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FROM RANDOMNESS TO PERIODICITY — THE EFFECT OF POLARIZATION IN THE MINORITY GAME STRATEGY SPACE*

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Properties and behavior of modified minority game are analyzed. It appears that results of the game depend strongly on the way how we draw strategies for players. The probability that given strategy will be chosen is determined by the polarization parameter P. This parameter differs between strategies that go along or against the existing market trend. Strong polarization of the space leads to the periodic dynamics and, finally, for negative P values to the domination of the single strategy in the system. When P is changed the variability of the process decreases, showing kind of the phase transition region. The dependence of the variability on the polarization parameter can be understood on the basis of the crowd–anticrowd theory.

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1. Minority game

Minority game is a simple model that shows quite a complex dynamical behavior. The game is based on the competition between a single agent and the whole community [1–3]. Each of N different players makes one of two decisions 1 or -1 — this is a single step of the game. A single player wins when its decision is in the minority. It means that majority of players fail at each turn. The decision of each player is according to one of strategies he owns. Each player has several strategies, that give concrete choice $s_i = \pm 1$ of its state, depending on the sequence of the last m winning decisions. For each strategy its individual score is counted. The score increases each time when the strategy would win in the given game step, irrespective of whether it is actually in use or not. Each player uses

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the strategy of the highest actual score. The total result of the game A is counted as a sum of all individual decisions, and as a function of time (number of game steps) exhibits very interesting behavior. It may model economic market, sociological or biological systems in which various objects compete for limited resources.

One memory state of length m consists of last m winning choices ± 1. There are 2^m possible memory states, and 2^{2^m} possible strategies. It can be shown that the important parameter is the number of a different strategy class, that is 2^m [3]. The total result of the game A, as a function of time, looks as an outcome of a random process. Its characteristics depend on the parameters of the game. The main control parameter that decides about the character of the process is the relative size of reduced strategy space $\alpha = 2^m/N$ [4–8]. Depending on the value of α system is in the frozen $\alpha > \alpha_{\rm c}$ or unfrozen symmetric phase $\alpha < \alpha_{\rm c}$. The critical value $\alpha_{\rm c}$ is a function of the number of strategies per agent, and for two strategies for each agent it has the value $\alpha_{\rm c}(2) = 0.3374\ldots$ In the classical formulation, strategies are drawn from a set of all possible strategies with uniform probability. Below we study the case when this probability depends on the strategy. It appears that the character of the play changes with this probability.

2. Polarization of strategy space

Giardina and Bouchaud [9] proposed a marked model, based on the minority game idea, but much more complex. Apart from some price formation mechanism, close to that of the real market, they proposed also some bias in the strategy space. They observed three different regimes of game behavior, depending on the value of two parameters: the first controlling the price formation, and the second changing the strategy space polarization. In this article we show that simple modification of the standard minority game is enough to obtain different game regimes. The strategy space is polarized in the following way. A memory vector $s_1, \ldots s_m$ of length m has polarization $\mu = \sum^m_{i=1} s_i$, where $s_i = \pm 1$ denotes the choice of minority in the i-th game step. The polarization μ thus characterizes trend visible in the last m outcomes of the game. Its value changes from $-m$ to m. The strategy is given by determination of a decision for each individual memory state. These decisions are given by the following probability

$$p(s_k) = \frac{1}{2}\left(1 + s_k \frac{P\mu_k}{m}\right), \tag{1}$$

where μ_k is the polarization of the k-th memory state. When strategies are drawn according to the defined above probability (1) with negative P values, there is a larger number of them that decide to play opposite to μ.

When the choice of the majority of players is opposite to μ, the winning state agrees with the sign of μ, thus supporting trend of the game. When P is positive the majority of drawn strategies will play in accordance with the trend, so the winning choice will be the opposite one. This means that positive values of P favor antipersistent dynamics. This may create periodic or quasi periodic game processes. Despite the sign of the parameter P, its any nonzero value changes the structure of drawn strategies. It is clear that the variability of the resulting process changes with P. The character of this change depends on the parameter α. Some of its tendencies can be deduced on a basis of crowd–anticrowd theory [10–13], as will be shown below.

