Vol. 36 (2005) ACTA PHYSICA POLONICA B No 8

IS DISCRETIZATION OF THE STOCHASTIC CONTINUOUS-TIME PROCESSES A REASON FOR THE NON-LINEAR LONG-TERM AUTOCORRELATIONS OBSERVED IN HIGH-FREQUENCY FINANCIAL TIME-SERIES?*

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(Received April 15, 2005)

By using regular time-steps we define discrete-time random walks and flights on subordinate (directed) Continuous-Time Hierarchical (or Weierstrass-Mandelbrot) Walks and Flights, respectively. The obtained results can be considered as a kind of warning that indicates some persistent, non-linear, long-term autocorrelations (artifacts) accompanying the recording of empirical high-frequency financial (and probably other types of) time-series by regular time-steps, indeed.

PACS numbers: 05.45.Tp, 02.50.–r, 05.40.–a

1. Motivation

We consider a possible reason for non-linear, long-term autocorrelations present in empirical and our synthetic high-frequency (HF) financial time-series. The autocorrelations present in empirical time-series, which were assumed by physicists as a stylized fact, were studied by them since more than one decade [1–4]. In distinction the synthetic time-series were obtained by us from the recently developed one-dimensional Continuous-Time Hierarchical Walks (CTHW) [5, 6] and analogous Continuous-Time Hierarchical Flights (CTHF) [5]. It seems that the power-law autocorrelations discovered by discretization of the time-series obtained within the CTHW and analogous log-normal ones found for the CTHF, have a persistent character, *i.e.* they seem to be unavoidable artifacts for the HF time series yet not only of financial type.

* Presented at the First Polish Symposium on Econo- and Sociophysics, Warsaw, Poland, November 19–20, 2004.

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2. The model

In this section we consider the above mentioned two types of the hierarchical (Weierstrass–Mandelbrot) models which cover two types of representations of empirical high-frequency financial time-series and hence two types of the corresponding non-linear autocorrelations (power-law and log-normal ones which we call indeed the 'long-term autocorrelations' observed in the same time-windows).

2.1. Self-affine CTRW formalism and main result

The present, generalized version of the CTRW model is the combined one defined by the non-separable hierarchical (or Weierstrass–Mandelbrot) walk which can be occasionally (randomly) intermitted by momentary localizations (WWRIL); the localizations themselves are also described by the Weierstrass–Mandelbrot (or hierarchical) process. It should be noted that the steps of the walk as well as the momentary localizations are uncorrelated. For example, the typical part of synthetic trajectory of the walker, which can be obtained within the WWRIL by simulation, is briefly shown in Fig. 1; note that the horizontal intervals represent localizations while tangent one the walks. This approach makes it possible to study by (hierarchical) stochastic (Monte Carlo) simulations the whole spatial-temporal region,

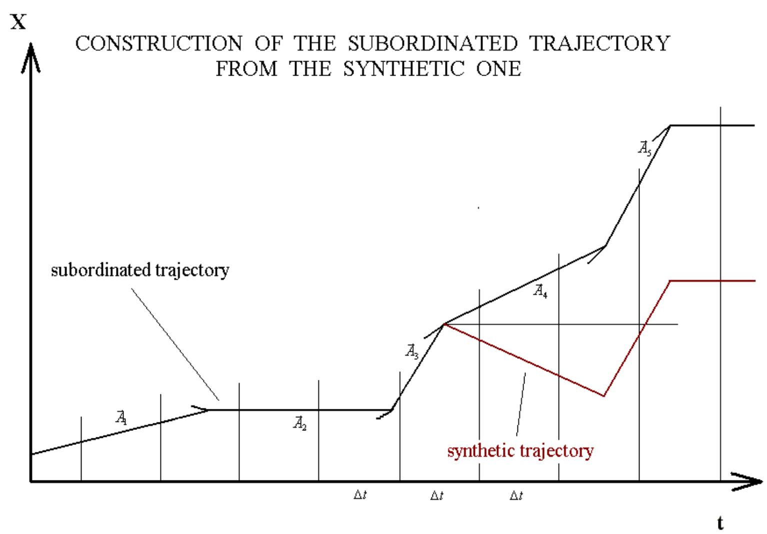

Fig. 1. Synthetic (lower) and subordinated (upper) trajectories in continuous and discretized time (where discretization step is denoted by Δt). As it is seen, the subordinated trajectory (which is identical with the synthetic one until the beginning of vector $\vec{A}_4$) can expand only in the positive X-direction.

while analytically it is possible to study only the initial, pre-asymptotic and asymptotic ones but not very important intermediate region. We found that the following two-stage procedure of extracting the non-linear long-term autocorrelations between single steps of the walker from the WWRIL (which originally, as synthetic trajectory in continuous-time, does not exhibit this type of correlations) is necessary: *(i)* discretization of time, *(ii)* transformation from the synthetic to the subordinated (one-sided or dual) walk; in particular, this second stage requires explanation.

