HAMILTONIAN STRUCTURES FOR PAIS–UHLENBECK OSCILLATOR

Katarzyna Bolonek[†], Piotr Kosiński[‡]

Department of Theoretical Physics II, University of Łódź
Pomorska 149/153, 90-236 Łódź, Poland

The Hamiltonian structures for quartic oscillator are considered. All structures admitting quadratic Hamiltonians are classified.

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1. Introduction

There is a long-held belief that in quantum theory of gravitation space-time must change its nature at distances comparable to the Planck scale. In order to model such a situation one can invoke the Heisenberg uncertainty rules. In their standard form they make the notion of classical phase space meaningless on quantum level while configuration space retains its meaning. However, one can further demand that the coordinates are noncommuting operators which implies some uncertainty relations making also the notion of the point in space-time no longer sensible. The simplest way to do this is to impose the commutation rules

$$[x^\mu,\ x^\nu] = i\Theta^{\mu\nu},$$

where $\Theta^{\mu\nu}$ is a constant c-number tensor. Recently, there has been much activity concerning field theories on such noncommutative space-time [1,2]. They appear to have some attractive properties. On the other hand their quantization seems to be more subtle problem than in the standard case. In fact, the noncommutative space-time can be replaced by its commutative

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counterpart provided one simultaneously replaces ordinary product of field variables by "star product" defined by

$$\Phi_1(x) \star \Phi_2(x) = e^{\frac{i}{2}\Theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}} \Phi_1(x)\Phi_2(y)\mid_{x=y} .$$

Therefore, once $\Theta^{0i} \neq 0$, the resulting Lagrangian contains time derivatives of arbitrary order; the theory is nonlocal in time. This makes the quantization procedure much more complicated. Indeed, within the standard framework, the first step to quantize a given classical theory is to put it in the Hamiltonian form. There exists the general algorithm which allows to construct the Hamiltonian formalism for higher-derivative [3–5] and nonlocal [6–8] theories. However, its main drawback is that the Hamiltonian is not bounded from below; the quantization can be formally carried out but the resulting theory has serious disadvantages like, for example, the nonexistence of stable ground (vacuum) state. This is the price one has to pay for the generality of Ostrogradski formalism. From this point of view it seems reasonable to pose the question whether, for a given specific system, there exist alternative canonical formalisms more adequate for quantization purposes. It can happen that, due to the peculiar properties of the system under consideration, there exists canonical formalism which, being quantized, produce quantum theory with more desirable properties than Ostrogradski approach. Our main motivation is to show, on the simplest example, that such a situation is possible; namely, that, in some cases, there exists a variety of Hamiltonians and the corresponding symplectic structures including those leading to the nice quantum theory (with stable ground state, *etc.*)

The Ostrogradski instability is shared by all theories described by the Lagrangians containing time derivatives of at least second order. Moreover, the instability phenomenon seems to be not directly related to the nonlinear character of underlying dynamics. Therefore, the simplest model to be considered is the celebrated Pais–Uhlenbeck quartic oscillator [9], linear theory of fourth order.

We will study here alternative Hamiltonian formalisms for quartic oscillator. The starting point is the obvious observation that the general solution to Lagrangian equation depends on four arbitrary constants. This implies that the corresponding Hamiltonian system should have two degrees of freedom. By inspecting the explicit form of solutions we find that there are always at least two independent globally defined constants of motion which, in addition, are quadratic in dynamical variables. On the other hand, the Hamiltonian must be also a constant of motion. Therefore, we can write out the most general Ansatz for quadratic Hamiltonian. By demanding the canonical equations to be equivalent to the initial Lagrangian one we find the relevant Poisson structures. In principle, the family of candidates for Hamiltonian functions is much wider. First, one could take an arbitrary function

of the two above-mentioned constants of motion. Moreover, for some values of parameters the dynamics is superintegrable, *i.e.* admits third independent globally defined constant of motion; then the most general Hamiltonian is a function of three integrals of motion. However, more complicated Hamiltonians result in more complicated or even singular symplectic structures. This implies that the relation between basic dynamical variable, its time derivatives and Darboux coordinates is a complicated nonlinear one and it is not clear whether it can be promoted to quantum theory.

Let us conclude this section with the following remark. Our construction is neither a pure application nor an extension of Ostrogradski algorithm. First, it cannot be applicable for all quartic systems. This can be seen by noting that we need here second (independent of the Ostrogradski Hamiltonian) globally defined integral of motion. This implies that Ostrogradski dynamics is integrable which does not seem to be automatically true, in spite of the fact that the canonical equations for Ostrogradski Hamiltonian have a very specific form. Second, the family of Hamiltonians constructed here includes in some cases the positive-definite ones, the property not shared by Ostrogradski Hamiltonian.

The details of our construction are presented in Sec. 2 while Sec. 3 is devoted to concluding remarks. Appendix contains some additional remarks concerning the problem of embedding the fourth-order system into Lagrangian system of two degrees of freedom.

