

COUNTING STATIONARY POINTS OF RANDOM LANDSCAPES AS A RANDOM MATRIX PROBLEM*

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Finding the mean of the total number N_{tot} of stationary points for N -dimensional random Gaussian landscapes can be reduced to averaging the absolute value of characteristic polynomial of the corresponding Hessian. First such a reduction is illustrated for a class of models describing energy landscapes of elastic manifolds in random environment, and a general method of attacking the problem analytically is suggested. Then the exact solution to the problem [Y.V. Fyodorov, *Phys. Rev. Lett.* **92**, 240601 (2004) and *Phys. Rev. Lett.* **93**, 149901(E) (2004)] for a class of landscapes corresponding to the simplest, yet nontrivial “toy model” with N degrees of freedom is described. For $N \gg 1$ our asymptotic analysis reveals a phase transition at some critical value μ_c of a control parameter μ from a phase with finite landscape complexity: $N_{\text{tot}} \sim e^{N\Sigma}$, $\Sigma(\mu < \mu_c) > 0$ to the phase with vanishing complexity: $\Sigma(\mu > \mu_c) = 0$. This is interpreted as a transition to a glass-like state of the matter.

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Characterising geometry of a complicated landscape is an important problem motivated by numerous applications in physics, image processing and other fields of applied mathematics [2]. The simplest, yet non-trivial task is to find the mean number of all stationary points (minima, maxima and saddles), which is a relevant question in statistical physics of disordered (glassy) systems [3–7], and more recently in string theory [8].

Assuming that the landscape is described by a sufficiently smooth random function \mathcal{H} of N real variables $\mathbf{x} = (x_1, \dots, x_N)$ the problem amounts

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to finding all solutions of the simultaneous stationarity conditions $\partial_k \mathcal{H} = 0$ for all $k = 1, \dots, N$, with ∂_k standing for the partial derivative $\frac{\partial}{\partial x_k}$. The total number $N_{\text{tot}}^{(D)}$ of its stationary points in any spatial domain D is given by $N_{\text{tot}}^{(D)} = \int_D \rho(\mathbf{x}) d^N \mathbf{x}$, with $\rho(\mathbf{x})$ being the corresponding density of the stationary points. The ensemble-averaged value of such a density can be found according to the so-called Kac–Rice formula as

$$\rho_{\text{av}}(\mathbf{x}) = \left\langle \left| \det (\partial_{k_1, k_2}^2 \mathcal{H}) \right| \prod_{k=1}^N \delta(\partial_k \mathcal{H}) \right\rangle, \quad (1)$$

where $\delta(x)$ stands for the Dirac’s δ -function.

Whenever the underlying physical problem necessitates to deal with an absolute value of the determinant, its presence considered to be a serious technical challenge, see [11] and references therein. In particular, an intensive work and controversy persists in calculating the so-called thermodynamic complexity of the free energy for the standard Sherrington–Kirkpatrick model of spin glasses [7] or its generalisations [6]. Several heuristic schemes based on various versions of the replica trick were proposed in the literature recently to deal with the problem, see discussion and further references in [7]. Despite some important insights, the present status of the methods is not yet completely satisfactory. On the other hand, keeping absolute value of the determinant is instrumental for dealing with the systems whose energy function displays many extrema in the configuration space. A famous example is the notoriously difficult case of the random field Ising model, where disregarding the absolute value induces an additional symmetry leading to the so-called “dimensional reduction” prediction [12], which proved to be wrong [13].

In this paper we are specifically addressing the problem of stationary point counting for energy functionals describing an elastic d -dimensional manifold embedded in a random media of $N + d$ dimensions, see Fig. 1. This may serve either as a model of domain walls in the random field Ising model [14], or it is directly related to directed polymers in a random environment.

