DENSITY PERTURBATIONS IN OPEN MODELS OF EARLY UNIVERSE WITH POSITIVE COSMOLOGICAL CONSTANT

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The analytical solutions of the density perturbation equation in the Friedman–Lemaître–Robertson–Walker (FLRW) open cosmological models with radiation and positive cosmological constant are provided. The perturbations are of two types: the first propagating as acoustical waves, and the second of non-wave nature. It is shown that there occurs dispersion on curvature and cosmological constant for acoustical perturbations. The wave solutions have anomalous dispersion.

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1. Introduction

The propagating mode of density perturbations during the radiational era is closely related to the temperature perturbations of background radiation. In a flat universe filled with radiation with a zero cosmological constant, density perturbations propagate as flat acoustical waves at a sound velocity $v = 1/\sqrt{3}$ (light velocity $c = 1$). The line dependence between frequency and wave number $\omega = k/\sqrt{3}$ proves that phase and group velocities are equal. The analytical approach of representing perturbations as acoustical waves during the radiational epoch in a flat universe was proposed by Sachs and Wolfe [1], and independently by Lukash [2], Chibisov and Mukhanov [3] within the Hamilton formalism in Field–Shepley variables [4]. The analogical result was obtained for various gauge-invariant formalisms [5] as well as for synchronous Lifshitz–Khalatnikov formalism [6, 7] using analytical solutions of perturbation equations [8].

In an open universe filled with radiation with the zero cosmological constant, acoustic waves propagate as a scalar field of the mass $m = 1$. The dispersion relation is nonlinear and there is a limiting frequency below which the propagation of waves is no longer possible [5], while in [9] it is shown that independently of particular formalism describing adiabatic perturbations within an expanding universe there are perturbation parameters satisfying the d'Alembert equation in some Robertson–Walker spacetime.

In a flat universe with radiation and positive cosmological constant density acoustic waves propagate as density waves, but are dispersed on the cosmological constant [10]. Thus it is of interest to investigate the behaviour of perturbations in open models with radiation and cosmological constant. In this work we find analytical solutions of gauge-invariant perturbation Eq. [11] in open FLRW models filled with radiation and with positive cosmological constant. With other gauge-invariant approaches to perturbations [12–19] the situation is similar, as there are explicit formulas for transforming the perturbation equation within any gauge-invariant formalism to a corresponding equation within any other formalism [5]. Here we demonstrate that within the solutions there is a class of perturbations propagating as acoustical waves. Besides, there appear perturbations of non-acoustic character. For acoustic perturbations we determine the dispersion relation, as well as the phase and group velocities. Furthermore, we analyse dispersion as the derivative of the group velocity in respect to wavelength and show that the dispersion of wave solutions is anomalous.

2. Background evolution

The time evolution of universe models with $K = 0, -1$ with radiation and positive cosmological constant is described by a Friedman equation with a scale factor $a(t)$ of the form [20]

$$\begin{aligned} a(t) &= \sqrt{\frac{3}{2\Lambda}}\sqrt{z(\tau)} \\ &= \sqrt{\frac{3}{2\Lambda}}\sqrt{K\left[1-\cosh\left(2\sqrt{\frac{\Lambda}{3}}\,t\right)\right] + \alpha\,\sinh\left(2\sqrt{\frac{\Lambda}{3}}\,t\right)}\,, \end{aligned} \tag{1}$$

where the parameter $\alpha = 2\sqrt{\mathcal{M}\Lambda/3}$, $\mathcal{M}$ is a motion constant providing the relation between the energy density ε and the scale factor $\mathcal{M} = \varepsilon\, a^4$, while τ is a non-dimensional parameter relating to the cosmological time of the Robertson–Walker metrics $\tau = 2\sqrt{\Lambda/3}\,t$ and $z(\tau) = K[1-\cosh(\tau)] + \alpha\sinh(\tau)$ (it is assumed that $8\pi G = 1$). Assuming $K = 0$ in Eq. (1) yields the scale factor for a flat model with $\Lambda > 0$. Conformal time η is related to time τ:

$$\eta(\tau) = \begin{cases} \gamma_2^{-1} F\left(\arccos\left[\dfrac{\alpha}{\sqrt{\alpha^2+2\gamma_2^2 z(\tau)}}\right], m_1\right), & \text{for } 0<\alpha<1\,, \\ \sqrt{2}\arctan\sqrt{z(\tau)}\,, \qquad \text{for } K=-1 & \text{and } \alpha=1\,, \\ \dfrac{1}{\sqrt{2\alpha}} F\left(\arccos\left[\dfrac{\alpha-z(\tau)}{\alpha+z(\tau)}\right], m_2\right), & \text{for } \alpha>1\,, \end{cases} \tag{2}$$

where $\gamma_2 = [-K+\sqrt{K^2-\alpha^2}]^{\frac{1}{2}}/\sqrt{2}$, $m_1 = \left[2\sqrt{K^2-\alpha^2}/(-K+\sqrt{K^2-\alpha^2})\right]^{\frac{1}{2}}$, $m_2 = [(\alpha+K)/2\alpha]^{\frac{1}{2}}$, and $F(\varphi, m)$ is an elliptic integral of the first kind [21,22]. By reversing the relation (2) the scale factor (1) can be expressed as a function of conformal time. Then the background evolution $a(\eta)$ for $\alpha \neq 1$ is described by Jacobi elliptic functions, and for $\alpha = 1$ by a trigonometric function