3. Results for the nonzero polarization parameter *P*

Variability of the signal is calculated as

$$\sigma = \left(\left\langle \frac{1}{K} \sum_{i=1}^{K} \left(A_i - \frac{1}{K} \sum_{j=1}^{K} A_i \right)^2 \right\rangle \right)^{\frac{1}{2}} , \tag{2}$$

where K is the number of the time evolution points in the single game. A_i is the game total result and it is equal to the sum over all decisions of players in the i-th game step. The variability σ is averaged over forty different game realizations, denoted in (2) by $\langle\ \rangle$. Results for low values of α are presented in Figs. 1 and 2, and for high α in Fig. 3. All data below are calculated in the case of two strategies per one player.

When $\alpha < \alpha_{\rm c}$ the variability decreases as a function of $|P|$ down to the value close to zero, and then slightly increases again (see Fig. 1). This behavior is almost symmetric for positive and negative values of P, however for small values of m, *i.e.* for $m = 1$ or $m = 2$ the whole curve is slightly moved in the direction of negative values of P. The variability rescaled by the square root of the player number $\sigma/\sqrt{N}$ lies on the universal curve for each given value of α. For higher values of α this curve becomes more and more symmetric, and then for $\alpha \approx \alpha_{\rm c}$ it again becomes nonsymmetrical. Even if the value of σ does not depend on the sign of the parameter P, the signal A as a function of time looks differently for positive and negative values of P. It can be seen in Fig. 2, that for positive value of P the game is almost periodic, with strongly anticorrelated signal. For large, negative P signal has positive correlation, and even when it changes periodically it has a tendency to stay at one level for longer time. Both processes can be compared with the typical signal time dependence for $P = 0$ (Fig. 2). For $\alpha > \alpha_{\rm c}$ variability σ increases with $|P|$ (Fig. 3) for P larger than -0.25. The main features of the variability behavior may be explained by the use of the crowd–anticrowd theory.

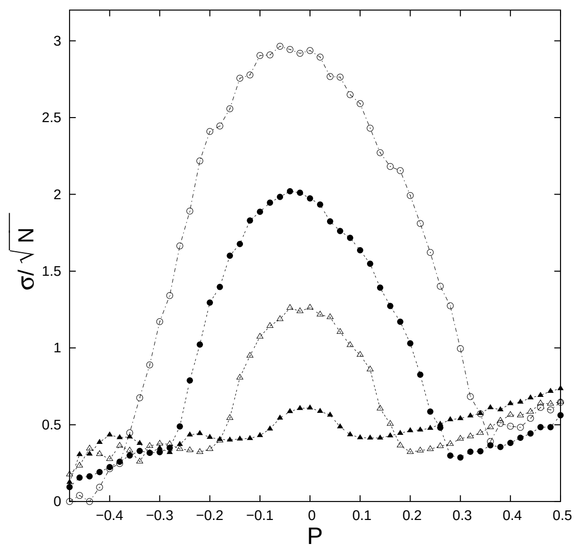

Fig. 1. Variability as a function of polarization parameter P for $\alpha < \alpha_c$. Open points are plotted for $m = 3, N = 200$, solid points for $m = 4, N = 200$, open triangles for $m = 5, N = 200$, and solid triangles for $m = 6, N = 200$.

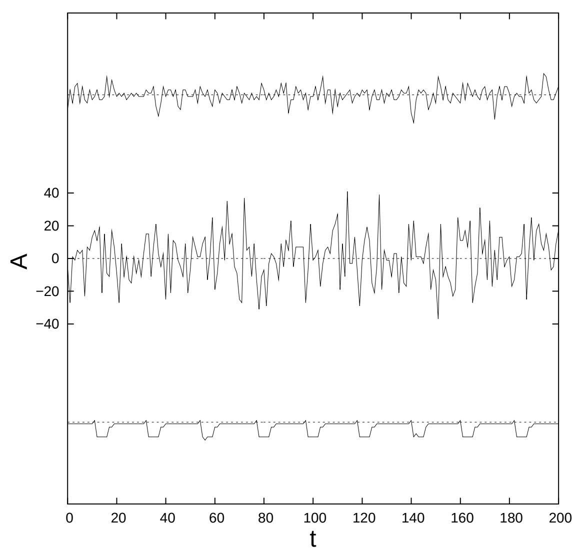

Fig. 2. Total signal A of the game as a function of time for different values of P. Top line is plotted for $m = 5, N = 300, P = 0.4$, center line for $m = 5, N = 300, P = 0$, and the bottom line for $m = 5, N = 300, P = -0.3$. The top and bottom lines are moved along the vertical axis, their local zero levels are plotted in dashed lines.