2. Hamiltonian structures

Our starting point is the following Lagrangian

$$L = \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2 - \frac{m\lambda}{2}\ddot{q}^2 \,. \tag{1}$$

For $\lambda = 0$ one gets the harmonic oscillator of mass m and frequency ω.
The relevant dynamical equation reads

$$\lambda q^{(\mathrm{IV})} + \ddot{q} + \omega^2 q = 0 \tag{2}$$

or, equivalently

$$\lambda \left(\frac{d^2}{dt^2} + \omega_1^2\right)\left(\frac{d^2}{dt^2} + \omega_2^2\right) q = 0 \,; \tag{3}$$

here

$$w_{1,\,2}^2 \equiv \frac{1 \pm \sqrt{1 - 4\lambda\omega^2}}{2\lambda} \,. \tag{4}$$

The form of solution to Eq. (3) depends on $w_{1,\,2}^2$. There are the following possibilities:

(i) $0 < \lambda < \frac{1}{4\omega^2}$; then $\omega_{1,\,2}^2 > 0$ and $\omega_1^2 \neq \omega_2^2$;

(ii) $\lambda = 0$; the harmonic oscillator case;

(iii) $\lambda < 0$; then $\omega_1^2 < 0$, $\omega_2^2 > 0$, $\omega_1^2 + \omega_2^2 \neq 0$;

(iv) $\lambda = \frac{1}{4\omega^2}$; then $\omega_1^2 = \omega_2^2 = 2\omega^2$, *i.e.* we are dealing with degeneracy;

(v) $\lambda > \frac{1}{4\omega^2}$; both $\omega_{1,\,2}^2$ are complex;

$$w_{1,\,2}^2 = \frac{1 \pm i\sqrt{4\lambda\omega^2 - 1}}{2\lambda}, \qquad \omega_1 = \bar{\omega}_2 \equiv \omega_0. \tag{5}$$

We shall consider these cases separately.

(1) <u>The oscillatory regime (i)</u>

The general solution reads

$$q(t) = A_1 \cos(\omega_1 t + \alpha_1) + A_2 \cos(\omega_2 t + \alpha_2). \tag{6}$$

It depends on four arbitrary constants $A_{1,\,2}$, $\alpha_{1,\,2}$ which can be found knowing q, $\dot{q}$, $\ddot{q}$ and $\dddot{q}$ at any given time. Consequently, there are at most four independent locally defined integrals of motion; however, at least one of them must depend explicitly on time. Two integrals can be readily found by computing $A_{1,\,2}^2$ from Eq. (6) and its first three time derivatives. In this way one obtains the global integrals (normalized for further convenience)

$$J_1 = \frac{m}{\sqrt{2}(\omega_1^4 - \omega_2^4)} \left((\dddot{q} + \omega_1^2 \dot{q})^2 + \omega_2^2 (\ddot{q} + \omega_1^2 q)^2 \right),$$

$$J_2 = \frac{m}{\sqrt{2}(\omega_1^4 - \omega_2^4)} \left((\dddot{q} + \omega_2^2 \dot{q})^2 + \omega_1^2 (\ddot{q} + \omega_2^2 q)^2 \right). \tag{7}$$

For generic values of parameters no additional independent globally defined integral (which does not depend explicitly on time) exists; our system is integrable but not superintegrable. However, for λ, ω^2 such that $\frac{\omega_1}{\omega_2}$ is rational, $\frac{\omega_1}{\omega_2} = \frac{k}{l}$, it becomes superintegrable. The additional integral can be constructed as follows [10]. One writes $\sin(l\alpha_1 - k\alpha_2) = \sin(l(\omega_1 t + \alpha_1) - k(\omega_2 t + \alpha_2))$; the latter is expressible polynomially in $\sin(\omega_{1,\,2} t + \alpha_{1,\,2})$, $\cos(\omega_{1,\,2} t + \alpha_{1,\,2})$ which, in turn, can be computed from

Eq. (6) and its first three time derivatives. In what follows we are interested in generic values of λ. Therefore, we consider $J_{1,2}$ to be the only relevant integrals.

As usual, the integrals of motion are related to some symmetries. Using Noether theorem suitably generalized to higher-derivative theories one finds the symmetries responsible for the existence of $J_{1,2}$. They read

$$q \rightarrow q + \varepsilon(\dddot{q} \pm (\omega_1^2 - \omega_2^2)\dot{q}) \,. \tag{8}$$

We can now construct the Hamiltonian formalism. There exists standard procedure called Ostrogradski formalism [3, 4] which works for any higher-derivative theory. However, for a particular dynamics there can exist a variety of suitable Hamiltonian structures.

For the reason explained in Sec. 1 we restrict ourselves to quadratic Hamiltonians. Keeping in mind that the Hamiltonian itself is a constant of motion and, moreover, its rescaling is equivalent to the time rescaling, one can write the following Ansatz

$$H(\beta) = J_1 \cos \beta + J_2 \sin \beta \,, \qquad -\pi \leq \beta < \pi \,. \tag{9}$$

Using

$$q^{(n)} = \{q^{(n-1)}, H\}, \qquad n = 1, \, 2, \, 3 \tag{10}$$

one finds the following one-parameter family of Poisson structures

$$\begin{aligned}
\{q, \, \dot{q}\} &= \gamma \left(\frac{1}{\cos \beta} + \frac{1}{\sin \beta} \right) , \\
\{q, \, \ddot{q}\} &= 0 \,, \\
\{q, \, \dddot{q}\} &= -\gamma \left(\frac{\omega_2^2}{\cos \beta} + \frac{\omega_1^2}{\sin \beta} \right) , \\
\{\dot{q}, \, \ddot{q}\} &= \gamma \left(\frac{\omega_2^2}{\cos \beta} + \frac{\omega_1^2}{\sin \beta} \right) , \\
\{\dot{q}, \, \dddot{q}\} &= 0 \,, \\
\{\ddot{q}, \, \dddot{q}\} &= \gamma \left(\frac{\omega_2^4}{\cos \beta} + \frac{\omega_1^4}{\sin \beta} \right) ,
\end{aligned} \tag{11}$$

with

$$\gamma \equiv \frac{1}{\sqrt{2}m\lambda(\omega_1^2 - \omega_2^2)} \,. \tag{12}$$

Let us note the following:

(a) The Poisson structure exists for all β except $\beta = -\pi$, $-\frac{\pi}{2}$, 0, $\frac{\pi}{2}$ (this can be easily understood from Eq. (6) — both A_1 A_2 are needed to characterize fully the motion); consequently, there exists four disjoint sectors for β : $(-\pi, -\frac{\pi}{2})$, $(-\frac{\pi}{2}, 0)$, $(0, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \pi)$. However, the transformation $H \to -H$, $q_i \leftrightarrow p_i$ leaves Hamiltonian equations invariant. Therefore, it is sufficient to consider the sectors $(-\frac{\pi}{2}, 0)$ and $(0, \frac{\pi}{2})$.