Application of the time discretization procedure with time-step (horizon) Δt automatically introduces discretization of the space variable by assuming $\Delta X(t_n) = X(t_n + \Delta t) - X(t_n),\ n = 0, 1, 2, \ldots$, as the corresponding spatial single-steps (here $X(t_n)$ and $X(t_n + \Delta t)$) are the positions of the walker on the synthetic continuous-time random walk trajectory at successive discrete time instants $n = 0, 1, 2, \ldots$. This is how the discretized synthetic trajectory was defined (*cf.* Fig. 1) while the subordinated trajectory (also shown there) can develop only in the positive X-direction as it was done by transition from

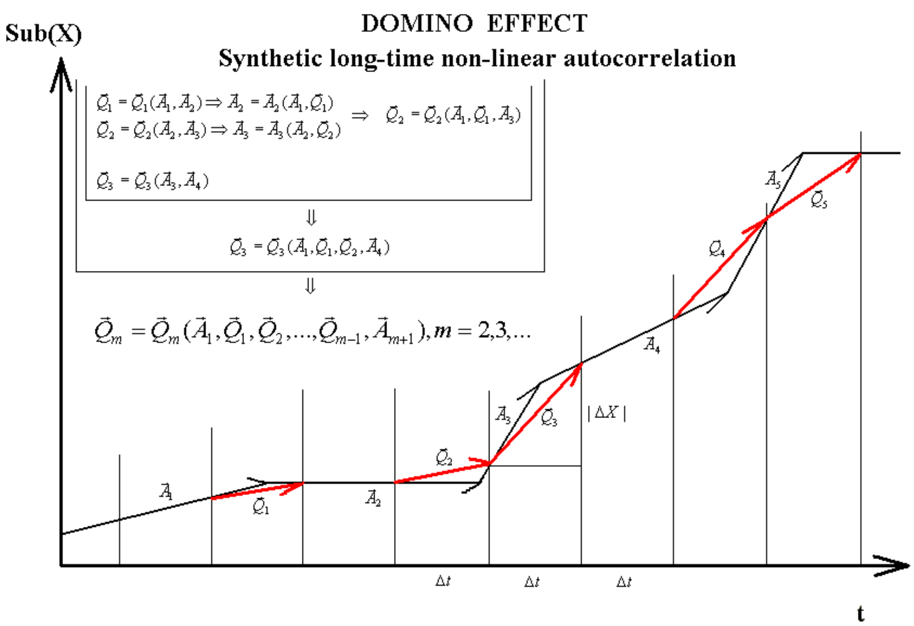

Fig. 2. Plot of a single realization of a basic synthetic, subordinate (directed) continuous-time trajectory (defined by the sequence of vectors $\vec{A}_1$, $\vec{A}_2$, $\vec{A}_3$, $\vec{A}_4$, $\vec{A}_5, \ldots$) and synthetic, discretized one (defined by a sequence of adequately chosen, characteristic vectors $\vec{Q}_1, \ldots, \vec{Q}_2$, $\vec{Q}_3, \ldots, \vec{Q}_4$, $\vec{Q}_5, \ldots$). The vertical axis denoted as Sub(X) (*i.e.* subordinate X) is defined as: $\mathrm{Sub}(X(n\,\Delta t)) = \sum_{j=0}^{n-1} \mid \Delta X(j\,\Delta t) \mid$.

$\Delta X(t_n)$ (connected with the discrete synthetic trajectory) to its absolute value $\mid \Delta X(t_n) \mid$. As it is seen, this one-sided random walk obeys the feature of non-negative of their increments; in the case of Lévy processes this would relate to Lévy subordinates considered in [9, 10].

