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VALUE AT RISK IN THE PRESENCE OF THE POWER LAWS*

Piotr Jaworski

Institute of Mathematics, Warsaw University
Banacha 2, 02-097 Warszawa, Poland

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The aim of this paper is to determine the Value at Risk (VaR) of the portfolio consisting of several long positions in risky assets. We consider the case when the tail parts of distributions of logarithmic returns of these assets follow the power law of the same degree and the lower tail of associated copula C follows the power law of degree 1. We provide the asymptotic formula for Value at Risk and determine the optimal portfolio. We show that the part of the capital invested in the i-th asset should be equal to the conditional probability that the drop of the value of the i-th asset will be smaller than the others under the condition that the value of the all assets will be smaller than c times their initial value ($c \ll 1$).

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1. Introduction

1.1. Motivation

Decision making in finance is decision making under uncertainity. The outcome of present decisions depends on quantities (like future stock prices or exchange rates), which are yet unknown. The usual approach is to represent such quantities by random variables. As a consequence, the outcome of the decision (*e.g.* the future value of the investment) is a random variable too. The possible random variability adds the *risk dimension* to the problem. A natural question is how to measure risk. In this paper we deal with Value at Risk, nowdays one of the most popular risk measures.

Furthermore, in order to determine accurately the risk exposure one has to deal with the complexity of the problem. Usually the outcome is a function of several random quantities, so it is nescessary to describe properly their interdependences. In this paper we base on copulas, which are scaleless dependency measures of random variables.

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1.2. Copulas

We recall that a function

$$C : \langle 0,1\rangle^d \longrightarrow \langle 0,1\rangle\,,$$

is called a copula (see [20] §2.10) if for every $u = (u_1,\dots,u_d)$ and $v = (v_1,\dots,v_d)$ $(u_i, v_i \in \langle 0,1\rangle)$

$$\begin{aligned} (\exists i \ \ u_i = 0) \ &\Rightarrow\ C(u) = 0\,,\\ (\exists j\ \forall i \neq j \ \ u_i = 1) \ &\Rightarrow\ C(u) = u_j\,,\\ (\forall i \ \ \ u_i \le v_i) \ &\Rightarrow\ V_C(u,v) \ge 0\,, \end{aligned}$$

where $V_C(u,v)$ is the C-volume of the rectangle with lower vertex u and upper vertex v–$I(u,v)$.

$$V_C(u,v) = \varDelta^1_{v_1-u_1}\dots\varDelta^d_{v_d-u_d} C(u_1,\dots,u_d)\,,$$

where

$$\begin{aligned} \varDelta^k_h C(t_1,\dots,t_d) \ &= \ C(t_1,\dots t_{k-1}, t_k + h, t_{k+1},\dots t_d)\\ &\quad - C(t_1,\dots t_{k-1}, t_k, t_{k+1},\dots t_d)\,. \end{aligned}$$

Let $\mathcal{X}_i$, $i = 1,\dots,d$ be random variables defined on the same probability space $(\varOmega,\mathcal{M},\mathbb{P})$. The joint cumulative distribution $F_{\mathcal{X}}$ can be described using an appropriate copula $C_{\mathcal{X}}$ (see [20] Th. 2.10.11):

$$F_{\mathcal{X}}(x) = C_{\mathcal{X}}(F_{\mathcal{X}_1}(x_1),\dots,F_{\mathcal{X}_d}(x_d))\,,$$

where $F_{\mathcal{X}_i}$ are cumulative distributions of $\mathcal{X}_i$. Note that strictly increasing transformations of random variables $\mathcal{X}_i$ do not affect the copula. Indeed, if

$$\mathcal{X}'_i = f_i(\mathcal{X}_i), \ \ i = 1,\dots,d\,,$$

where f_i are strictly increasing (and so invertible), then

$$\begin{aligned} F_{\mathcal{X}'}(x) \ &= \ F_{\mathcal{X}}(f_1^{-1}(x_1),\dots,f_d^{-1}(x_d))\\ &= \ C_{\mathcal{X}}(F_{\mathcal{X}_1}(f_1^{-1}(x)),\dots,F_{\mathcal{X}_d}(f_d^{-1}(x_d)))\\ &= \ C_{\mathcal{X}}(F_{\mathcal{X}'_1}(x_1),\dots,F_{\mathcal{X}'_d}(x_d))\,. \end{aligned}$$

Therefore, if one is interested in tail dependence of random variables rather than in their *individual* distribution, then the proper choice is to study the copula. The more so, since the copula is uniquely determined at every point u such, that the equations $F_{\mathcal{X}_i}(x_i) = u_i$ have solutions.