$$a(\eta) = \begin{cases} \sqrt{\dfrac{3}{\Lambda}}\,\gamma_1\, \mathrm{sc}\,(\gamma_2\,\eta,\, m_1)\,, & \text{for } 0<\alpha<1, \\ \sqrt{\dfrac{3}{2\Lambda}}\tan\dfrac{\eta}{\sqrt{2}}, & \text{for } \alpha=1 \text{ and } K=-1, \\ \sqrt{\dfrac{3\alpha}{2\Lambda}}\,\dfrac{\mathrm{sn}(\sqrt{2\alpha}\,\eta,\, m_2)}{1+\mathrm{cn}(\sqrt{2\alpha}\,\eta, m_2)}, & \text{for } \alpha>1, \end{cases} \tag{3}$$

where parameter $\gamma_1 = [-K-\sqrt{K^2-\alpha^2}]^{\frac{1}{2}}/\sqrt{2}$. If the parameter $\alpha \to 1^{\mp}$ in (3), then the value of scale factor for $\alpha = 1$ is obtained. The range of conformal scale factor (3) is given as the following intervals depending on the value of parameter α

$$\eta \in \begin{cases} \left(0,\ \gamma_2^{-1}K\,(m_1)\right), & \text{for } 0<\alpha<1, \\ \left(0,\ \pi/\sqrt{2}\right), & \text{for } \alpha=1 \text{ and } K=-1, \\ \left(0,\ \sqrt{2/\alpha}\,K\,(m_2)\right), & \text{for } \alpha>1, \end{cases} \tag{4}$$

where $K(m)$ is an elliptic integral of the first kind [21,22].

3. Density perturbations

3.1. Equation describing density perturbations

The equation for gauge-invariant adiabatic density perturbations with cosmological constant in metric time t is of the form [11] (see Eq. (73) there):

$$\begin{aligned}\ddot{\Delta} &+ \left(2+3c_{\rm s}^2-6w\right) H\dot{\Delta}\\ &- \left[\left(\frac{1}{2}+4w-\frac{3}{2}w^2-3c_{\rm s}^2\right)\varepsilon+(5w-3c_{\rm s}^2)\Lambda+12K(c_{\rm s}^2-w)a^{-2}\right]\Delta\\ &- c_{\rm s}^2\,\nabla^2\Delta=0\,,\end{aligned}\tag{5}$$

where $c_{\rm s}^2=\dot{p}/\dot{\varepsilon}$, $w=p/\varepsilon$, $H=\dot{a}/a$. The gauge-invariant variable $\Delta=a^{(3)}\nabla_a\mathcal{D}^a$ and $\mathcal{D}_a = (a/\varepsilon)^{(3)}\nabla_a\varepsilon$, while ${}^{(3)}\nabla_a$ is a space gradient orthogonal to 4-velocity. Using a spherical coordinate system $\{r,\ \theta,\ \phi\}$ and assuming the expansion of density in an analogous manner as in the scalar field theory, we can put down

$$\Delta(t,r,\theta,\phi)=\sum_{l,m}\int\Big[\mathcal{A}_{klm}u_{klm}(t,r,\theta,\phi)+\mathcal{A}^*_{klm}u^*_{klm}(t,r,\theta,\phi)\Big]d\boldsymbol{k}\,,\tag{6}$$

where the modes $u_{klm}(t,r,\theta,\phi)=\mu_k(t)Z_{klm}(r,\theta,\phi)$ are the product of time amplitude $\mu_k(t)$ and hyperspheric harmonics $Z_{klm}(r,\theta,\phi)$. Then the time component of Eq. (5) for radiation ($\varepsilon=(1/3)p$) with a cosmological constant and curvature K takes the form

$$\left[\frac{k^2-7K}{6z(\tau)}-\frac{1}{2}\left(\frac{\dot{z}(\tau)}{z(\tau)}\right)^2\right]\mu_k(\tau)+\frac{1}{2}\frac{\dot{z}(\tau)}{z(\tau)}\dot{\mu}_k(\tau)+\ddot{\mu}_k(\tau)=0\,.\tag{7}$$

While the space component of Eq. (5) satisfies the Helmholtz equation with space curvature [23]

$$\nabla^2 Z_{klm}(r,\theta,\phi)+(k^2-K)Z_{klm}(r,\theta,\phi)=0\,,\tag{8}$$

where harmonics $Z_{klm}(r,\theta,\phi)=\Pi_{kl}(r)Y_{lm}(\theta,\phi)$, $\Pi_{kl}(r)$ is the radial component of harmonics, and $Y_{lm}(\theta,\phi)$ are spherical harmonics. For realistic wave numbers ($k^2>0$) functions $Z_{klm}(r,\theta,\phi)$ oscillate in space, constituting the base of orthonormal functions square-integrable within a Lobachevsky space, while for wave numbers $k^2\in[-1,\ 0)$ functions $Z_{klm}(r,\theta,\phi)$ constitute a supplementary series [24, 25].