(b) One easily checks that

$$\det \left[\{q^{(m)}, q^{(n)}\}^3_{m, n=0} \right] = \left(\frac{\omega_1^2 - \omega_2^2}{\cos \beta \, \sin \beta} \right)^2 \tag{13}$$

which is nonvanishing. We conclude that the Poisson structures (11) are sympletic.

(c) For any admissible β

$$\begin{aligned}
\{\ddot{q} + \omega_1^2 q, \, \ddot{q} + \omega_2^2 q\} &= 0, \\
\{\dddot{q} + \omega_1^2 \dot{q}, \, \dddot{q} + \omega_2^2 \dot{q}\} &= 0, \\
\{\ddot{q} + \omega_1^2 q, \, \dddot{q} + \omega_2^2 \dot{q}\} &= 0, \\
\{\ddot{q} + \omega_2^2 q, \, \dddot{q} + \omega_1^2 \dot{q}\} &= 0.
\end{aligned} \tag{14}$$

(d) The structures corresponding to different $\beta' s$ are different (*i.e.* not canonically equivalent). Indeed, $q, \dot{q}, \ddot{q}, and \, \dddot{q}$ are well-defined functions of canonical variables. Therefore, the canonical transformations cannot change the numerical values of the Poisson brackets. On the other hand, due to $\omega_1^2 \neq \omega_2^2$, $\sin \beta$ and $\cos \beta$ are uniquely fixed once the r.h.s. of Eqs. (11) are known.

The canonical variables are found by passing to Darboux coordinates. There is a freedom in defining such a transformation — one can always perform an additional symplectic (in our case — also linear) transformation. We shall impose a further constraint $p_i \sim \dot{q}_i$, $i = 1, 2$. Using (c) one finds the following canonical variables:

— for the $(0, \frac{\pi}{2})$ sector:

$$\begin{aligned}
q_1 &= \delta \sqrt{\cos \beta}(\ddot{q} + \omega_1^2 q), \\
p_1 &= m\delta \sqrt{\cos \beta}(\dddot{q} + \omega_1^2 \dot{q}), \\
q_2 &= \delta \sqrt{\sin \beta}(\ddot{q} + \omega_2^2 q), \\
p_2 &= m\delta \sqrt{\sin \beta}(\dddot{q} + \omega_2^2 \dot{q}), \\
q &= \frac{1}{\sqrt{\sqrt{2}\lambda \left(\omega_1^2 - \omega_2^2\right)}} \left(\frac{q_1}{\sqrt{\cos \beta}} - \frac{q_2}{\sqrt{\sin \beta}} \right),
\end{aligned}$$

$$H(\beta) = \left(\frac{p_1^2}{2m} + \frac{m\omega_2^2}{2}q_1^2 \right) + \left(\frac{p_2^2}{2m} + \frac{m\omega_1^2}{2}q_2^2 \right),$$

$$\delta \equiv \sqrt{\frac{\sqrt{2}\lambda}{\omega_1^2 - \omega_2^2}}. \tag{15}$$

— for the $(-\frac{\pi}{2}, 0)$ sector:

$$q_1 = \delta\sqrt{\cos\beta}(\ddot{q} + \omega_1^2 q),$$
$$p_1 = m\delta\sqrt{\cos\beta}(\dddot{q} + \omega_1^2\dot{q}),$$
$$q_2 = \delta\sqrt{-\sin\beta}(\ddot{q} + \omega_2^2 q),$$
$$p_2 = -m\delta\sqrt{-\sin\beta}(\dddot{q} + \omega_2^2\dot{q}),$$
$$q = \frac{1}{\sqrt{\sqrt{2}\lambda\left(\omega_1^2 - \omega_2^2\right)}} \left(\frac{q_1}{\sqrt{\cos\beta}} - \frac{q_2}{\sqrt{-\sin\beta}} \right),$$
$$H(\beta) = \left(\frac{p_1^2}{2m} + \frac{m\omega_2^2}{2}q_1^2 \right) - \left(\frac{p_2^2}{2m} + \frac{m\omega_1^2}{2}q_2^2 \right). \tag{16}$$

The formulae (15), (16) have a nice interpretation. The q variable is one of the coordinates of two-dimensional quadratic system for which q_1 and q_2 are normal coordinates [11]. However, for $\beta \in (-\frac{\pi}{2}, 0)$ the energy of one of the normal oscillations enters with negative sign. Actually, in each sector all systems look the same except the formula for q in terms of normal coordinates $q_{1,2}$.