From the probabilistic point of view every copula C is a joint cumulative distribution function of some probability measure μ_C on the unit rectangle with uniform margins. But, in certain cases the copula $C_{\mathcal{X}}$ is a joint cumulative distribution of some random variables defined on the same probability space as $\mathcal{X}_i$. Indeed let $\mathcal{P}_i$, $i = 1, \ldots, d$ be random variables defined by

$$\mathcal{P}_i = F_{\mathcal{X}_i}(\mathcal{X}_i)\,.$$

PROPOSITION **1.1.** *If the cumulative distributions $F_{\mathcal{X}_i}$ are continuous then:*

1. *$\mathcal{P}_i$ have uniform distributions on $\langle 0,1\rangle$;*
2. *The copula $C_{\mathcal{X}}$ is uniquely determined;*
3. *The d-dimensional cumulative distribution $F_{\mathcal{P}}$ coincides with the copula $C_{\mathcal{X}}$*

$$F_{\mathcal{P}}(p) = C_{\mathcal{X}}(p)\,.$$

Proof. The first two points are obvious. The third one can be proved in the same way as the third point of proposition 1 in [16].

1.3. Main results

Last years "Value at Risk" (VaR) became one of the most popular measures of risk in the "practical" mathematical finance (see for example Refs. [5,6,14,17,19,21,22]). Roughly speaking the idea is to determine the biggest amount one can lose on certain confidence level $1-\alpha$.

We shall deal with the following simple case. An investor has in his portfolio d risky assets which are highly dependent.

Let $S_{i,0}$ and $S_{i,1}$ be prices at the beginning and at the end of the period. Let ω_i be the part of the capital invested in the i-th asset. So the final value of the investment equals

$$W_1 = W_0 \sum \omega_i \frac{S_{i,1}}{S_{i,0}}\,.$$

If the distribution functions are continuous then, for the confidence level $1-\alpha$, VaR is determined by the condition

$$P(W_0 - W_1 \leq \mathrm{VaR}_{1-\alpha}) = 1-\alpha\,,$$

i.e. the probability that the loss will be greater than $\mathrm{VaR}_{1-\alpha}$, is smaller than $1-\alpha$.

In the beginnings the reserchers dealt with the case $\alpha = 0.05$. They assumed that the joint distribution of the returns is normal. Therefore, to calculate VaR, it was enough to estimate the means, variances and covariances ([22]). Later, the Basle Committee on Banking Supervision forced a switch to $\alpha = 0.01$ ([1,2]). The models based on the Gaussian law became inadequate. It was nescessary to take into consideration the power-like tails of the distributions of returns and to describe the dependence of returns of different assets by means of copulas (see [4, 8, 9]). We shall follow this line of research.

Let s_i be the logarithmic returns

$$s_i = \ln\left(\frac{S_{i,1}}{S_{i,0}}\right).$$

We assume that there exists such positive constant $\bar{x}$, that:

- s_i have the continuous cumulative distributions with power-like lower tails with the same index $\gamma > 2$ (this range covers the empirical exponents — see [18] §9.3, [4] §2.3.1 or [7, 11–13]). For $x < -\bar{x}$

$$F_i(x) = P(s_i \le x) = a_i(-x)^{-\gamma}.$$

- The lower tail part of the copula of s_i's is equal to a positive homogeneous function of degree 1 (compare [8, 15, 16]). For $q = (q_1, \dots q_d)$, such that $0 \le q_i \le a_i \bar{x}^{-\gamma}$ $C(q) = L(q)$, where

$$L : \langle 0, +\infty)^d \longrightarrow \langle 0, +\infty), \quad L(tq) = tL(q), \quad \text{for} \quad 0 \le t.$$

- The measure μ_L associated to L is absolutely continuous with respect to the Lebesque measure and it has a density which is continuous on the complement of the origin.