We introduce another parameter $\kappa=\left[(k^2-4K)/3\alpha\right]^{1/2}$, relating two arbitrary parameters $\{\alpha,k\}$ of perturbation Eq. (7). For $K=0$ this parameter turns into parameter κ from [5]. Using the relation (2) between the independent parameter t and conformal time η the perturbation Eq. (7) can be split into five equations

$$\left[\alpha\kappa^2 - K - \frac{2\gamma_1^2}{\mathrm{cn}^2(\gamma_2\eta,\, m_1)} - \frac{2\gamma_2^2}{\mathrm{sn}^2(\gamma_2\eta,\, m_1)}\right]\mu_k(\eta) + \mu_k''(\eta) \;=\; 0\,,$$
$$\text{for } 0<\alpha<1 \text{ and } \kappa>1, \tag{9}$$

$$\left[\kappa^2 - K - \frac{4}{\sin^2(\sqrt{2}\,\eta)}\right]\mu_k(\eta) + \mu_k''(\eta) \;=\; 0\,,$$
$$\text{for } \alpha=1 \text{ and } \kappa>1, \tag{10}$$

$$\left[\alpha\kappa^2 + 2\alpha + K - \frac{4\alpha}{\mathrm{sn}^2\left(\sqrt{2\alpha}\,\eta,\, m_2\right)}\right]\mu_k(\eta) + \mu_k''(\eta) \;=\; 0\,,$$
$$\text{for } \alpha>1 \text{ and } \kappa\in(0,1)\cup(1,\infty), \tag{11}$$

$$\left[3\alpha + K - \frac{4\alpha}{\mathrm{sn}^2\left(\sqrt{2\alpha}\,\eta,\, m_2\right)}\right]\mu_k(\eta) + \mu_k''(\eta) \;=\; 0\,,$$
$$\text{for } \alpha>1 \text{ and } \kappa=1, \tag{12}$$

$$\left[1 - K - \frac{4}{\sin^2(\sqrt{2}\,\eta)}\right]\mu_k(\eta) + \mu_k''(\eta) \;=\; 0\,,$$
$$\text{for } \alpha=1 \text{ and } \kappa=1\,. \tag{13}$$

Within equations (9)–(13) there occur following asymptotic transitions with parameters $\alpha \to 1^{\mp}$ and $\kappa \to 1^{+}$. When in Eq. (9) the asymptotic transition $\alpha \to 1^{-}$ is performed, then we obtain (10); using (11) in Eq. (11) yields equation $\alpha \to 1^{+}$, and $\kappa > 1$ in (12) gives (13). When in Eq. (10) $\kappa > 1$ is performed for $\kappa \to 1^{+}$, we obtain Eq. (13), while using $\kappa \to 1^{+}$ in (11) gives (12).

3.2. Qualitative analysis of solutions for density perturbation equations

Eqs. (9)–(13) in conformal time have a normal form (involving no first derivatives) $q(\eta)\mu_k(\eta)+\mu_k''(\eta) = 0$. Function $q(\eta)$ of perturbation equation is symmetric with the maximum at the middle of conformal time interval (4). The maximal value $q_{\max}$ is determined by the parameters $\{\alpha, \kappa\}$. When $q_{\max} > 0$, then the solutions of perturbation equations are of oscillatory character, and when $q_{\max} \leq 0$, we have non-oscillatory ones [26]. Thus for Eq. (9) the maximal value is $q_{\max} = \alpha\kappa^2 - 2\alpha + K$. Solving equation $q_{\max} = 0$, we get the relation $\kappa = \kappa(\alpha)$, represented in Fig. 1 with a dashed line. The maximal value of the coefficient in Eq. (10) is $q_{\max} = \kappa^2 - K - 4$. For $\kappa > \sqrt{4+K}$, we get oscillatory solutions, while for $\kappa \leq \sqrt{4+K}$ they become non-oscillatory. For Eq. (11) $q_{\max} = \alpha\kappa^2 - 2\alpha + K$. The curve $q_{\max} = 0$ is plotted in Fig. 1 with a dashed line. For the other Eqs. (12)–(13), $q_{\max} < 0$, so we have non-oscillatory solutions only. Equation $q_{\max} = 0$ is symmetric in respect to the sign inversion $\kappa \to -\kappa$, so it is sufficient to consider just one half-plane $\kappa > 0$. Between the curves $k = i$ and $k = 0$ we have the supplementary series, while above $k = 0$ — the main series. Below $k = i$, there is a non-physical domain. The curve $q_{\max} = 0$ for $\alpha = 1/6$ intersects the curve $\kappa = [(-4K)/3\alpha]^{1/2}$, which means that within the interval $\alpha \in (0, 1/6)$ the curve $\kappa = [(-4K)/3\alpha]^{1/2}$ separates the region of oscillatory solutions from the region of non-oscillatory ones in space, while for $\alpha > 1/6$ the curve $q_{\max} = 0$ separates the region of oscillatory solutions from the region of non-oscillatory ones in time. For the solutions for which $q_{\max} > 0$, we determine in Sec. 5 the interval of conformal time with oscillatory time solutions.

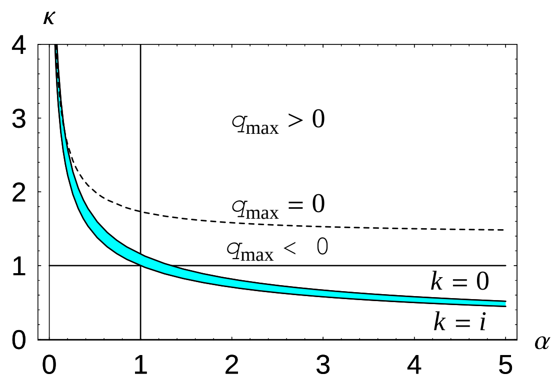

Fig. 1. Relation $q_{\max} = 0$ ($\kappa = \kappa(\alpha)$) for Eqs. (9), (10), and (11). The shaded region is the region of supplementary series.