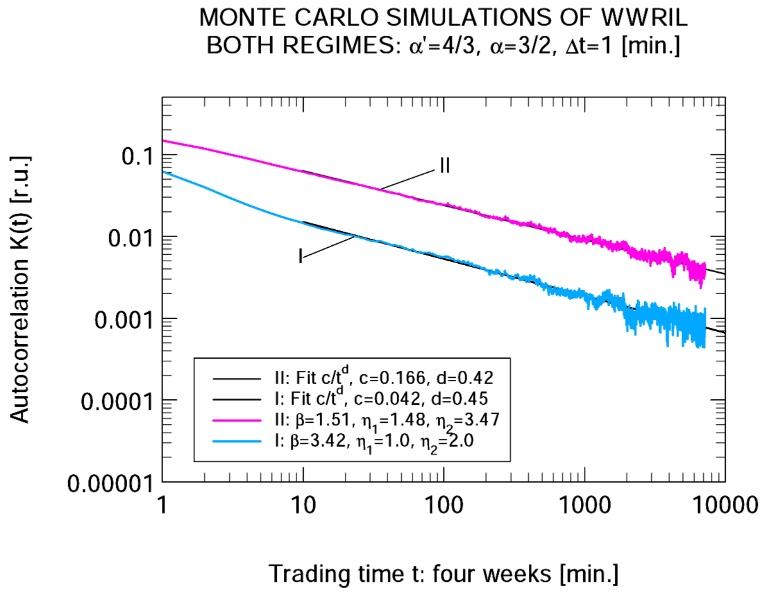

Fig. 3. Autocorrelation of the centered absolute variations of the stock price (or the walker centered absolute variations of the single step displacement defined by $\mid \Delta X(t) \mid - \langle\mid \Delta X(t) \mid\rangle$, where time $t = n\,\Delta t$, $n = 0, 1, 2, \ldots$, and $\Delta X(t) = X(t+\Delta t) - X(t)$), given by $K(t) = \langle\mid \Delta X(0) \cdot \Delta X(t) \mid\rangle - \langle\mid \Delta X(0) \mid\rangle \cdot \langle\mid \Delta X(t) \mid\rangle$ for the synthetic high-frequency time-series. This quantity was obtained by our time-discretization procedure within the Weierstrass–Mandelbrot walks randomly intermitted by localizations (WWRIL) for: (I) Gaussian, and (II) non-Gaussian regimes of the stock price. The slopes of both curves (defined by exponent d for almost three decades) differ but slightly (*viz.* for case I: $d \approx 0.42$, and for case II: $d \approx 0.45$). The dynamic exponents, η_1 and η_2, define the evolutions of the second and fourth moments of the stock prices (displacements) $X(t)$ (and they depend on the partial dynamic exponents α', α, β). The temporal partial dynamic exponents α' and α describe the localization and time-dependence of the walking state, respectively. The spatial exponent β defines space penetration within the walking state.

The basic continuous-time series obtained from these stochastic simulations is shown in Figs. 1 and 2 by a sequence of vectors $\vec{A}_1$, $\vec{A}_2$, $\vec{A}_3$, $\vec{A}_4$, $\vec{A}_5, \ldots$, connecting the turning points of a single realization of a subordinate random walk trajectory expanding in positive X-direction as we study only the absolute values of the stock price variation $\mid \Delta X \mid$. This simulation is supported by the waiting-time distribution which is the main quantity of our two-state (walking-localization) model. The states of the model are again characterized by their own waiting-time distributions (which give indeed the main distribution in the form of a weighted sum). Each single-state waiting-time distribution is a hierarchical, geometrically weighted superposition of partial waiting-time distributions, describing the regular spatial-temporal processes (connected with single hierarchy generations) which are already easy to simulate [6, 11].

The synthetic (derivative), discrete time-series was obtained by discretization of the original (basic) continuous-time series at a fixed time horizon Δt (shown in Fig. 2 by the sequence of characteristic vectors $\vec{Q}_1, \ldots, \vec{Q}_2$, $\vec{Q}_3, \ldots, \vec{Q}_4$, $\vec{Q}_5, \ldots$).

As it is seen, the turning points of the basic continuous-time series are, in general, incommensurate with the analogous points supplied by the discrete time series. The autocorrelation function $K(t)$ (defined in the caption to Fig. 3) has been studied versus time just for this discrete time-series.

As it is shown in Fig. 3, the autocorrelation $K(t)$ exhibits a power-law relaxation over more than three decades both for the Gaussian and non-Gaussian processes.

3. Further results and concluding remarks

Hitherto, we studied the representation of financial tick data by the continuous-time Weierstrass–Mandelbrot walk trajectory while in this section we consider the same set of data points represented by the continuous-time Weierstrass–Mandelbrot flight trajectory. In the latter case, the displacement of the walker or the price variation is shown by the vertical vector (instantaneous jumps) and not by the tangent one (the walk having a finite velocity such as, for example, that shown by vectors $\vec{A}_j,\ j = 1, 2, \ldots,$ in Figs. 1 and 2).

In Fig. 4 synthetic tick data (full circles) together with synthetic and subordinated continuous-time discretized trajectories are shown. As it is seen, vector $\vec{Q}_{n+1}$ depends on vector $\vec{Q}_n$ *i.e.*, the domino effect is also possible for this type of representation of this tick data.