In a discretized (lattice) version, the simplest form of the energy functional can be written as

$$\mathcal{H} = \frac{\mu}{2} \sum_i \mathbf{X}_i^2 + \frac{t}{2} \sum_{\langle i, j \rangle} (\mathbf{X}_i - \mathbf{X}_j)^2 + \sum_i V_i[\mathbf{X}_i], \quad \mu \geq 0, t \geq 0. \quad (2)$$

Here the index i serves to number points in a d -dimensional lattice approximating the internal space of the elastic manifold, and the N -component

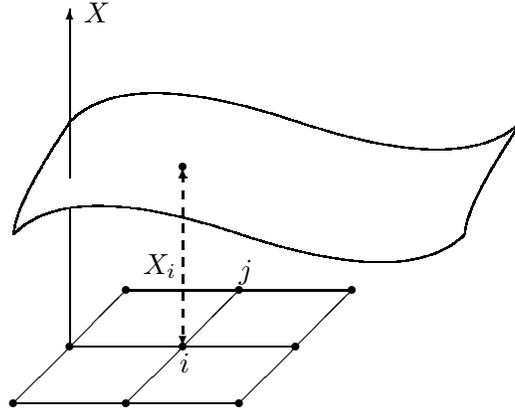


Fig. 1. Sketch of an elastic manifold for dimensions $d = 2$, $N = 1$.

vectors $\mathbf{X}_i = (X_i^{(1)}, \dots, X_i^{(N)})$ parametrise transverse deviations of the manifold from its reference position $\mathbf{X}_i = 0$ at a given site of the lattice. The first term in (2) is a parabolic confining potential serving to stabilise the reference plane $X_i = 0$, second term stands for the elastic energy interaction between the nearest neighbours in the lattice, and the last one is considered to be a random Gaussian function of both i and X , with zero mean and the variance specified by the pair correlation function chosen in the form:

$$\langle V_i[\mathbf{X}_i] V_j[\mathbf{X}_j] \rangle = NV^2 \delta_{ij} f\left(\frac{1}{2N}(\mathbf{X}_i - \mathbf{X}_j)^2\right), \quad (3)$$

where $f(x)$ is any smooth function, with suitable decay at infinity. The stationarity conditions then amount to the set of simultaneous equations:

$$\frac{\partial \mathcal{H}}{\partial X_i^{(l)}} = \mu X_i^{(l)} + t \sum_j (X_i^{(l)} - X_j^{(l)}) + \frac{\partial V_i}{\partial X_i^{(l)}} = 0, \quad l = 1, \dots, N \quad (4)$$

so that the mean total number of the stationary points of \mathcal{H} amounts according to the Kac–Rice formula to

$$\begin{aligned} \langle \mathcal{N}_{\text{tot}} \rangle &= \int_{-\infty}^{\infty} \prod_i \prod_{l=1}^N dX_i^{(l)} \left\langle \prod \delta\left(\frac{\partial \mathcal{H}}{\partial X_i^{(l)}}\right) \right\rangle \\ &\times \left\langle \left| \det \left(\mu \delta_{ij} \delta_{l_1 l_2} + t \delta_{l_1 l_2} \Delta_{ij} + \delta_{ij} \frac{\partial^2 V_i}{\partial X_i^{(l_1)} \partial X_i^{(l_2)}} \right) \right| \right\rangle. \quad (5) \end{aligned}$$

In the above expression we introduced the matrix Δ_{ij} for a discrete Laplacian operator Δ corresponding to the underlying lattice and defined via:

$$\Delta_{mn} = \frac{\partial^2}{\partial X_m \partial X_n} \left[\frac{1}{2} \sum_{\langle i,j \rangle} (X_i - X_j)^2 \right] \quad (6)$$

and used that the first and the second derivatives of $V[\mathbf{X}]$ are (locally) statistically independent due to Gaussian nature of the disorder, so that the two factors in the integrand can be averaged independently, which is an immense simplification. In fact, the averaging of the product of δ -functions in (5) is a rather straightforward task, using independence of the random potential in different lattice sites and its overall Gaussian nature. Moreover, the mean value of the modulus of the determinant is apparently independent of the coordinates $X_i^{(l)}$ which allows to perform the integration explicitly. After necessary manipulations, we arrive at an important intermediate expression

$$\langle \mathcal{N}_{\text{tot}} \rangle = \frac{\left\langle \left| \det \left[\delta_{ij} \left(\mu \delta_{l_1 l_2} + \frac{\partial^2 V_i}{\partial X_i^{(l_1)} \partial X_i^{(l_2)}} \right) + t \delta_{l_1 l_2} \Delta_{ij} \right] \right| \right\rangle}{\det^N (\mu \delta_{ij} + t \Delta_{ij})}. \quad (7)$$