4. Solutions

Eqs. (9)–(13) have analytical solutions given with special, or elementary functions. The solutions fall into several classes, depending on the value of the pair (α, κ). However, for the purposes of perturbation propagating study it is useful to divide all the solutions into the solutions travelling as acoustic waves with variable amplitude and frequency, and the non-wave solutions. The qualitative discussion of various solution classes depending on the value of the pair $\{\alpha, \kappa\}$ has been provided above. Below we present the analytical solutions and graphical plots for each class, as well as the solution asymptotics at the initial singularity and the behaviour of the solutions in the final stages of time evolution. The asymptotic transition between the analytical solutions are investigated depending on the value of α and κ.

4.1. Solutions for $0 < \alpha < 1$ and $\kappa > 1$

When $\alpha \in (0,\ 1)$, then $\kappa > 1$, and the solutions represent the temporal part of acoustic wave with amplitude and frequency changing with time

$$\mu_k(\eta) \ = \ \sqrt{\frac{\kappa^2}{1+\kappa^2} + \frac{\alpha}{4(1+\kappa^2)\gamma_1^2}\,\mathrm{cs}^2(\gamma_2\eta,\ m_1) + \frac{\gamma_1^2}{\alpha(1+\kappa^2)}\,\mathrm{sc}^2(\gamma_2\eta,\ m_1)} \times \exp[i\,\Theta(\eta)]\,. \tag{14}$$

The time amplitude of this solution is given through Jacobi elliptic functions, while the temporal part of the phase $\Theta(\eta)$ is a nonlinear function of conformal time, represented by a linear expression involving conformal time and two elliptic integrals of the third kind, given by the following formula

$$\begin{aligned}\Theta(\eta) \ &= \ \gamma_0\,\eta + \gamma_3^-\,\Pi\left(\mathrm{am}\left(\gamma_2\eta,\ m_1\right),\ \beta_-^2,\ m_1\right) \\ &\quad + \gamma_3^+\,\Pi\left(\mathrm{am}\left(\gamma_2\eta,\ m_1\right),\ \beta_+^2,\ m_1\right),\end{aligned} \tag{15}$$

where coefficients

$$\begin{aligned}\gamma_0 \ &= \ -4\frac{\alpha\gamma_1^2\sqrt{\kappa^4-1}\sqrt{\alpha\kappa^2+K}}{\alpha^2+4\gamma_1^4-4\alpha\gamma_1^2\kappa^2}\,, \\ \gamma_3^{\mp} \ &= \ \mp\frac{4\gamma_1^3\sqrt{\alpha\kappa^2+K}}{\alpha(2\gamma_1^2-\alpha(\kappa^2\mp\sqrt{\kappa^4-1}))}\,, \\ \beta_{\mp}^2 \ &= \ \frac{-2\gamma_1^2+\alpha(\kappa^2\mp\sqrt{\kappa^4-1})}{\alpha(\kappa^2\mp\sqrt{\kappa^4-1})}\,.\end{aligned}$$

Function $\Pi(\varphi, \beta^2, m)$ is an elliptic integral of the third kind, and $\mathrm{am}(u, m)$ is the amplitude of elliptic Jacobi integral [21, 22]. Second solution can be obtained by complex coupling of the first solution (14) The graphic plot of particular real and imaginary parts of solution (14), (15) is shown in Fig. 2.

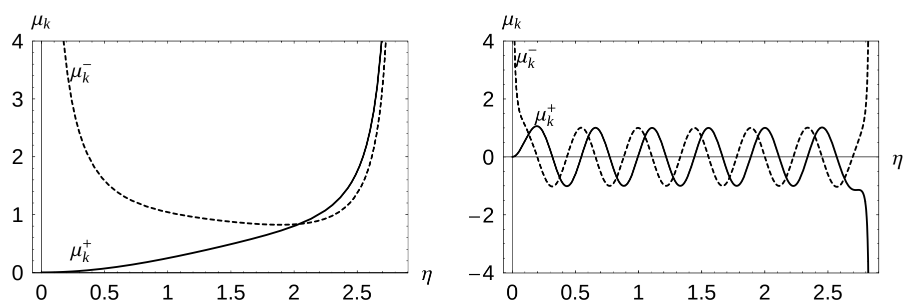

Fig. 2. Solutions $\mu_k^{\mp}(\eta)$ for $\alpha = 1/2$ and $\kappa = \{9/5, 20\}$.

4.2. Solutions for $\alpha = 1$ and $\kappa > 1$

If the solution of Eq. (10) is put into a complex form, it can be expressed by a quite simple formula, similar to the solution not involving the cosmological constant from [5]

$$\mu_k(\eta) = \left[-i + \sqrt{2}\,\omega^{-1}\cot(\sqrt{2}\,\eta)\right]\exp[i\omega\eta]\,, \tag{16}$$

where $\omega^2 = \kappa^2 - K$, and $\kappa = \left[(k^2 - 4K)/3\right]^{1/2}$. Another solution can be obtained by complex coupling of the first solution, or putting $\omega \to -\omega$. The real component of this solution is regular in $\eta = 0^+$, while the imaginary component is singular in $\eta = 0^+$ and $\eta = \pi/\sqrt{2}^-$.