Let us consider in some detail the sector $(-\frac{\pi}{2}, 0)$. Passing to the Lagrangian

$$L(\beta) = \left(\frac{m}{2}\dot{q}_1^2 - \frac{m\omega_2^2}{2}q_1^2 \right) - \left(\frac{m}{2}\dot{q}_2^2 - \frac{m\omega_1^2}{2}q_2^2 \right) \tag{17}$$

is a regular procedure. On the other hand, under the canonical transformation

$$\tilde{q}_1 = \frac{1}{\sqrt{\lambda(\omega_1^2 - \omega_2^2)}}(q_1 - q_2),$$

$$\tilde{q}_2 = \frac{1}{m\sqrt{\lambda(\omega_1^2 - \omega_2^2)}}(p_1 + p_2),$$

$$\tilde{p}_1 = \sqrt{\frac{\lambda}{\omega_1^2 - \omega_2^2}}(\omega_1^2 p_1 + \omega_2^2 p_2),$$

$$\tilde{p}_2 = m\sqrt{\frac{\lambda}{\omega_1^2 - \omega_2^2}}(\omega_2^2 q_1 - \omega_1^2 q_2) \tag{18}$$

the Hamiltonian attains the Ostrogradski form

$$H(\beta) = \tilde{p}_1 \tilde{q}_2 - \frac{\tilde{p}_2^2}{m\lambda} - \frac{m}{2} \tilde{q}_2^2 + \frac{m\omega^2}{2} \tilde{q}_1^2 \,. \tag{19}$$

Note that Eq. (19) defines now a singular Hamiltonian in the sense that momenta are not expressible in terms of velocities and coordinates; in fact, $\dot{\tilde{q}}_i = \frac{\partial H}{\partial \tilde{p}_i}$ imply $\dot{\tilde{q}}_2 = -\frac{\tilde{p}_2}{m\lambda}$, but also $\dot{\tilde{q}}_1 = \tilde{q}_2$, so $\tilde{p}_1$ cannot be expressed in terms of $\tilde{q}_i$, $\dot{\tilde{q}}_i$. Therefore, some care is needed when passing to the Lagrangian formalism which results in additional variable — the Lagrange multiplier enforcing $\dot{\tilde{q}}_1 = \tilde{q}_2$; as a result $\tilde{q}_1$ obeys Eq. (2). However, $\tilde{q}_1$ coincides with q only for $\beta = -\frac{\pi}{2}$; for other values of β q and $\tilde{q}_1$, are different linear combinations of normal coordinates $q_{1,2}$.

Using Eqs. (16) one can express the Lagrangian (17) in terms of q variable (up to a total derivative)

$$\begin{aligned} L(\beta) &= \frac{m\delta^2}{2}(\cos\beta + \sin\beta)\,\dddot{q}^2 \\ &\quad -m\delta^2(\omega_1^2\left(\cos\beta + \frac{1}{2}\sin\beta\right) + \omega_2^2\left(\sin\beta + \frac{1}{2}\cos\beta\right))\ddot{q}^2 \\ &\quad +\frac{m\delta^2}{2}((\omega_1^2 + 2\omega_2^2)\omega_1^2\cos\beta + (\omega_2^2 + 2\omega_1^2)\omega_2^2\sin\beta)\dot{q}^2 \\ &\quad -\frac{m\delta^2}{2}\omega_1^2\omega_2^2(\omega_1^2\cos\beta + \omega_2^2\sin\beta)q^2 \end{aligned} \tag{20}$$

which leads to the following equation of motion

$$\left((\cos\beta + \sin\beta)\frac{d^2}{dt^2} + (\omega_1^2\cos\beta + \omega_2^2\sin\beta)\right)\left(\frac{d^2}{dt^2} + \omega_1^2\right)\left(\frac{d^2}{dt^2} + \omega_2^2\right)q = 0\,. \tag{21}$$

We see from Eq. (21) that there appears a new mode $\omega^2 = \frac{\omega_1^2\cos\beta + \omega_2^2\sin\beta}{\cos\beta + \sin\beta}$ unless $\beta = -\frac{\pi}{4}$. This is not surprising. First, let us stress that the theory defined by Eqs. (16) and (17) solves the problem of finding the Hamiltonian system containing Eq. (2) (or, equivalently, Eq. (3)) as one of dynamical equations. Indeed, Eq. (2) is the direct consequence of the definition of q in terms of $q_{1,2}$ and the basic dynamical equations the latter obey. Moreover, due to $\omega_1 \neq \omega_2$, in order to determine time-dependence of q one has to know both q_1 and q_2 which implies one has to impose four initial conditions on q; therefore, the theory given by Eqs. (16) and (17) describes the general solution to Eq. (2) for arbitrary value of β. In fact, the present formulation does not differ very much in spirit from Ostrogradski formalism. In the latter, one of the canonical equations implies that the substitution $q_1 \rightarrow$

q, $q_2 \to \dot{q}$ is consistent while in the former Hamiltonian equations imply the consistency of the rule $q_{1,2} \sim \ddot{q} + \omega_{1,2}^2 q$. For this reason, making the substitution $q_i \sim \ddot{q} + \omega_i^2 q$ in (17) one gets consistent equation in spite of the fact that the number of independent variables is reduced. On the other hand this is not a point transformation which in general results in new modes (see Appendix).

Finally, let us compare our findings with those of Ref. [9]. Again, it is a matter of simple computation to verify that the formalism developed in Sec. 2.1 of [9] corresponds to $\beta = -\frac{\pi}{4}$.

