conditions, the neutrons would exhibit a typical velocity of about 2187 m.s^{-1} and the oscillations should take place with a half period of about 17 ps only. That means that neutrons could disappear in the other spacetime sheet (with a maximum probability amplitude of 1/2) after covering a distance of about 37 nm.

9. Conclusion

In this paper, Dirac- and Pauli-like equations valid for a two-sheeted spacetime have been derived and studied. Our model can be adequately used to mimic two-branes universe at low energies and thus could be relevant for the study of branes phenomenology. The results of this work are almost identical to those obtained in a previous paper [18] where a similar idea of quantum two-sheeted spacetime was developed using NCG. The mathematics used in this paper present the advantage of being more appealing since it is more simple. Nevertheless, several minor differences between both approaches have been noticed. Contrarily to the non-commutative approach, it is shown that a transfer of matter can occur from one brane to the other one even for free particles. This is a specificity of the "finite difference" analysis of the present paper. Concerning the effect of electromagnetic field, no significant difference between both approaches have been noted. We have demonstrated the existence of an oscillatory behavior of the particles in the presence of magnetic potentials but a freezing of the oscillations in presence of scalar potentials. In this paper, a possible experimental set-up, relevant to force particles like neutrons to oscillate, has been proposed. It is suggested that some specific configurations of electromagnetic fields, accessible with our present technology, could be adequately used to achieve this goal.

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