WEALTH CONDENSATION AND "CORRUPTION" IN A TOY MODEL*

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(Received August 22, 2005)

We discuss the wealth condensation mechanism in a simple toy economy in which individual agent's wealths are distributed according to a Pareto power law and the overall wealth is fixed. The observed behaviour is the manifestation of a transition which occurs in Zero Range Processes (ZRPs) or "balls in boxes" models. An amusing feature of the transition in this context is that the condensation can be induced by *increasing* the exponent in the power law, which one might have naively assumed penalised greater wealths more.

PACS numbers: 02.50.–r, 05.70.Fh

* Presented at the Conference on Applications of Random Matrices to Economy and Other Complex Systems, Kraków, Poland, May 25–28, 2005.

1. Introduction

The rich really are different, as was first remarked by Pareto in the 1890s [1]. He was interested in quantifying the distribution of elites of various sorts and found numerous examples of observables such as "wealth" (in some suitable measure such as income) following a power law distribution at the top end of the spectrum. In contrast the bulk of the income distribution in most societies follows a log-normal distribution, which is often called Gibrat's law.

More explicitly the distribution of large wealths may be written as

$$p(w) \sim w^{-1-\alpha} \quad \text{for } w \gg w_0 \,. \tag{1}$$

with α typically between 1–2. This distribution is referred to as Pareto's distribution and the exponent α as the Pareto index. The wealth distribution of lesser mortals can be fitted by

$$p(w) = \frac{1}{w\sqrt{2\pi\sigma^2}} \exp\left[-\frac{\log^2(w/w_0)}{2\sigma^2}\right], \tag{2}$$

where $\beta = 1/\sqrt{2\sigma^2}$ is called the Gibrat [2] index.

Thinking as physicists it is natural to ask what kinds of processes might lead to the observed distributions. For a single pot of money, say $w(t)$, a multiplicative stochastic process $w(t+1) = a(t)w(t)$ leads directly to a log-normal distribution. Iterating $w(t+1) = a(t) \cdot a(t-1) \cdots w(0)$ and taking a log gives:

$$\log w(t+1) = \log a(t) + \log a(t-1) + \cdots + \log w(0) \,. \tag{3}$$

Applying the central limit theorem then gives us the log-normal distribution for w.

Getting a power law is a little harder, but Levy and Solomon [3] observed that putting in a "poverty bound" w_{m}

$$\langle \log a(t) \rangle < 0, \quad 0 < w_{\mathrm{m}} < w(t) \tag{4}$$

meant that the drift to $\log w(t \to \infty) \to -\infty$ is balanced by the reflection on the reflecting barrier located at $0 < w_{\mathrm{m}}$ and gives a power-law distribution. Alternatively one can include additive noise, $w(t+1) = a(t)w(t) + b(t)$. These give Pareto indices determined by $\alpha = 1/(1 - w_{\mathrm{m}})$ and $\langle a^{\alpha} \rangle = 1$, respectively.

One can attempt to extend this approach by considering N pots of money in interaction with one another, as was done by Solomon *et al.* who used a generalised Lotka–Volterra process [4]

$$
\begin{aligned}
w_i(t+1) - w_i(t) &= \left[\varepsilon_i(t)\sigma_i + c_i(w_1, w_2, \cdots, w_N, t)\right] w_i(t) \\
&+ a_i \sum_j b_j w_j(t)
\end{aligned}
\tag{5}
$$

and Bouchaud and Mézard [5] who constructed a flow-like model

$$
\frac{dW_i(t)}{dt} = \eta_i(t) W_i(t) + \sum_{j(\neq i)} J_{ij}(t) W_j(t) - \sum_{j(\neq i)} J_{ji}(t) W_i(t)\,. \tag{6}
$$

This incorporates interactions between different agents with individual wealths $W_i(t)$ as well as multiplicative noise $\eta_i(t)$. The model is amenable to a mean-field treatment in which all the $J_{ij}(t)$ are equal and time-independent and similarly for the $\eta_i(t)$. In this case the probability of a normalised wealth $w_i = W_i/\overline{W}$, with $\overline{W} = N^{-1} \sum_i W_i$, is given by the Pareto-like distribution

$$
p(w) \sim \frac{\exp - \frac{\alpha - 1}{w}}{w^{1+\alpha}}\,, \tag{7}
$$

where $\alpha = 1 + J/\sigma^2 > 1$ and σ^2 is the variance of the Gaussian distribution of η. This is clearly greater than one for the variant of the model described above, but it may be adjusted by introducing various plausibly motivated additional terms. With a little artistic licence one might call the case $\alpha < 1$ a liberal economy and $\alpha > 1$ a social economy.