(2) <u>The case $\lambda < 0$ *(iii)*</u>

Let us pass to the case *(iii)*. Putting $\omega_1^2 = - \mid \omega_1 \mid^2$ one gets

$$\left(\frac{d^2}{dt^2} - \mid \omega_1 \mid^2 \right) \left(\frac{d^2}{dt^2} + \omega_2^2 \right) q = 0 \tag{22}$$

with the general solution

$$q = A e^{\mid \omega_1 \mid t} + A' e^{-\mid \omega_1 \mid t} + B \cos(\omega_2 t + \beta) . \tag{23}$$

As in the previous case one easily finds two integrals.

$$
\begin{aligned}
I_1 &= \frac{m}{\sqrt{2}(\mid \omega_1 \mid^4 - \omega_2^4)} ((\dddot{q} - \mid \omega_1 \mid^2 \dot{q})^2 + \omega_2^2 (\ddot{q} - \mid \omega_1 \mid^2 q)^2) , \\
I_2 &= \frac{m}{\sqrt{2}(\mid \omega_1 \mid^4 - \omega_2^4)} ((\dddot{q} + \omega_2^2 \dot{q})^2 - \mid \omega_1 \mid^2 (\ddot{q} + \omega_2^2 q)^2) .
\end{aligned} \tag{24}
$$

However, there exists also the third globally defined integral. The reason for that is that there is now only one angle variable which has to be cyclic. The additional integral can be found by computing $\ln(e^{\mid \omega_1 \mid t}) \equiv \mid \omega_1 \mid t + \ln A$ and $\cos(\omega_2 t + \beta)$. Then $C \equiv \arccos(\cos(\omega_2 t + \beta)) - \frac{\omega_2}{\mid \omega_1 \mid} \ln(A e^{\mid \omega_1 \mid t})$ is time-independent and $\cos C$ can be computed from $q, \dot{q}, \ddot{q}$ and $\dddot{q}$. The resulting expression is rather complicated and will not be considered here.

We proceed along the same lines as in the first case. Define the Hamiltonian

$$H(\beta) = I_1 \cos \beta + I_2 \sin \beta , \qquad -\pi \leq \beta < \pi . \tag{25}$$

The family of admissible Poisson structures reads

$$
\begin{aligned}
\{q, \dot{q}\} &= \gamma \left(\frac{1}{\cos \beta} + \frac{1}{\sin \beta} \right) , \\
\{q, \ddot{q}\} &= 0 ,
\end{aligned}
$$

$$\{q, \ddot{q}\} = \gamma \left(\frac{|\omega_1|^2}{\sin\beta} - \frac{\omega_2^2}{\cos\beta} \right),$$

$$\{\dot{q}, \ddot{q}\} = -\gamma \left(\frac{|\omega_1|^2}{\sin\beta} - \frac{\omega_2^2}{\cos\beta} \right),$$

$$\{\dot{q}, \dddot{q}\} = 0,$$

$$\{\ddot{q}, \dddot{q}\} = \gamma \left(\frac{|\omega_1|^4}{\sin\beta} + \frac{\omega_2^4}{\cos\beta} \right),$$

$$\gamma = \frac{1}{-\sqrt{2}m\lambda(|\omega_1|^2 + \omega_2^2)}. \tag{26}$$

These structures can be obtained from Eq. (11) making the replacement $\omega_1^2 \to -|\omega_1|^2$. Again we have four sectors β and it is sufficient to consider two of them only:

— for $\beta \in (0, \frac{\pi}{2})$ one gets

$$q_1 = \delta\sqrt{\cos\beta}(\ddot{q} - |\omega_1|^2 q),$$

$$p_1 = m\delta\sqrt{\cos\beta}(\dddot{q} - |\omega_1|^2 \dot{q}),$$

$$q_2 = \delta\sqrt{\sin\beta}(\ddot{q} + \omega_2^2 q),$$

$$p_2 = m\delta\sqrt{\sin\beta}(\dddot{q} + \omega_2^2 \dot{q}),$$

$$q = \frac{1}{\sqrt{-\sqrt{2}\lambda(|\omega_1|^2 + \omega_2^2)}} \left(\frac{q_1}{\sqrt{\cos\beta}} - \frac{q_2}{\sqrt{\sin\beta}} \right),$$

$$H(\beta) = \left(\frac{p_1^2}{2m} + \frac{m\omega_2^2}{2}q_1^2 \right) + \left(\frac{p_2^2}{2m} - \frac{m|\omega_1|^2}{2}q_2^2 \right),$$

$$\delta \equiv \sqrt{\frac{-\sqrt{2}\lambda}{|\omega_1^2| + \omega_2^2}}. \tag{27}$$

— for $\beta \in (-\frac{\pi}{2}, 0)$ one gets

$$q_1 = \delta\sqrt{\cos\beta}(\ddot{q} - |\omega_1|^2 q),$$

$$p_1 = m\delta\sqrt{\cos\beta}(\dddot{q} - |\omega_1|^2 \dot{q}),$$

$$q_2 = \delta\sqrt{-\sin\beta}(\ddot{q} + \omega_2^2 q),$$

$$p_2 = -m\delta\sqrt{-\sin\beta}(\dddot{q} + \omega_2^2 \dot{q}),$$

$$q = \frac{1}{\sqrt{-\sqrt{2}\lambda(|\omega_1|^2 + \omega_2^2)}} \left(\frac{q_1}{\sqrt{\cos\beta}} - \frac{q_2}{\sqrt{-\sin\beta}} \right),$$

$$H(\beta) = \left(\frac{p_1^2}{2m} + \frac{m\omega_2^2}{2}q_1^2 \right) - \left(\frac{p_2^2}{2m} - \frac{m|\omega_1|^2}{2}q_2^2 \right). \tag{28}$$

Again the conclusion is that the q variable is a linear combination of normal coordinates for some quadratic system. The only difference as compared with the previous case is that the forces are in part repelling.