Under these assumptions we show in Section 5:

THEOREM **1.1.** *For α close enough to 0*

$$\mathrm{VaR}_{1-\alpha} = W_0 - W_0 \prod_{i=1}^{d} \omega_i^{g_i} \exp\left(-\left(\frac{L(a)}{\alpha}\right)^{1/\gamma}\right)\left(1 + O\left(\alpha^{1/\gamma}\right)\right),$$

where g_i are equal to elasticities of L

$$g_i = \frac{a_i}{L(a)} \frac{\partial L}{\partial a_i}(a).$$

The leading part of the above formula will be called the asymptotic VaR

$$A\mathrm{VaR}_{1-\alpha} = W_0 \left(1 - \prod_{i=1}^{d} \omega_i^{g_i} \,\exp\left(-\left(\frac{L(a)}{\alpha}\right)^{1/\gamma}\right)\right).$$

In the next section we show that in order to minimize the asymptotic VaR one should take ω_i equal g_i. Furthermore g_i's can be expressed in terms of the conditional probability:

$$g_i = P(s_j \le s_i | s_j \le -z)\,, \qquad j = 1, \ldots, d\,, \quad \text{for} \quad z > \bar{x}\,.$$

2. Auxiliary results

Since the density of μ_L is continuous outside the origin, L is differentiable outside the origin. Moreover, since L is homogeneous of degree 1 its first derivatives are homogeneous of degree 0, *i.e.* for $t > 0$ and $q \neq 0$

$$\frac{\partial L}{\partial q_i}(tq) = \frac{\partial L}{\partial q_i}(q)\,.$$

Next, let $\Delta_i(a)$ be the pyramid with a rectangular base

$$\Delta_i(a) = \left\{ q : 0 \le \frac{q_j}{a_j} \le \frac{q_i}{a_i} \le 1 \right\}.$$

Lemma 2.1. *For* $q \neq 0$

$$\mu_L(\Delta_i(q)) = q_i \frac{\partial L}{\partial q_i}(q)\,.$$

Proof. We show the formula for $i = d$, for all the others the proof is similar.

$$\begin{aligned}
\mu_L(\Delta_d(q)) &= \int_0^{q_d} \int_0^{s_d q_{d-1}/q_d} \cdots \int_0^{s_d q_1/q_d} \frac{\partial^n L}{\partial_{q_d} \ldots \partial_{q_1}}(s)\, ds_1 \ldots ds_d \\
&= \int_0^{q_d} \frac{\partial L}{\partial q_d}\left(\frac{s_d q_1}{q_d}, \ldots, \frac{s_d q_{d-1}}{q_d}, s_d\right)\, ds_d \\
&= \int_0^{q_d} \frac{\partial L}{\partial q_d}(q_1, \ldots, q_{d-1}, q_d)\, ds_d \\
&= q_d \frac{\partial L}{\partial q_d}(q)\,.
\end{aligned}$$

LEMMA **2.2.** *For $z > \bar{x}$ and $0 < i \leq d$*

$$P(s_j \leq s_i | s_j \leq -z,\ j = 1, \dots d) = \frac{a_i}{L(a)} \frac{\partial L}{\partial a_i}(a).$$

Proof.

$$P(s_j \leq s_i | s_j \leq -z,\ j = 1, \dots d) = \frac{P(s_j \leq s_i \leq -z,\ j = 1, \dots d)}{P(s_j \leq -z,\ j = 1, \dots d)}$$

$$= \frac{\mu_L(F_j^{-1}(q_j) \leq F_i^{-1}(q_1) \leq -z)}{L(F_1(-z), \dots, F_d(-z))}$$

$$= \frac{\mu_L(\varDelta_i(z^{-\gamma} a))}{L(z^{-\gamma} a)} = \frac{\mu_L(\varDelta_i(a))}{L(a)} = \frac{a_i}{L(a)} \frac{\partial L}{\partial a_i}(a)\,.$$

3. The probability of an excess of a loss over a security level

Let $c = e^{-z}$, where $z > \bar{x}$, be a fixed security level, then

$$P\left(\frac{W_1}{W_0} \leq e^{-z}\right) = P\left(\sum_{i=1}^{d} \omega_i e^{s_i} \leq e^{-z}\right) = \mu_C(V_z)\,,$$

where

$$V_z = \left\{ q : \sum_{i=1}^{d} \omega_i \exp\left(F_i^{-1}(q_i)\right) \leq e^{-z} \right\}$$

$$= \left\{ q : \sum_{i=1}^{d} \exp\left(z + F_i^{-1}(q_i) + \ln(\omega_i)\right) \leq 1 \right\}\,.$$