To proceed further one should be able to calculate the ensemble average of the modulus of the characteristic determinant of a lattice Laplacian Δ perturbed by a kind of random potential. This problem in full generality is apparently rather difficult, and we will only discuss here a promising approach inspired by an analogy with the field of the Anderson localisation of a quantum particle by static disorder. Indeed, in the latter field the problem of finding the ensemble average of products of one-particle Green's functions of the lattice operators with disorder was most successfully tackled in the framework of the so-called supersymmetry approach due to Efetov [15], see also a recent review article by Zirnbauer [16]. Following this analogy, we suggest a technique of evaluating the absolute value of the determinants for a real symmetric matrix $\mu + H$ via the following useful identity [1, 17]:

$$|\det(\mu + H)| = \lim_{\varepsilon \rightarrow 0} \frac{[\det(\mu + H)]^2}{\sqrt{\det(\mu + i\varepsilon + H)} \sqrt{\det(\mu + i\varepsilon + H)}}. \quad (8)$$

The two factors in the denominator of the right hand side of (8) can be represented in terms of the Gaussian integrals absolutely convergent as long as $\varepsilon > 0$. Further representing the determinantal factors in the numerator in terms of the Gaussian integral over anticommuting (Grassmann) variables we thus get a *bona fide* supersymmetric [15] object to be analysed. Actual implementation of this method is far from being trivial and we leave it as an

interesting problem for future investigations. Nevertheless, it is important to stress that a possibility to perform the ensemble average explicitly exists whenever matrix entries of H are Gaussian-distributed. Similar strategy may be even employed when H is a stochastic differential operator with certain Gaussian part, as in the notoriously difficult case of the random field Ising model [12]. For this reason it is natural to hope that the suggested method may appear of certain utility for a broad class of disordered models.

In the rest of the present paper we discuss briefly an explicit and rigorous solution [1] of the counting problem for a simple, yet nontrivial limiting case of a “zero-dimensional” manifold ($d = 0$) whose energy functional is of the form

$$\mathcal{H} = \frac{\mu}{2} \sum_{k=1}^N x_k^2 + V(x_1, \dots, x_N) \quad (9)$$

with a random Gaussian-distributed potential $V(\mathbf{x})$ characterised by a particular pair correlation function f as in (3). This expression follows from (2) after neglecting the elastic coupling: $t = 0$, and as such just serves to describe behaviour of a single particle in a random potential. Despite seeming simplicity, the problem of studying this “toy model” [14] dynamics and thermodynamics is still rather non-trivial, with many features typical of complex, glassy systems. In particular, it is known to display a very nontrivial glassy behaviour at low enough temperatures — an unusual off-equilibrium relaxation dynamics attributed to a complex structure of the energy landscape. Although particular dynamical as well as statical properties may differ substantially for different functions $f(x)$ (e.g. “long-ranged” vs “short-range” correlated potentials, see [10]), the very fact of glassy relaxation is common to all of them. In fact, the same model admits an alternative interpretation as a spin-glass, with x_i being looked at as “soft spins” in a quadratic well interacting via a random potential V [9]. From this point of view it is most interesting to concentrate on the limit of large number of “spins” $N \gg 1$. The experience accumulated from working with various types of spin-glass models [6] suggests that for the energy landscape to be complex enough to induce a glassy behaviour the total number of stationary points $N_{\text{tot}}(\mu)$ should grow exponentially with N as $N_{\text{tot}}(\mu) \sim \exp N \Sigma(\mu)$. The quantity $\Sigma(\mu) > 0$ in such a context is natural to call the *landscape complexity*. On the other hand, it is completely clear that the number of stationary points should tend to $N_{\text{tot}} = 1$ for very large μ when the random part is negligible in comparison with the deterministic one. In fact, when $N \rightarrow \infty$ we will find that a kind of sharp transition to the phase with vanishing complexity occurs at some finite critical value μ_c , so that $\Sigma(\mu) = 0$ as long as $\mu > \mu_c$, whereas $\Sigma(\mu) > 0$ for $\mu < \mu_c$ and tends to zero quadratically when $\mu \rightarrow \mu_c$. Such a transition is just the glass transition observed earlier in a framework of a different approach in [9, 10].