(3) The degenerate case *(iv)*

Consider the double frequency case *(iv)*:

$$\left(\frac{d^2}{dt^2} + 2\omega^2\right)^2 q = 0 . \tag{29}$$

Then

$$q(t) = A_1 \cos(\sqrt{2}\omega t + \alpha_1) + A_2 t \cos(\sqrt{2}\omega t + \alpha_2) . \tag{30}$$

The relevant integrals of motion are (again suitably normalized)

$$
\begin{aligned}
I_1 &= \frac{m}{\omega^4}\left((\dddot{q} + 2\omega^2\dot{q})^2 + 2\omega^2(\ddot{q} + 2\omega^2 q)^2\right), \\
I_2 &= \frac{m}{\omega^2}\left(2(\dddot{q} + 2\omega^2\dot{q})\dot{q} - \ddot{q}^2 + 4\omega^4 q^2\right).
\end{aligned}
\tag{31}
$$

Again, our system admits third integral which is globally defined but complicated and will not be considered.

We put

$$H(\beta) = I_1 \cos\beta + I_2 \sin\beta \quad -\pi \le \beta < \pi \tag{32}$$

and find

$$
\begin{aligned}
\{q,\dot{q}\} &= -\frac{\cos\beta}{2m\sin^2\beta}, \\
\{q,\ddot{q}\} &= 0, \\
\{q,\dddot{q}\} &= \frac{(2\cos\beta + \sin\beta)\omega^2}{2m\sin^2\beta}, \\
\{\dot{q},\ddot{q}\} &= -\frac{(2\cos\beta + \sin\beta)\omega^2}{2m\sin^2\beta}, \\
\{\dot{q},\dddot{q}\} &= 0, \\
\{\ddot{q},\dddot{q}\} &= -\frac{2(\cos\beta + \sin\beta)\omega^4}{m\sin^2\beta} .
\end{aligned}
\tag{33}
$$

There are now two sectors, $(-\pi, 0)$ and $(0, \pi)$ and it is sufficient to consider only one, say $\beta \in (0,\ \pi)$. Note that for $\beta = \frac{\pi}{2}$, $H = I_2$; this is possible because I_2 depends on both A_1 and A_2.

Let us define new variables

$$q_1 = \frac{\sqrt{\sin\beta}}{\omega^2}(\ddot{q} + 2\omega^2 q),$$

$$q_2 = \frac{\cos\beta}{\sqrt{\sin\beta}\,\omega^2}(\ddot{q} + 2(1 + tg\beta)\omega^2 q),$$

$$p_1 = \frac{m\cos\beta}{\sqrt{\sin\beta}\,\omega^2}(\dddot{q} + 2(1 + tg\beta)\omega^2 \dot{q}),$$

$$p_2 = \frac{m\sqrt{\sin\beta}}{\omega^2}(\dddot{q} + 2\omega^2 \dot{q}). \tag{34}$$

Then the Hamiltonian takes the form

$$H = \frac{p_1 p_2}{m} + m\omega^2(2q_1 q_2 - q_1^2) \tag{35}$$

while q is the linear combination of basic variables

$$q = \frac{\cos\beta}{2\sqrt{\sin\beta}}\left(-\frac{q_1}{\sin\beta} + \frac{q_2}{\cos\beta}\right). \tag{36}$$

The Hamiltonian does not depend explicitly on β and the only β-dependence comes from the expression for q in terms of basic variables q_1, q_2.

(4) <u>Complex frequencies *(v)*</u>

Finally, let us consider the complex frequencies case. Formally, one can use the results of *(i)* and define the integrals

$$J_1 = \frac{m}{\sqrt{2}(\omega_0^4 - \bar{\omega}_0^4)}((\ddot{q} + \omega_0^2 \dot{q})^2 + \bar{\omega}_0^2(\ddot{q} + \omega_0^2 q)^2),$$

$$J_2 = \frac{m}{\sqrt{2}(\omega_0^4 - \bar{\omega}_0^4)}((\ddot{q} + \bar{\omega}_0^2 \dot{q})^2 + \omega_0^2(\ddot{q} + \bar{\omega}_0^2 q)^2). \tag{37}$$

They are no longer real but rather obey

$$\bar{J}_2 = -J_1. \tag{38}$$

The one-parameter Ansatz for the real Hamiltonian reads

$$H(\beta) = i(e^{i\beta}J_1 + e^{-i\beta}J_2). \tag{39}$$

The relevant Poisson structure is given by

$$\{q, \dot{q}\} = -i\gamma(e^{i\beta} + e^{-i\beta}),$$

$$\{q, \ddot{q}\} = 0,$$
$$\{q, \dddot{q}\} = i\gamma(\omega_0^2 e^{i\beta} + \bar{\omega}_0^2 e^{-i\beta}),$$
$$\{\dot{q}, \ddot{q}\} = -i\gamma(e^{i\beta}\omega_0^2 + e^{-i\beta}\bar{\omega}_0^2),$$
$$\{\dot{q}, \dddot{q}\} = 0,$$
$$\{\ddot{q}, \dddot{q}\} = -i\gamma(\omega_0^4 e^{i\beta} + \bar{\omega}_0^4 e^{-i\beta}),$$
$$\gamma \equiv \frac{1}{\sqrt{2}m\lambda(\omega_0^2 - \bar{\omega}_0^2)}. \tag{40}$$

Now, all values of $-\pi \le \beta < \pi$ are admissible. Again, we could consider only half of this domain, say $0 \le \beta < \pi$, but there is no point to do this as we are dealing with one sector only. Define

$$q_1 = \frac{1}{\varepsilon}(\ddot{q} + \omega_0^2 q),$$
$$q_2 = \frac{1}{\bar{\varepsilon}}(\ddot{q} + \bar{\omega}_0^2),$$
$$p_1 = \frac{m}{\varepsilon}(\dddot{q} + \omega_0^2 \dot{q}),$$
$$p_2 = \frac{m}{\bar{\varepsilon}}(\dddot{q} + \bar{\omega}_0^2 \dot{q}),$$
$$\varepsilon^2 \equiv -im\gamma(\omega_0^2 - \bar{\omega}_0^2)^2 e^{-i\beta}. \tag{41}$$