Note that all summands are smaller than 1. So we get the following estimation for coefficient q_i of a point q from V_z:

$$q_i \leq F_i(-z - \ln(\omega_i)) = a_i (z + \ln(\omega_i))^{-\gamma}\,.$$

Therefore,

$$V_z \subset I\left(0, \left(\frac{a_i}{(z + \ln(\omega_i))^{\gamma}}\right)\right) \subset \bigcup_{i=1}^{d} \varDelta_i((z + \ln(\omega_i))^{-\gamma} a)\,.$$

On the other hand the sum of positive weights ω_i is 1. Therefore, if for $i = 1, 2 \dots, d$

$$q_i \leq F_i(-z) = a_i z^{-\gamma}\,,$$

then q belongs to V_z *i.e.* we obtain that

$$I(0, z^{-\gamma}a) \subset V_z\,.$$

Moreover, the vertex $z^{-\gamma}a$ belongs to the border of V_z. Indeed

$$\sum_{i=1}^{d} \omega_i \exp\left(F^{-1}(z^{-\gamma}a_i)\right) = \sum_{i=1}^{d} \omega_i e^{-z} = e^{-z}.$$

In such a way we get the following estimates:

PROPOSITION **3.1.** *For* $z > \bar{x}$

$$z^{-\gamma}L(a) \le \mu_C(V_z) \le \sum_{i=1}^{d} (z + \ln(\omega_i))^{-\gamma} a_i \frac{\partial L}{\partial q_i}(a)\,.$$

Proof. Since $I(0, z^{-\gamma}a) \subset V_z$,

$$\mu_C(V_z) \ge \mu_C(I(0, z^{-\gamma}a)) = C(z^{-\gamma}a) = L(z^{-\gamma}a) = z^{-\gamma}L(a).$$

On the other hand $V_z \subset \cup_{i=1}^{d} \Delta_i((z + \ln(\omega_i))^{-\gamma}a)$, hence

$$\begin{aligned} \mu_C(V_z) &\le \sum_{i=1}^{d} \mu_C(\Delta_i((z + \ln(\omega_i))^{-\gamma}a)) \\ &= \sum_{i=1}^{d} \mu_L(\Delta_i((z + \ln(\omega_i))^{-\gamma}a)) \\ &= \sum_{i=1}^{d} (z + \ln(\omega_i))^{-\gamma} a_i \frac{\partial L}{\partial q_i}(a)\,. \end{aligned}$$

4. Value at risk

We start with the asymptotical improvement for the formula for $\mu_C(V_z)$.

PROPOSITION **4.1.** *For* $z > \bar{x} + \max\left(|\ln(\omega_i)|\right)$

$$\mu_C(V_z) = z^{-\gamma}L(a)\Big(1 - \frac{\gamma}{z} \sum_{i=1}^{d} \ln(\omega_i) g_i(a) + O\left(z^{-2}\right)\Big)\,,$$

where

$$g_i = \frac{a_i}{L(a)} \frac{\partial L}{\partial a_i}(a)\,.$$

Proof. First we show that

$$\mu_L(\Delta_1((z+\ln(\omega_1))^{-\gamma}a)\setminus V_z)=O(z^{-\gamma-2}).$$

We put $q'=(q_2,\dots q_d)$ and $a'=(a_2,\dots,a_d)$. Since L is homogeneous, the same is true for the associated measure μ_L

$$\mu_L(\Delta_1((z+\ln(\omega_1))^{-\gamma}a)\setminus V_z)=z^{-\gamma}\mu_L\left(\Delta_1\left(\left(1+\frac{\ln(\omega_1)}{z}\right)^{-\gamma}a\right)\setminus z^{\gamma}V_z\right).$$

We have

$$\begin{aligned}&\Delta_1\left(\left(1+\frac{\ln(\omega_1)}{z}\right)^{-\gamma}a\right)\setminus z^{\gamma}V_z\\&=\Delta_1\left(\left(1+\frac{\ln(\omega_1)}{z}\right)^{-\gamma}a\right)\setminus I\left(0,\left(\left(1+\frac{\ln(\omega_1)}{z}\right)^{-\gamma}a_1,a'\right)\right)\\&\quad\cup\left(I\left(0,\left(1+\frac{\ln(\omega_1)}{z}\right)^{-\gamma}a_1,a'\right)\setminus z^{\gamma}V_z\right).\end{aligned}$$

The density g_L is bounded outside the rectangle $I(0,a)$, hence it is enough to show that the Euclidean volumes of both components are small.