For small $N = 1, 2$ statistics of stationary points of (9) were investigated long ago in a classical study of specular light reflection from a random sea surface [3] and addressed several times ever since in various physical contexts, see [4, 5]. Let us demonstrate that the case of arbitrary N can be very efficiently studied by reducing it to a problem typical for the well-developed theory of large random matrices [18]. Indeed, adopting formula (7) to the present case we see that the total number of stationary points in the whole space is given by:

$$N_{\text{tot}}(\mu) = \frac{1}{\mu^N} \langle |\det(\mu - H)| \rangle, \quad (10)$$

where $-\hat{H}$ stands for the matrix of second derivatives of the potential: $H_{k_1 k_2} \equiv \partial_{k_1, k_2}^2 V$. We see that the problem basically amounts to evaluating the ensemble average of the absolute value of the characteristic polynomial $\det(\mu I_N - H)$ (a.k.a. spectral determinant) of a particular random matrix H . Statistical properties of the potential V result in the following second-order moments of the entries H_{ij} $\{(i, j) = 1, \dots, N\}$:

$$\langle H_{il} H_{jm} \rangle = \frac{J^2}{N} [\delta_{ij} \delta_{lm} + \delta_{im} \delta_{lj} + \delta_{il} \delta_{jm}], \quad (11)$$

where we denoted $J^2 = f''(0)$. This allows one to write down the density of the joint probability distribution (JPD) of the matrix H explicitly as

$$\mathcal{P}(H) dH \propto dH \exp \left\{ -\frac{N}{4J^2} \left[\text{Tr}(H^2) - \frac{1}{N+2} (\text{Tr}H)^2 \right] \right\}, \quad (12)$$

where $dH = \prod_{1 \leq i < j \leq N} dH_{ij}$ and the proportionality constant can be easily found from the normalisation condition and will be specified later on. It is evident that such a JPD is invariant with respect to rotations $H \rightarrow O^{-1} H O$ by orthogonal matrices $O \in O(N)$, but it is nevertheless different from the standard one typical for the so-called Gaussian Orthogonal Ensemble (GOE) [18]. However, introducing one extra Gaussian integration it is in fact straightforward to relate averaging over the JPD (12) to that over the standard GOE. In particular,

$$\langle |\det(\mu - H)| \rangle = \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} e^{-N \frac{t^2}{2}} \langle |\det[(\mu + Jt) - H_0]| \rangle_{\text{GOE}}, \quad (13)$$

where the averaging over H_0 is performed with the GOE-type measure: $dH_0 C_N \exp \left\{ -\frac{N}{4J^2} \text{Tr} H_0^2 \right\}$, with $C_N = N^{1/2} / [(2\pi J^2 / N)^{N(N+1)/4} 2^{N/2}]$ being the relevant normalisation constant.

To evaluate the ensemble averaging in (13) in the most economic way one can exploit explicitly the mentioned rotational $O(N)$ -invariance and at the first step in a standard way [18] reduce the ensemble averaging to the integration over eigenvalues $\lambda_1, \dots, \lambda_N$ of the matrix H_0 . After a convenient rescaling $\lambda_i \rightarrow J\sqrt{2/N}\lambda_i$ the resulting expression acquires the following form:

$$\begin{aligned} & \langle |\det [(\mu + Jt) - H_0]| \rangle_{\text{GOE}} \\ & \propto \int_{-\infty}^{\infty} d\lambda_1 \dots \int_{-\infty}^{\infty} d\lambda_N \prod_{i < j}^N |\lambda_i - \lambda_j| \prod_{i=1}^N \left| \sqrt{\frac{N}{2}}(m + t) - \lambda_i \right| e^{-\frac{1}{2}\lambda_i^2}, \end{aligned} \quad (14)$$

where we denoted $m = \mu/J$. One may notice that the above N -fold integral can be further rewritten as a $N + 1$ fold integral:

$$\begin{aligned} & e^{\frac{N}{4}(m+t)^2} \int_{-\infty}^{\infty} d\lambda_1 \dots \int_{-\infty}^{\infty} d\lambda_{N+1} \\ & \times \prod_{i=1}^{N+1} e^{-\frac{1}{2}\lambda_i^2} \delta \left(\sqrt{\frac{N}{2}}(m + t) - \lambda_{N+1} \right) \prod_{i < j}^{N+1} |\lambda_i - \lambda_j|. \end{aligned} \quad (15)$$