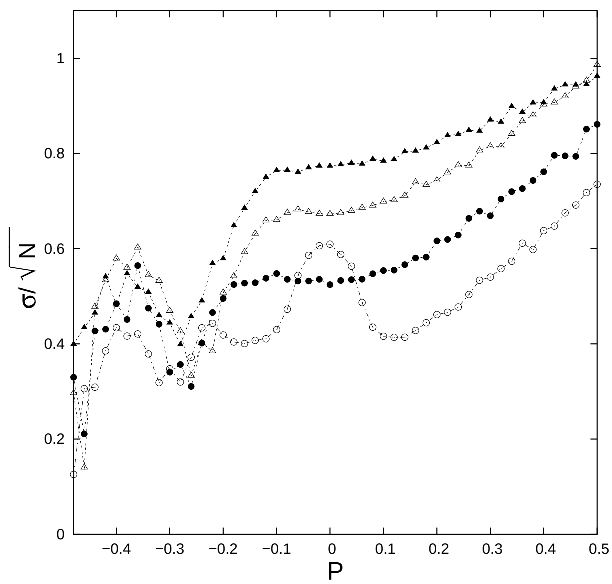

Fig. 3. Variability as a function of polarization parameter P for $\alpha > \alpha_c$. Open points are plotted for $m = 6, N = 200$, solid points for $m = 7, N = 200$, open triangles for $m = 8, N = 200$, and solid triangles for $m = 8, N = 100$.

4. Crowd–anticrowd theory

The behavior of the variability can be described on the basis of so called crowd–anticrowd theory [10–13]. The theory is built on the observation that the variability is a function of difference between two opposite strategies, and that the income from each pair of the opposite strategy cancels out, hence all we get in the variability comes from unpaired strategies. We can write

$$\sigma^2 = \sum_R (n_R - n_{R'})^2 \, , \tag{3}$$

where n_R is the number of players that use strategy from the class R and $n_{R'}$ is the number of players using opposite strategy to R, such that almost always makes decision contrary to that of R. As it was mentioned before, we are speaking here not about just strategies, but about classes of strategies. Within each class the strategies do not differ much, they differ by one or several decisions. It can be shown that there is 2^m such different classes [2], and then 2^m opposite strategies. Together we have the set of 2^{m+1} strategies. The assumption (3) means that the only classes that are correlated are opposite ones. For all other pairs correlations average out, because sometimes strategies lead to opposite, and sometimes to the same, decisions. To use the formula (3), some assumption about a possible distribution of numbers n_R and $n_{R'}$ has to be done. There are two different distributions that are used. First is the uniform distribution of n_R and $n_{R'}$ — $p(n) = 1/2^{m+1}$. In the second, we assume that the most numerous strategy is opposite to the least numerous one and so on. We order n_R in decreasing sequence. In the general case, where $P = 0$, numbers n in decreasing order are

$$n(R) = \left(1 - \frac{R-1}{2^{m+1}}\right)^2 - \left(1 - \frac{R}{2^{m+1}}\right)^2 , \tag{4}$$

which has been written for the case of two strategies per one player. The formula above is easy to explain, assuming independent choices of the strategies. Using this formula with the assumption of the largest number n representing opposite strategy to that of smallest n, we get [12]

$$\sigma_{\mathrm{low}} = \frac{N}{\sqrt{3}2^{\frac{m}{2}+1}} \left[1 - 2^{-2(m+1)}\right]^{\frac{1}{2}} , \tag{5}$$

whereas the first assumption about the uniform distribution, gives in the limit of large N

$$\sigma'_{\mathrm{low}} = \frac{N}{\sqrt{3}2^{\frac{m+3}{2}}} [1 - 2^{-2(m+1)}]^{\frac{1}{2}} . \tag{6}$$

Simulation data lie between these lines.