Fig. 3 shows the behaviour of solutions for two particular values of κ. For $\omega > \sqrt{7/3}$ the solution is equivalent to an acoustic wave propagating as a scalar field of the mass $m = -7K$ with variable amplitude and fixed frequency. It can be said that the positive cosmological constant contributed to the mass of the scalar field.

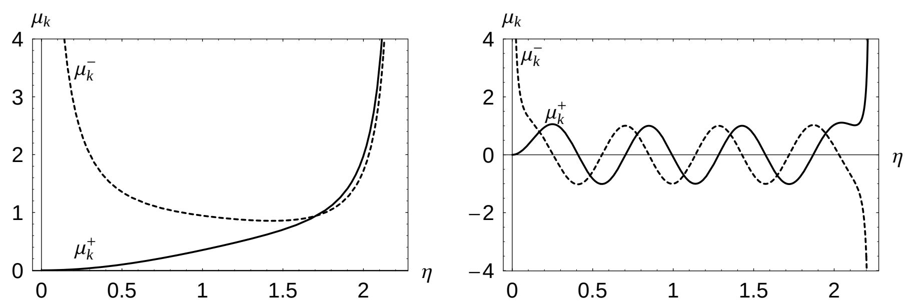

Fig. 3. Solutions $\mu_k^{\mp}(\eta)$ for two values $\kappa = \{3/2, 11\}$.

4.3. Solutions for $\alpha > 1$ and $\kappa > 1$

In this case the solutions of Eq. (11) can be expressed in the exponential form, which explicitly gives their amplitude and phase

$$\mu_k(\eta) = \left[\frac{\kappa^2 - 1}{\kappa^2 + 1} + \frac{2}{(\kappa^2 + 1)\,\mathrm{sn}^2\left(\sqrt{2\alpha}\,\eta,\ m_2\right)}\right]^{1/2}\exp[i\,\Theta(\eta)]\,, \tag{17}$$

where the time component of phase $\Theta(\eta)$ is a nonlinear function of conformal time:

$$
\begin{aligned}
\Theta(\eta) &= \sqrt{\frac{(K+\alpha\kappa^2)(1+\kappa^2)}{2\alpha(\kappa^2-1)}} \qquad (18)\\
&\times \left[\sqrt{2\alpha}\,\eta - \Pi\left(\mathrm{am}\left(\sqrt{2\alpha}\,\eta,\, m_2\right), -\frac{1}{2}(\kappa^2-1),\, m_2\right)\right].
\end{aligned}
$$

The solution is singular at both ends of the conformal time interval. Near the initial singularity ($\eta = 0$), where radiation is dominant, the solution $\mu_k(\eta)$ behaves like a solution with zero cosmological constant [5]. In later stages of conformal time $\eta \simeq \sqrt{2/\alpha}\,K(m_2)$ the model follows an exponential pattern of perturbation growth [27]. The graphic plot of the particular solutions is provided in Fig. 4.

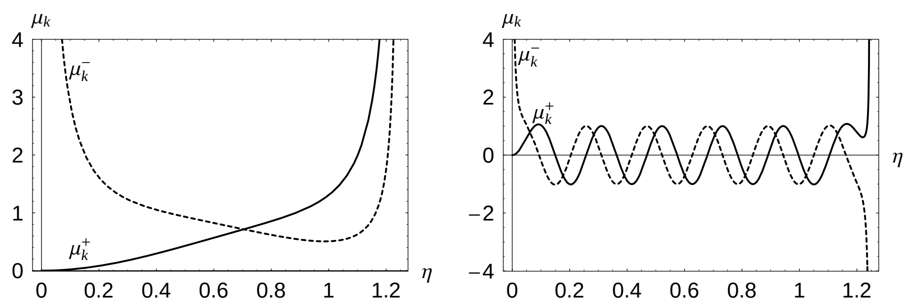

Fig. 4. Solutions $\mu_k^{\mp}(\eta)$ for $\alpha = 4$ and two values $\kappa = \{3/2, 15\}$.

4.4. Solutions for $\alpha > 1$ and $\kappa = 1$

In this case the solution is of extremely simple form

$$
\mu_k^-(\eta) = \frac{1}{\mathrm{sn}(\sqrt{2\alpha}\,\eta,\ m_2)}, \qquad (19)
$$

$$
\mu_k^+(\eta) = \frac{1}{\mathrm{sn}(\sqrt{2\alpha}\,\eta,\ m_2)}\left[\sqrt{2\alpha}\,\eta - E\left(\mathrm{am}\left(\sqrt{2\alpha}\,\eta,\ m_2\right),\ m_2\right)\right]. \qquad (20)
$$

Solution μ_k^- is singular at $\eta = 0^+$, while μ_k^+ is regular and growing for $\eta = 0^+$. Performing the asymptotic transition of $\alpha \to 1^+$ in solution (19) yields solution (23), and the analogous transition in (20) turns it into the zero solution. The solutions are plotted in Fig. 5.