The Hamiltonian takes the form

$$H(\beta) = \left(\frac{p_1^2}{2m} + \frac{m\bar{\omega}_0^2}{2}q_1^2\right) + \left(\frac{p_2^2}{2m} + \frac{m\omega_0^2}{2}q_2^2\right) \tag{42}$$

while the expression for q reads

$$q = \frac{\varepsilon q_1 - \bar{\varepsilon} q_2}{\omega_0^2 - \bar{\omega}_0^2}. \tag{43}$$

The canonical variables are not real. In fact, $\bar{q}_1 = q_2$, $\bar{p}_1 = p_2$. The real canonical variables are obtained by taking the real and imaginary parts

$$q_1 = \frac{1}{\sqrt{2}}(Q_1 + iQ_2),$$
$$q_2 = \frac{1}{\sqrt{2}}(Q_1 - iQ_2),$$
$$p_1 = \frac{1}{\sqrt{2}}(P_1 - iP_2),$$
$$p_2 = \frac{1}{\sqrt{2}}(P_1 + iP_2). \tag{44}$$

Then

$$H(\beta) = \left(\frac{P_1^2}{2m} + \frac{m(\omega_0^2 + \bar{\omega}_0^2)}{4} Q_1^2 \right) - \left(\frac{P_2^2}{2m} + \frac{m(\omega_0^2 + \bar{\omega}_0^2)}{4} Q_2^2 \right)$$
$$+ \frac{im}{2}(\bar{\omega}_0^2 - \omega_0^2)Q_1 Q_2 \, . \tag{45}$$

Further change of variables transforming $H(\beta)$ into the sum of dilatation and rotation is also possible [9] (*cf.* Sec. 3).

3. Concluding remarks

Let us summarize our results. We have found essentially one-parameter families of inequivalent quadratic Hamiltonian structures in all cases *(i)*, *(iii)*, *(iv)*, *(v)*. In the first two cases these families consist of four disjoint sectors while there are only two sectors in the *(iv)* case and one in the *(v)* case. In each sector the Hamiltonian can be put into the parameter-independent form; the structures belonging to any sector differ in the way the q-variable is expressed in terms of basic variables.

Due to the symmetry $H \leftrightarrow -H, q_i \leftrightarrow p_i$, one can reduce by two the number of sectors we have to consider. Therefore, in the case *(i)* one has basically two sectors. The Hamiltonian is, respectively, the sum or difference of two independent harmonic oscillators. Our q variable is a linear combination of two basic coordinates q_1, q_2. Taking into account the possibility of rescaling the Hamiltonian and performing simple canonical transformation $q_1 \rightarrow \pm q_i, p_i \rightarrow \pm p_i$ one concludes from Eq. (15), (16) that q can be arbitrary linear combination of q_1, q_2 except that both coefficients are nonvanishing.

Similar results hold for the case *(iii)*. The only difference is that now one oscillator describes the repelling linear force. In the degenerate case *(iv)* there is essentially one sector (if one again takes into account the symmetry $q_i \leftrightarrow p_i$, $H \leftrightarrow -H$). The Hamiltonian takes less familiar form (35) while q is given by (36).

In the complex case *(v)* there is one sector even without taking into account the above-mentioned symmetry. The Hamiltonian (45) is now the difference of two harmonic oscillators coupled by the interaction term proportional to the product of coordinate variables. Due to the fact that the kinetic energy is not positive, definite passing to normal coordinates is now impossible.

Obviously, the Hamiltonian structures considered in Ref. [9] are the particular elements of our families ($\beta = -\frac{\pi}{4}$ for *(i)* and *(iii)*, $\beta = \frac{\pi}{2}$ for *(iv)* and $\beta = -\pi$ for *(v)*).

Finally, note that, apart from the Hamiltonian, there is always an additional quadratic integral of motion. Therefore, we expect that in all cases the

separation of variables is possible. This is obvious for the first two families. In the degenerate case one defines [9]

$$
\begin{aligned}
q_1 &= q_1', \\
q_2 &= q_1' - \frac{1}{m\omega}p_2', \\
p_1 &= p_1' - m\omega q_2', \\
p_2 &= m\omega q_2'.
\end{aligned} \tag{46}
$$

Then H, Eq. (35) takes the form

$$
H = -\omega(q_1'p_2' - q_2'p_1') - m\omega^2(q_1'^2 + q_2'^2) \tag{47}
$$

which separates in polar coordinates.

Finally, consider the case of two complex conjugated frequencies squared. Making an Ansatz [9]

$$
\begin{aligned}
q_1 &= \frac{1}{2\sqrt{\bar\omega_0}}((\tilde q_1 - \tilde p_2) - i(\tilde p_1 - \tilde q_2)), \\
q_2 &= \frac{1}{2\sqrt{\omega_0}}((\tilde q_1 - \tilde p_2) + i(\tilde p_1 - \tilde q_2)), \\
p_1 &= \frac{1}{2}\sqrt{\bar\omega_0}((\tilde q_2 + \tilde p_1) - i(\tilde p_2 + \tilde q_1)), \\
p_2 &= \frac{1}{2}\sqrt{\omega_0}((\tilde q_2 + \tilde p_1) + i(\tilde p_2 + \tilde q_1))
\end{aligned} \tag{48}
$$

one obtains

$$
H = -\left(\frac{\omega_0 + \bar\omega_0}{2}\right)(\tilde q_1\tilde p_2 - \tilde q_2\tilde p_1) + \frac{i}{2}(\omega_0 - \bar\omega_0)(\tilde q_1\tilde p_1 + \tilde q_2\tilde p_2), \tag{49}
$$

i.e. the Hamiltonian becomes a commuting sum of angular momentum and dilatation and separates in polar coordinates.