For higher values of $\alpha > \alpha_{\mathrm{c}}$ calculations are a bit different. There are so many available classes of strategies that only some of them may be used, hence there we assume that each strategy used is played only by one player, hence there are N strategies in play. If such procedure is applied one gets the third formula for variability, valid for higher α range

$$\sigma_{\mathrm{high}} = \frac{\sqrt{N}}{2} \left[1 - \frac{N}{2^{m+1}}\right]^{\frac{1}{2}} . \tag{7}$$

In order to make use of the above theory in our case the frequency numbers (4) have to be changed. If probabilities (1) are used then the number of drawn strategies depends on their polarization. For example for $P > 0$ there are more strategies with decision in agreement with the trend, thus the winning strategies are the other ones. Such tendency means that the strategies that are less numerous are the first to be chosen by players. With such assumption we have new numbers of the strategy rank. In the simplest case of $m = 1$ we have $2^{m+1} = 4$ different classes

$$\begin{aligned}
n_1 &= (1-P)^2[(1-P)^2 + 4(1-P^2) + 2(1+P)^2] , \\
n_2 &= (1-P^2)[3(1-P^2) + 2(1+P)^2] , \\
n_3 &= (1-P^2)[(1-P^2) + (1+P)^2] , \\
n_4 &= (1+P)^4 .
\end{aligned} \tag{8}$$

Now the procedure of pairing $R = 1; R' = 4$ and $R = 2; R' = 3$ may be applied, and after using (3) we obtain a polynomial of 8-th rank, which is plotted in Fig. 4.

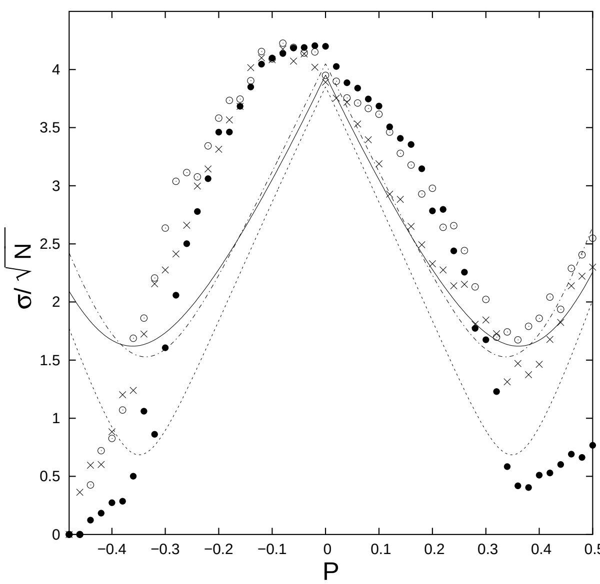

Fig. 4. Crowd–anticrowd calculations with varying parameter P. Solid points represent $m = 3, N = 400$ data, open points $m = 2, N = 200$, and crosses $m = 1, N = 100$. It can be seen how all these data, all calculated for the same parameter α accumulate close to one universal curve. Solid curve is plotted for $m = 1$, by pairing first and the last in the rank occupancies. Dashed line is obtained by the same method, but for $m = 2$. Dash-dotted line is calculated for $m = 1$, and for uniform distribution of pairs.

It can be seen that even with so strong assumption we are able to recover main features of the dependence of volatility σ on polarizability P. It can be also seen that more detailed analysis is needed to obtain more precise description. When the second approach is used -— equal pair probability, the resulting curve lies slightly above the first one. The data for α above the critical value needs more general definition of n_R.

5. Conclusions

Classical minority games have been modified by introduction of the polarization parameter P dependent probability, with which given strategy is drawn. As a result, there are more trend supporting strategies in the play, or more strategies that are against existing trend. It appears that variability of the game signal strongly depends on the parameter P. This dependence is qualitatively described by the use of the rough version of crowd–anticrowd theory. To obtain more exact quantitative results more detailed analysis is needed. Interestingly, both signs of the parameter P lead to the oscillating signal, where positive or negative correlations are present, depending on the sign of P. The analysis of the dynamics in this model would answer the question what decides about oscillating or random type of the signal.

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