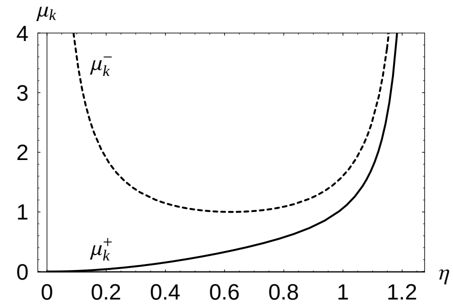

Fig. 5. Two particular solutions (19) and (20) $\mu_k^{\mp}(\eta)$ for $\alpha = 4$.

4.5. Solutions for $\alpha > 1$ and $0 < \kappa < 1$

The analytical solutions of Eq. (11) are non-oscillatory

$$\mu_k^-(\eta) = \frac{\sqrt{2-(1-\kappa^2)\,\mathrm{sn}^2\left(\sqrt{2\alpha}\,\eta,\ m_2\right)}}{\mathrm{sn}\left(\sqrt{2\alpha}\,\eta,\ m_2\right)} \exp\left[\sqrt{\frac{(K+\alpha\kappa^2)(1+\kappa^2)}{2\alpha(1-\kappa^2)}} \times \left[\sqrt{2\alpha}\,\eta - \mathit{\Pi}\left(\mathrm{am}\left(\sqrt{2\alpha}\,\eta,\ m_2\right),\ \frac{1}{2}\left(1-\kappa^2\right),\ m_2\right)\right]\right], \quad (21)$$

$$\mu_k^+(\eta) = \frac{\sqrt{2-(1-\kappa^2)\,\mathrm{sn}^2\left(\sqrt{2\alpha}\,\eta,\ m_2\right)}}{\mathrm{sn}\left(\sqrt{2\alpha}\,\eta,\ m_2\right)} \sinh\left[-\sqrt{\frac{(K+\alpha\kappa^2)(1+\kappa^2)}{2\alpha(1-\kappa^2)}} \times \left[\sqrt{2\alpha}\,\eta - \mathit{\Pi}\left(\mathrm{am}\left(\sqrt{2\alpha}\,\eta,\ m_2\right),\ \frac{1}{2}\left(1-\kappa^2\right),\ m_2\right)\right]\right]. \quad (22)$$

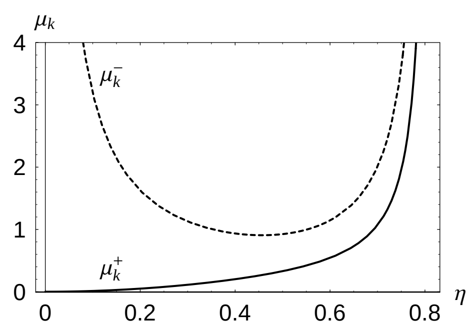

Fig. 6. Solutions $\mu_k^{\mp}(\eta)$ for $\alpha = 10$ and $\kappa = \frac{1}{2}$.

Solution $\mu_k^-(\eta)$ near the singularity is falling off, while near $\eta = \sqrt{2/\alpha}\, K\,(m_2)$ it is growing. On the other hand, solution $\mu_k^+(\eta)$ near the singularity is growing, while near $\eta = \sqrt{2/\alpha}\, K\,(m_2)$ it is falling off. Furthermore μ_k^+ is regular and growing for $\eta = 0^+$. The solutions are plotted in Fig. 6.

4.6. Particular solutions for $\alpha = 1$ and $\kappa = 1$

In this case the solution of Eq. (13) is of extremely simple form

$$\mu_k^-(\eta) = \frac{1}{\sin(\sqrt{2}\,\eta)}\,, \tag{23}$$

$$\mu_k^+(\eta) = -\cos(\sqrt{2}\,\eta) + \sqrt{2}\,\frac{\eta}{\sin(\sqrt{2}\,\eta)}\,. \tag{24}$$

The mode μ_k^- is singular at both ends of the conformal time interval $\eta = \{0^+, \frac{\pi}{\sqrt{2}}^-\}$, while μ_k^+ is regular for $\eta = 0^+$ and monotonously growing. Only solution (23) can be obtained by asymptotic transition from solution (19). Performing an asymptotic transition $\kappa \to 1^+$ in solution (16) one obtains just the solution $\mu_k^-(\eta)$. Analogously as above, the imaginary component of solution (16) after asymptotic transition reduces to the trivial solution $\mu_k^-(\eta) = 0$. The behaviour of modes $\mu_k^\pm$ is shown graphically in Fig. 7.

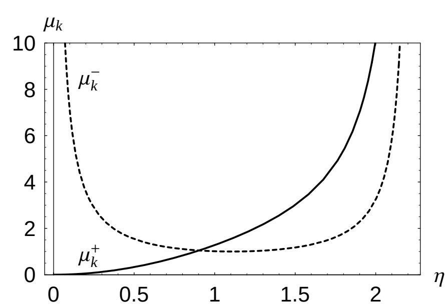

Fig. 7. Specific solutions $\mu_k^\mp(\eta)$ for $\alpha = 1$ and $\kappa = 1$.

4.7. Asymptotic solutions

As one can readily notice, the asymptotic form for $\eta = 0^+$ of the time-dependent coefficient of $\mu_k(\eta)$ in Eqs. (9)–(13) is of the same form, independently of parameters $\{\alpha, \kappa\}$, as well as the same as in the flat model [10]. Thus the asymptotic solutions near $\eta = 0^+$ do not depend on $\{\alpha, \kappa\}$ and are of the form

$$\lim_{\eta \to 0^+} \mu_k^- \approx \frac{1}{\eta}\,, \qquad \lim_{\eta \to 0^+} \mu_k^+ \approx \eta^2\,,$$

which means that for $\eta = 0^+$ the solution μ_k^- is singular, and μ_k^+ is regular.