Such a representation makes it immediately evident that, in fact, the expectation value of the modulus of the determinant in question is simply proportional to the mean spectral density $\nu_{N+1}[m + t]$ (a.k.a one-point correlation function $R_1^{(N+1)} \left[\sqrt{N/2}(m + t) \right]$, see [18]) of the same GOE matrix H_0 but of enhanced size $(N + 1) \times (N + 1)$:

$$\begin{aligned} & \langle |\det [(\mu + Jt)I_N - H_0]| \rangle_{\text{GOE}} \propto e^{\frac{N}{4}(m+t)^2} \nu_{N+1}[(m + t)], \\ & \nu_N[\lambda] = \frac{1}{N} \langle \text{Tr} \delta(\lambda I_N - H_0) \rangle_{\text{GOE}}. \end{aligned} \quad (16)$$

The last relation provides the complete solution of our original problem for any value of N , since the one-point function $R_1^{(N+1)} [x]$ is known in a closed form [18] for any value of N in terms of the Hermite polynomials $H_k(x)$.

Being interested mainly in extracting the complexity

$$\Sigma(\mu) = \lim_{N \rightarrow \infty} N^{-1} \ln N_{\text{tot}}(\mu),$$

we have to perform an asymptotic analysis of the mean eigenvalue density, and substitute the resulting expression into the integral (13). In the latter we can exploit the saddle-point method for asymptotic analysis. For $0 < m < 1$

the relevant saddle point is $t_s = m$ satisfying $0 < \lambda_s = t_s + m < 2$, and validating the use of the semicircular spectral density $\nu[\lambda_s] = \frac{1}{2\pi} \sqrt{4 - \lambda_s^2}$ in the calculation. This yields

$$\langle |\det(\mu I_N - H)| \rangle \propto e^{\frac{N}{2}(m^2-1)} \sqrt{1-m^2}, \quad 0 < m < 1. \quad (17)$$

For $m > 1$, however, it turns out that one has to use exponentially small (“instanton”) value for the spectral density:

$$\nu[\lambda] \propto \exp \left\{ -N \left[\frac{1}{4} \lambda \sqrt{\lambda^2 - 4} - \ln \frac{\lambda + \sqrt{\lambda^2 - 4}}{2} \right] \right\}, \quad \lambda > 2, \quad (18)$$

where we only kept factors relevant for calculating the complexity in the limit of large N . The corresponding saddle-point value t_s in the t -integral is given by the solution of the equation $m = \frac{1}{2}(\lambda_s + \sqrt{\lambda_s^2 - 4})$ for the variable $\lambda_s = t_s + m$. The solution is easily found to be simply $\lambda_s = m + m^{-1}$ (note that $\lambda_s > 2$ ensuring consistency of the procedure) which yields the resulting value for the modulus of the determinant to be given by

$$\langle |\det(\mu I_N - H)| \rangle \propto e^{N \ln m}, \quad m > 1. \quad (19)$$

Invoking our basic relation Eq. (10) for N_{tot} we see that the landscape complexity $\Sigma(\mu)$ of the random potential function (9) is given by:

$$\Sigma(\mu) = \frac{1}{2} \left(\frac{\mu^2}{J^2} - 1 \right) - \ln \left(\frac{\mu}{J} \right), \quad \mu < \mu_c = J, \quad (20)$$

$$\Sigma(\mu) = 0, \quad \mu > \mu_c = J. \quad (21)$$

Earlier works referred to the critical value $\mu_c = J$ as, on one hand, signalling the onset of a nontrivial glassy dynamics [10], and, on the other hand, corresponding to the point of a breakdown of the replica-symmetric solution [9]. Our calculation provides an independent support to the point of view attributing both phenomena to extensive number of stationary points in the energy landscape. At the critical value the complexity vanishes quadratically: $\Sigma(\mu \rightarrow \mu_c) \propto (\mu_c - \mu)^2 / \mu_c^2$.

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