The Hamiltonian formalism provides the first step toward quantization. The standard approach based on Ostrogradski formalism and Dirac procedure [12, 13] provides a consistent quantum theory. However, its serious drawback is that the quantum Hamiltonian is unbounded from below. One is not surprised that the Hamiltonian is unbounded from below if the classical motion is unbounded (the cases *(iii)–(v)* above). On the contrary, in the case *(i)* the motion is bounded while the Ostrogradski Hamiltonian is again unbounded. We have shown that in this case there exists the whole family of Hamiltonians which, after quantization, yield stable ground state.

We have to stress that in all cases under consideration the quantization procedure is quite simple because the Hamiltonians are built with the help of operators well-known from ordinary quantum mechanics: oscillator Hamiltonian, angular momentum, dilatation operator *etc.*

Appendix

Let us discuss in more detail the problem of embedding the fourth-order dynamical system into Lagrangian system with two degrees of freedom. First let us note the following. Assume we have the first-order Lagrangian

$$L = L(q, \dot{q}). \tag{50}$$

Let us make the following substitution

$$
\begin{aligned}
q &= q(x, \dot{x}, \ddot{x}), \\
\tilde{L}(x, \dot{x}, \ddot{x}, \dddot{x}) &= L(q(x, \dot{x}, \ddot{x}), \dot{q}(x, \dot{x}, \ddot{x}, \dddot{x})).
\end{aligned}
\tag{51}
$$

Then one easily derives the following identity

$$
\begin{aligned}
&\frac{\partial \tilde{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial \tilde{L}}{\partial \ddot{x}}\right) - \frac{d^3}{dt^3}\left(\frac{\partial \tilde{L}}{\partial \dddot{x}}\right) \\
&= \left(\frac{\partial q}{\partial x} - \frac{d}{dt}\frac{\partial q}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial q}{\partial \ddot{x}}\right)\left(\frac{\partial L}{\partial q} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}})\right).
\end{aligned}
\tag{52}
$$

We see that, in general, the new equation of motion contains additional solutions except the case $\frac{\partial q}{\partial \dot{x}} = 0$, $\frac{\partial q}{\partial \ddot{x}} = 0$; in the latter case (51) describes point transformation leading to the equivalent dynamics.

Consider now the system of two decoupled degrees of freedom,

$$L = L_1(q_1, \dot{q}_1) + L_2(q_2, \dot{q}_2). \tag{53}$$

Assume that

$$q_i = q_i(q, \dot{q}, \ddot{q}), \qquad i = 1, 2 \tag{54}$$

be the substitution, in terms of one variable, consistent with the equations of motion. By the latter we mean that substituting (54) into both equations

$$\frac{\partial L_i}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L_i}{\partial \dot{q}_i}\right) = 0, \quad i = 1, 2 \tag{55}$$

produce the same equation for q:

$$\frac{\partial L_i}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L_i}{\partial \dot{q}_i}\right) = \alpha_i F\left(q, \dot{q}, \ddot{q}, \dddot{q}, q^{(\text{IV})}\right) \tag{56}$$

with some constants $\alpha_{1,2}$. Then (52) implies for the Lagrangian

$$\tilde{L}(q, \dot{q}, \ddot{q}, \dddot{q}) = L_1(q_1, \dot{q}_1) + L_2(q_2, \dot{q}_2) \tag{57}$$

the following identity

$$\frac{\partial \tilde{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial \tilde{L}}{\partial \ddot{q}}\right) - \frac{d^3}{dt^3}\left(\frac{\partial \tilde{L}}{\partial \dddot{q}}\right)$$

$$= \sum_{i=1}^{2}\left(\alpha_i \frac{\partial q_i}{\partial q} - \alpha_i \frac{d}{dt}\frac{\partial q_i}{\partial \dot{q}} + \alpha_i \frac{d^2}{dt^2}\frac{\partial q_i}{\partial \ddot{q}}\right) F(q, \dot{q}, \ddot{q}, \dddot{q}, q^{(\text{IV})}). \quad (58)$$

If it happens that α_i's are such that the second and third term on r.h.s. of (58) vanish, $\tilde{L}$ gives no additional solutions.

In our case

$$L = \alpha_1\left(\frac{m\dot{q}_1^2}{2} - \frac{m\omega_1^2 q_1^2}{2}\right) + \alpha_2\left(\frac{m\dot{q}_2^2}{2} - \frac{m\omega_2^2 q_2^2}{2}\right) \quad (59)$$

and the consistent substitution reads

$$q_1 \sim \ddot{q} + \omega_2^2 q,$$
$$q_2 \sim \ddot{q} + \omega_1^2 q.$$

Then q can be expressed in terms of q_1 and q_2 and one obtains the consistent embedding of q into two-dimensional system of first order. Moreover, for $\alpha_1 = -\alpha_2$ $\tilde{L}$ gives no additional mode. However, the procedure is consistent for any $\alpha_{1,2}$ provided $\alpha_1 \cdot \alpha_2 \neq 0$ (we must have two degrees of freedom in order to be able to express q algebraically in terms of basic dynamical variables).

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