5. Phase, phase and group velocities, and dispersion

For each of the three classes of oscillatory solutions, the discussion of phase, and phase and group velocities is presented here. Next we analyse the dispersion of velocities for each solution.

(i) $0 < \alpha < 1$ and $\kappa > 1$. Solution (14) has a nonlinear phase (15), shown in Fig. 8. The function exhibits a different behaviour within three distinct intervals of time η.

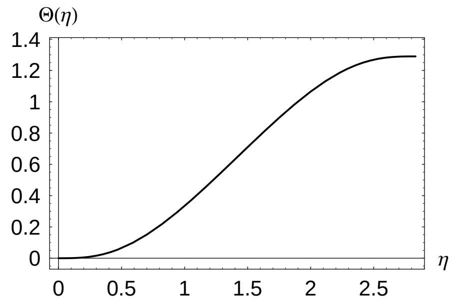

Fig. 8. Phase $\Theta(\eta)$ for $\alpha = \frac{1}{2}$ and $\kappa = 2$.

Initially the phase grows nonlinearly with conformal time, and then enters a linear growth, which ends in a nonlinear saturation. Introducing local frequency [28, 29] as the derivative of phase (15)

$$\begin{aligned} \omega(\eta,\kappa) &= \frac{\partial\Theta}{\partial\eta} \\ &= \frac{4\alpha\gamma_1^2\sqrt{K+\alpha\kappa^2}\sqrt{\kappa^4-1}\,\mathrm{cn}^2(\gamma_2\eta,m_1)\,\mathrm{sn}^2(\gamma_2\eta,m_1)}{\alpha^2-2\alpha(\alpha-2\gamma_1^2\kappa^2)\,\mathrm{sn}^2(\gamma_2\eta,m_1)-4\gamma_1^2(K+\alpha\kappa^2)\,\mathrm{sn}^4(\gamma_2\eta,m_1)} \end{aligned} \tag{25}$$

we get a nonlinear dispersion relation as a function of conformal time. The conformal time interval (η^-, η^+) with

$$\eta^{\mp} = \gamma_2^{-1} F\left(\left[\frac{\alpha\kappa^2+2\sqrt{K^2-\alpha^2}-K\mp\sqrt{(K+\alpha\kappa^2)^2-4\alpha^2}}{2(\alpha\kappa^2-K)}\right]^{1/2}, m_1\right),$$

delimited by the two zero points of coefficient $q(\eta)$ of perturbation Eq. (9), determines the time interval of propagation of perturbation in the acoustic pattern. Dispersion relation (25) enables us to determine the phase velocity

$$\begin{aligned}
v_{\rm ph} &= 4\alpha\gamma_1^2\sqrt{\frac{(K+\alpha\kappa^2)(\kappa^4-1)}{4K+3\alpha\kappa^2}} \\
&\quad\times \frac{\mathrm{cn}^2(\gamma_2\eta,m_1)\,\mathrm{sn}^2(\gamma_2\eta,m_1)}{\alpha^2-2\alpha(\alpha-2\gamma_1^2\kappa^2)\,\mathrm{sn}^2(\gamma_2\eta,m_1)-4\gamma_1^2(K+\alpha\kappa^2)\,\mathrm{sn}^4(\gamma_2\eta,m_1)}\,,
\end{aligned} \tag{26}$$

as well as the group velocity

$$\begin{aligned}
v_{\rm g} &= -\frac{4\gamma_1^2}{3}\sqrt{\frac{4K+3\alpha\kappa^2}{(K+\alpha\kappa^2)(\kappa^4-1)}} \\
&\quad\times \frac{\mathrm{cn}^2(\gamma_2\eta,m_1)\,\mathrm{sn}^2(\gamma_2\eta,m_1)}{\Big[\alpha^2-2\alpha(\alpha-2\gamma_1^2\kappa^2)\,\mathrm{sn}^2(\gamma_2\eta,m_1)-4\gamma_1^2(K+\alpha\kappa^2)\,\mathrm{sn}^4(\gamma_2\eta,m_1)\Big]^2} \\
&\quad\times\Big[\alpha^2(\alpha-2K\kappa^2-3\alpha\kappa^4) \\
&\quad+2\alpha\Big(-4K\gamma_1^2+\alpha^2(3\kappa^4-1)+4\gamma_1^2\Big(2K^2\kappa^2-2\alpha\kappa^2\Big(-K\gamma_1^2(1+\kappa^4)\Big)\Big) \\
&\quad\times\,\mathrm{sn}^2(\gamma_2\eta,m_1)+\alpha^2\kappa^2(1+\kappa^4)+K\alpha(1+3\kappa^4)\Big)\,\mathrm{sn}^4(\gamma_2\eta,m_1)\Big].
\end{aligned} \tag{27}$$