Introducing the inverse participation ratio $Y_2 = \sum_{i=1}^N w_i^2$ reveals very different behaviour depending on whether α is greater or less than one. Y_2 acts as an order parameter which distinguishes between cases where wealth is evenly distributed and cases where it is concentrated. In the evenly distributed case we would expect $w_i \sim 1/N$ then $Y_2 \sim 1/N$ so $Y_2 \to 0$ as $N \to \infty$. However, for an unequal distribution one (or more) w_i is extensive and $Y_2 \neq 0$ as $N \to \infty$. In the Bouchaud and Mézard model one finds $\langle Y_2 \rangle = 1 - \alpha$ for $\alpha < 1$.

If one calculates the average of the distribution (7), which corresponds to the average wealth of the individual, one sees that the basic difference between a social and a liberal economy is that it is finite in the former case and infinite in the latter. Thus, for $\alpha \leq 1$ one would, due to the non-integrable tail of the distribution, expect the appearance of rich individuals in the ensemble, with a wealth $N^{1/\alpha}$ times larger than the typical value. The authors of [5] interpreted this result as a condensation phenomenon.

2. The toy model

In the Bouchaud and Mézard model it is the non-integrable tail of the distribution which is in effect driving the condensation for $\alpha < 1$. It is interesting to ask what might happen in the situation where the total wealth of the economy W is fixed. It is not clear *a priori* whether condensation would continue to take place.

We address this question here by assuming that $p(w) \sim 1/w^{1+\alpha}$ characterises the single wealth-distributions in the ensemble with the individual wealths adding to $W = w_1 + \ldots + w_N$. For convenience, we assume that each individual wealth w_i is an integer given in units of the smallest available currency unit. The joint probability distribution of w_i's is thus:

$$P(w_1, ., w_N) \;=\; \frac{1}{Z(W,N)} \prod_i p(w_i)\, \delta\left(W - \sum_{i=1}^{N} w_i\right) , \tag{8}$$

where $Z(W,N)$ is the appropriate normalisation factor,

$$Z(W,N) \;=\; \sum_{\{w_i \geq 0\}} \prod_i p(w_i)\, \delta\left(W - \sum_{i=1}^{N} w_i\right) . \tag{9}$$

The "partition function" $Z(W,N)$ would of course be a trivial product of independent factors if it were not for the overall constraint.

This model is known variously as the balls-in-boxes or backgammon model [6, 7] where it has been applied to various condensation and glassy phenomena. The picture is of W balls distributed between N boxes with a prior probability of $p(w)$ balls in a box. $Z(W,N)$ also appears as the steady state solution to the Zero Range Process (ZRP) [8] which has been extensively studied in the context of non-equilibrium dynamics because of its relationship to the Asymmetric Exclusion Process (ASEP) [9]. This is a dynamical model which is defined by the hopping rates $u(w)$ for w balls in a box. In steady state these are related to the $p(w)$ by

$$u(w) = \frac{p(w-1)}{p(w)} . \tag{10}$$

One could think of tailoring a suitable $p(w)$ in a stationary distribution from a particular choice of rates. An appropriate choice for a power law $p(w)$ would be $u(w) \sim 1 + b/w$.

The model can be solved in the limit of an infinite number of boxes N and fixed density of balls per box $\rho = W/N$ (thermodynamical limit) by introducing the integral representation of the delta function

$$
\begin{aligned}
Z(N,\rho) = & \sum_{\{w_i \geq 0\}} \prod_i p(w_i) \\
& \times \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\lambda e^{-i\lambda(w_1+\cdots+w_N-\rho N)} \\
= & \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\lambda e^{i\lambda\rho N} \left(\sum_w p(w) e^{-i\lambda w} \right)^N \\
= & \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\lambda \exp\left(N(i\lambda\rho + K(i\lambda)\right), \qquad (11)
\end{aligned}
$$

where K is a generating function given by $K(\sigma) = \ln \sum_{w=1}^{\infty} p(w) e^{-\sigma w}$. Evaluating the integral using steepest descent gives

$$f(\rho) = \sigma_*(\rho)\rho + K(\sigma_*(\rho)) \ , \qquad (12)$$

where $\sigma_*(\rho)$ is a solution of the saddle point equation $\rho + K'(\sigma_*) = 0$ and $f(\rho)$ is a free energy density per box, $Z(W,N) = e^{Nf(\rho)+\cdots}$. For a suitable choice of the weights $p(w) \sim 1/w^{1+\alpha}$ the system displays a two phase structure as the density is varied with a critical density ρ_{cr}. When ρ approaches ρ_{cr} from below, σ_* approaches σ_{cr} from above. When ρ is larger than ρ_{cr} , σ becomes equal to the critical value σ_{cr} and the free energy is linear in ρ

$$f(\rho) = \sigma_{\mathrm{cr}}\rho + \kappa_{\mathrm{cr}} \ , \qquad (13)$$

where $\kappa_{\mathrm{cr}} = K(\sigma_{\mathrm{cr}})$. The change of regimes at ρ_{cr} corresponds to a condensation transition, in which an extensive fraction of the balls is in a single box. The critical value σ_{cr} is equal to the logarithm of the radius of convergence of the series in the generating function $K(\sigma)$. In particular, for purely power-like weights

$$p(w) = \frac{1}{\zeta(1+\alpha)} w^{-1-\alpha} \ , \quad w = 1, 2, \ldots, \qquad (14)$$