In Figs. 5 and 6 the autocorrelation function $K(t)$ exhibits log-normal autocorrelations after high-frequency time discretization (at time-horizon $\Delta t = 1$ min). It should be noted that these correlations can be mistaken locally for a power-law [7, 8]. Again these correlations have long-time, persistent character

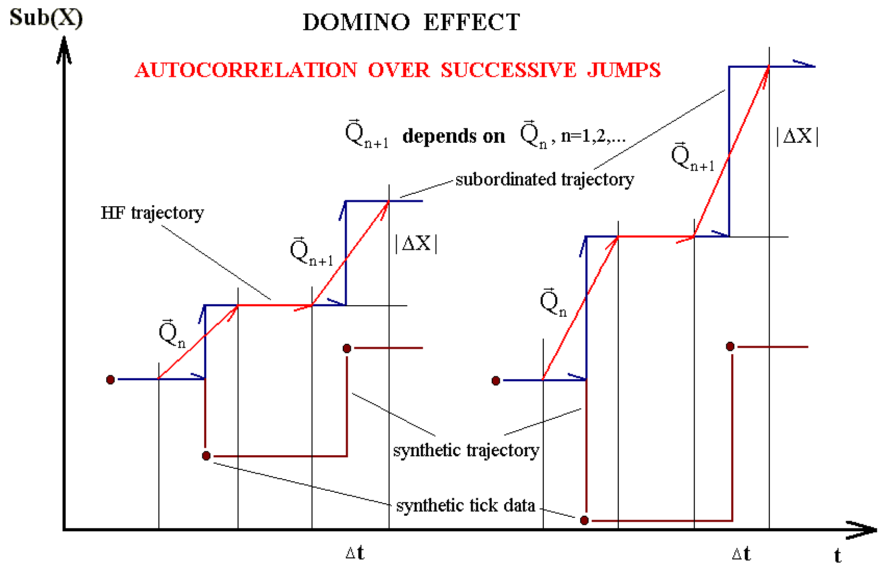

Fig. 4. Synthetic tick data (full circles) shown together with synthetic and subordinated continuous-time and discretized trajectories.

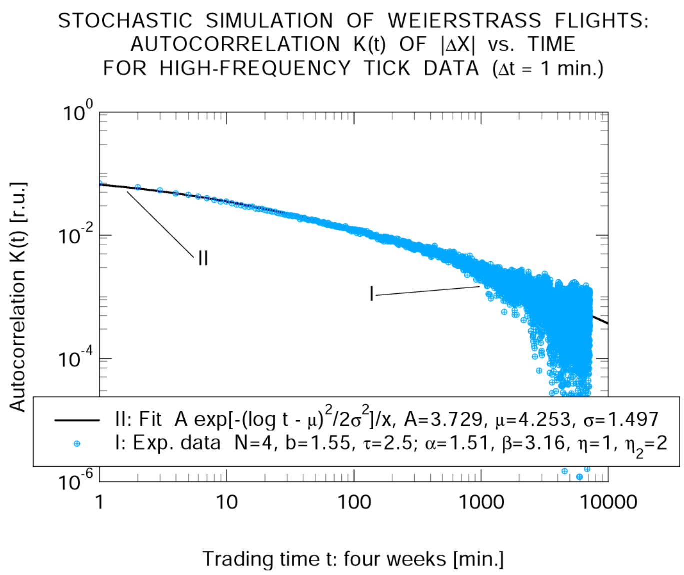

Fig. 5. The log-normal dependence of the autocorrelation function $K(t)$ (defined in the caption of Fig. 3) *vs.* time within four weeks time-window for the Gaussian price variations.

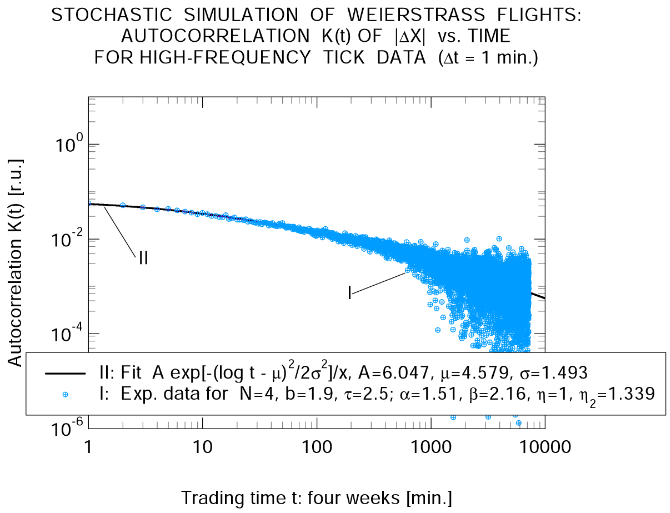

Fig. 6. The log-normal dependence of the autocorrelation function $K(t)$ (defined in the caption of Fig. 3) *vs.* time within four weeks time-window for the non-Gaussian price variations.

It should be emphasized that we studied a kind of random walk on a random walk and random walk on a random flight [12, 13] and obtained results which require further studies to better understand them, even in the context of arbitrary time-series.

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