(ii) $\alpha=1$ and $\kappa>1$. The temporal component of the phase of solution (16) is a linear function of conformal time of constant frequency ω. The dispersion relation

$$\omega=\sqrt{\frac{k^2-7K}{3}} \tag{28}$$

is a nonlinear function of wave number k, as well as of spatial curvature K. From the dispersion relation one can determine the phase velocity $v_{\rm ph}=\sqrt{(k^2-7K)/3}/k$ and the group velocity $v_{\rm g}=k/\sqrt{3(k^2-7K)}$. Their product is of fixed value $v_{\rm ph}v_{\rm g}=1/3$, analogously to the case of acoustic wave in models with negative curvature and without a cosmological constant [5]. The derivative of group velocity in respect to wavelength, called dispersion [30], is

$$\frac{dv_{\rm g}}{d\lambda}=-\frac{14\pi}{\sqrt{3}}\lambda\left(4\pi^2+7\lambda^2\right)^{-3/2}\,. \tag{29}$$

As the derivative is negative for all wavelengths ($\lambda=2\pi/k$), the dispersion is anomalous. It is graphically plotted in Fig. 9. A similar behaviour is encountered in the radiation model with negative curvature and zero cosmological constant.

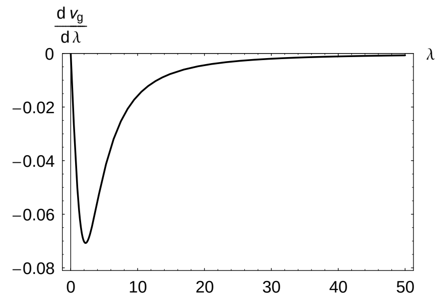

Fig. 9. Dispersion.

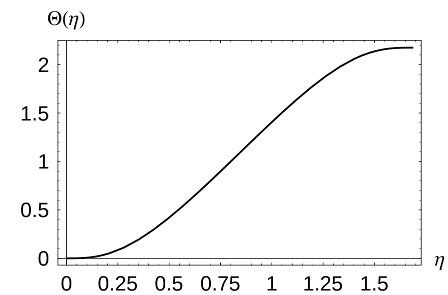

Fig. 10. Phase $\Theta(\eta)$ for $\alpha = 2$ and $\kappa = 2$.

(iii) $\alpha > 1$ and $\kappa > 1$. The temporal component of phase (18) of solution (17) is a nonlinear function of conformal time as plotted in Fig. 10.

In this case the phase reveals a qualitatively similar behaviour as for $\alpha \in (0,1)$ and $\kappa > 1$. The local frequency is a time derivative of the phase (18) and

$$\omega(\eta,\kappa) = \sqrt{\frac{(\kappa^2+1)(K+\alpha\kappa^2)}{\kappa^2-1}}\,\frac{(\kappa^2-1)\,\mathrm{sc}^2(\sqrt{2\alpha}\,\eta,m_2)}{2+(\kappa^2-1)\,\mathrm{sc}^2(\sqrt{2\alpha}\,\eta,m_2)} \tag{30}$$

is a nonlinear function of wave number k and conformal time η. The conformal time interval

$$(\eta_1,\eta_2)=\left(F\left(2\sqrt{\frac{\alpha}{K+\alpha(2+\kappa^2)}},m_2\right),\sqrt{2}K\frac{(m_2)}{\sqrt{\alpha}}-F\left(2\sqrt{\frac{\alpha}{K+\alpha(2+\kappa^2)}},m_2\right)\right),$$

delimited by zero points of the coefficient of $\mu_k(\eta)$ in perturbation Eq. (11), determines the time of propagating perturbation in the acoustic mode. Out of frequency (30) one can calculate the phase velocity

$$v_{\mathrm{ph}} = \sqrt{\frac{K+\alpha\kappa^2}{K+3\alpha\kappa^2}\,\frac{\kappa^2+1}{\kappa^2-1}}\,\frac{(\kappa^2-1)\,\mathrm{sn}^2(\sqrt{2\alpha}\,\eta,m_2)}{2+(\kappa^2-1)\,\mathrm{sn}^2(\sqrt{2\alpha}\,\eta,m_2)}\,, \tag{31}$$

as well as the group velocity

$$v_{\rm g} = \frac{1}{3\alpha}\sqrt{\frac{K+3\alpha\kappa^2}{(K+\alpha\kappa^2)(\kappa^4-1)}}\frac{\mathrm{sn}^2(\sqrt{2\alpha}\,\eta, m_2)}{[2+(\kappa^2-1)\,\mathrm{sn}^2(\sqrt{2\alpha}\,\eta, m_2)]^2}$$
$$\times\ [4K\kappa^2+2\alpha(3\kappa^4-1)-(\kappa^2-1)(2K+\alpha(1+2\kappa^2-\kappa^4))\,\mathrm{sn}^2(\sqrt{2\alpha}\,\eta, m_2)]\,. \tag{32}$$

6. Summary

All solutions of perturbation equations, describing perturbations in cosmological FLRW models with negative curvature, positive cosmological constant, and filled with radiation are presented. All solutions are analytical ones. The solutions are of oscillatory or non-oscillatory character, depending on the value of parameters $\{\alpha, \kappa\}$. For the wave solutions the phase and group velocities are determined, as well as the dispersion relations. The dispersion of wave solutions is investigated and revealed to be anomalous. Propagation of scalar perturbations in the presence of curvature and cosmological constant is fairly complex phenomenon. Dispersion effects should be taken into account to properly understand the CMBR fluctuation spectrum. Each solution is graphically plotted in respective diagrams.

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