The first one,

$$\Delta_1\left(\left(1+\frac{\ln(\omega_1)}{z}\right)^{-\gamma}a\right)\setminus I\left(0,\left(\left(1+\frac{\ln(\omega_1)}{z}\right)^{-\gamma}a_1,a'\right)\right),$$

is a pyramid of height $(1+\ln(\omega_1)/z)^{-\gamma}-1$ and a base

$$I\left(0,\left(1+\frac{\ln(\omega_1)}{z}\right)^{-\gamma}a'\right)\setminus I(0,a')\,.$$

Therefore, its volume is of order z^{-2}.

The estimation of the volume of the second one is more complicated.

$$\begin{aligned}&\mathrm{Vol}\left(I\left(0,\left(\left(1+\frac{\ln(\omega_1)}{z}\right)^{-\gamma}a_1,a'\right)\right)\setminus z^{\gamma}V_z\right)\\&=\ \mathrm{Vol}\Bigg\{(q):\sum_{i=1}^{d}\omega_1\exp\left(z\left(1-\left(\frac{q_i}{a_i}\right)^{-1/\gamma}\right)\right)\geq 1\,,\\&\quad\frac{q_1}{a_1}\leq\left(1+\frac{\ln(\omega_1)}{z}\right)^{-\gamma},\ \ 0\leq\frac{q_2}{a_2}\leq 1,\dots,0\leq\frac{q_d}{a_d}\leq 1\Bigg\}\end{aligned}$$

$$= a_1 a_2 \cdot \ldots \cdot a_d \cdot \mathrm{Vol}\Bigg\{ (q) : \sum_{i=1}^{d} \omega_1 \exp\left(z\left(1 - q_i^{-1/\gamma}\right)\right) \geq 1\,,$$

$$q_1 \leq \left(1 + \frac{\ln(\omega_1)}{z}\right)^{-\gamma}, 0 \leq q_2 \leq 1, \ldots, 0 \leq q_d \leq 1 \Bigg\}$$

$$= \prod a_i \int_0^1 \ldots \int_0^1 \Bigg(\left(1 + \frac{\ln(\omega_1)}{z}\right)^{-\gamma}$$

$$- \left(1 + \frac{\ln(\omega_1)}{z} - \frac{1}{z} \ln\left(1 - \sum_{i=2}^{d} \omega_i \exp\left(z\left(1 - q_i^{-1/\gamma}\right)\right)\right)\right)^{-\gamma} \Bigg) dq_2 \ldots dq_d\,.$$

The power function $x^{-\gamma}$ is convex, hence we get

$$\mathrm{Vol} \leq \prod a_i \int_0^1 \ldots \int_0^1 (-\gamma) \left(1 + \frac{\ln(\omega_1)}{z}\right)^{-\gamma-1}$$

$$\times \frac{1}{z} \ln\left(1 - \sum_{i=2}^{d} \omega_i \exp\left(z\left(1 - q_i^{-1/\gamma}\right)\right)\right) dq_2 \ldots dq_d$$

$$= \prod a_i \frac{\gamma}{z} \left(1 + \frac{\ln(\omega_1)}{z}\right)^{-\gamma-1}$$

$$\times \int_0^1 \ldots \int_0^1 - \ln\left(1 - \sum_{i=2}^{d} \omega_i \exp\left(z\left(1 - q_i^{-1/\gamma}\right)\right)\right) dq_2 \ldots dq_d\,.$$

The function $-\ln(x)$ is convex too, hence

$$\int_0^1 \ldots \int_0^1 - \ln\left(1 - \sum_{i=2}^{d} \omega_i \exp\left(z\left(1 - q_i^{-1/\gamma}\right)\right)\right) dq_2 \ldots dq_d$$

$$\leq \int_0^1 \ldots \int_0^1 \sum_{i=2}^{d} -\omega_i \ln\left(1 - \exp\left(z\left(1 - q_i^{-1/\gamma}\right)\right)\right) dq_2 \ldots dq_d$$

$$= \sum_{i=2}^{d} \omega_i \int_0^1 - \ln\left(1 - \exp\left(z\left(1 - q_i^{-1/\gamma}\right)\right)\right) dq_i\,.$$

Now it is enough to show that the integral

$$J(z) = \int_0^1 -\ln\left(1 - \exp\left(z\left(1 - x^{-1/\gamma}\right)\right)\right) dx$$

is of order z^{-1}. We change the variable. Let

$$y = -z\left(1 - q_i^{-1/\gamma}\right).$$

We obtain

$$J(z) = \frac{\gamma}{z}\int_0^\infty -\ln(1-e^{-y})\left(1+\frac{y}{z}\right)^{-\gamma-1} dy \le \frac{\gamma}{z}\int_0^\infty -\ln(1-e^{-y})dy\,.$$

Having integrate the last integral by parts we get an integral listed in [10] §534 example 11 which equals $\zeta(2)$ (Riemann zeta function).

In the same way we get the estimates for the other pyramids. We obtain

$$\begin{aligned}\mu_C(V_z) &= \sum_{i=1}^d (z+\ln(\omega_i))^{-\gamma} a_i \frac{\partial L}{\partial q_i}(a) + O(z^{-\gamma-2}) \\ &= z^{-\gamma}\sum_{i=1}^d \left(1 - \gamma\frac{\ln(\omega_i)}{z}\right) a_i \frac{\partial L}{\partial q_i}(a) + O(z^{-\gamma-2})\,.\end{aligned}$$

But L is homogeneous of degree 1, hence

$$L(a) = \sum_{i=1}^d a_i \frac{\partial L}{\partial a_i}(a)\,.$$

Therefore,

$$\mu_C(V_z) = z^{-\gamma} L(a)\left(1 - \frac{\gamma}{z}\sum_{i=1}^d \ln(\omega_i) g_i(a) + O\left(z^{-2}\right)\right),$$

where

$$g_i = \frac{a_i}{L(a)}\;\frac{\partial L}{\partial a_i}(a)\,.$$

Proof of theorem 1.1. Let

$$\alpha = P\left(\frac{W_1}{W_0} \le e^{-z}\right) = \mu_C(V_z)\,.$$

We estimate the dependence of z on α

$$\alpha = z^{-\gamma} L(a) \left(1 - \frac{\gamma}{z} \sum_{i=1}^{d} \ln(\omega_i) g_i + O\left(z^{-2}\right) \right).$$

Therefore,

$$z(\alpha) = \left(\frac{L(a)}{\alpha} \right)^{1/\gamma} - \sum_{i=1}^{d} g_i \ln(\omega_i) + O(\alpha^{1/\gamma}).$$

This finishes the proof

$$\mathrm{VaR}_{1-\alpha} = W_0 - W_0 e^{-z(\alpha)}.$$

5. The asymptotically optimal portfolio

Corollary **5.1.** *There is a unique "asymptotically" optimal portfolio* $\omega = (\omega_1, \ldots, \omega_d)$

$$\omega_i = \frac{a_i}{L(a)} \frac{\partial L}{\partial a_i}(a).$$

Proof. The portfolio ω which minimizes the asymptotic VaR is maximizing the function

$$G(\omega) = \prod_{i=1}^{d} \omega_i^{g_i}, \qquad g_i = \frac{a_i}{L(a)} \frac{\partial L}{\partial a_i}(a).$$

Since ω fulfills the constraint

$$\omega_1 + \ldots + \omega_d = 1,$$

there is a unique maximum at a point $(\omega_1, \ldots, \omega_d)$

$$\omega_i = \frac{g_i}{\sum g_j}.$$

Note that L is homogeneous of degree 1, hence

$$\sum_{j=1}^{d} g_j = 1 \qquad \text{and} \qquad \omega_i = g_i.$$

Furthermore, due to Lemma 2.2 we know that

$$g_i = P(s_j \leq s_i | s_j \leq -z,\ j = 1, \ldots d), \quad \text{for} \quad z > \bar{x},$$

therefore:

Corollary **5.2.** *The part of the capital invested in the i-th asset should be equal to the conditional probability that the drop of the value of the i-th asset will be smaller than of the others under the condition that the value of the all assets will be smaller than c times their initial value, where $c < \exp(-\bar{x})$.*

6. Conclusions

In this paper we deal with the portfolios consisting of several long positions in risky assets. Our aim was to check the impact of the diversification of the portfolio on its *joint* risk. We provide the approximate formula for Value at Risk and determine the optimal portfolio for the case when the tail parts of distributions of logarithmic returns of assets follow the power law of the same degree and the lower tail of associated copula C follows the power law of degree 1 (which we can observe in many empirical examples). Furthermore, we show that the composition of the optimal portfolio can be expressed in terms of the conditional probabilities which are much easier to estimate.

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