$\sigma_{\mathrm{cr}} = 0$. The normalisation factor is given by the Riemann Zeta function.

The transition to a condensed phase happens when W/N becomes larger than a critical density w_*, which is nothing but the mean wealth

$$w_* = \sum_w w\, p(w) \, . \qquad (15)$$

Since we can change the small w part of the distribution by tuning the appropriate macro-economical parameters without affecting the large w behaviour of $p(w)$, we have some control over where the threshold w_* will lie. We can define an *effective* probability distribution of wealth:

$$\widehat{p}(w) = \frac{1}{N} \left\langle \sum_i^N \delta(w_i - w) \right\rangle_P \tag{16}$$

which now, unlike the original $p(w)$, takes into account the finite total wealth W. Below threshold w_*, the system is in a phase in which the effective probability distribution $\widehat{p}(w)$ has an additional scale factor in comparison with the old distribution $p(w)$

$$\widehat{p}(w) \sim e^{-\sigma w} p(w)\,. \tag{17}$$

Here, σ depends only on the difference $W/N - w_*$. It vanishes at threshold, so that the old Pareto tails are restored at this point. Above threshold, the macro-economy responds to the increasing average wealth by creating a single individual with a wealth proportional to the total wealth W, namely $w_{\max} = W - Nw_*$

$$\widehat{p}(w) \sim p(w) + \frac{1}{N} \delta_{w, W-Nw_*}\,. \tag{18}$$

The behaviour of $\hat{p}(w)$ versus w is shown in Fig. 1, for index $\alpha = 3$, $N = 128, 512, 2048$ and a density $W/N > w_*$. At threshold the inverse

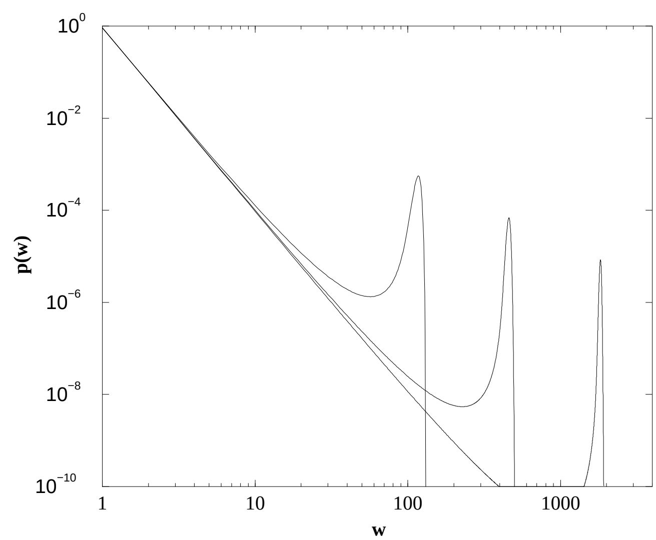

Fig. 1. Effective probability density of wealth. $\alpha = 3$, $N = 128, 512, 2048$ and $W/N > w_*$.

participation ratio

$$Y_2 = \frac{1}{N^2} \left\langle \sum_i w_i^2 \right\rangle_P = \frac{1}{N} \sum_w w^2 \widehat{p}(w) \ , \tag{19}$$

changes, in the large N limit, from 0 to $(W/N - w_*)^2$, signalling the appearance of a wealth condensation. Everything in excess of the critical wealth Nw_* ends up in the portfolio of a single individual. It can appear only in a social economy ($\alpha > 1$), because only in this case do we have a finite critical wealth per individual w_*. In a liberal economy, w_* is obviously infinite, meaning that the system remains always below threshold and there is never any condensation.

One interesting feature of the current model is that *increasing* the power α when the density of wealth is subcritical in a social economy can have the effect of pushing the model into the condensed phase, as can be deduced from plotting the critical density curve $\zeta(\alpha)/\zeta(1+\alpha)$, which is monotone decreasing, as seen in Fig. 2. If we are sitting below the line at some subcritical wealth density and increase α at constant density we can end up moving into the condensed phase, which is presumably not the intended effect of any increase in α, since rather than penalising greater wealths it initiates condensation.

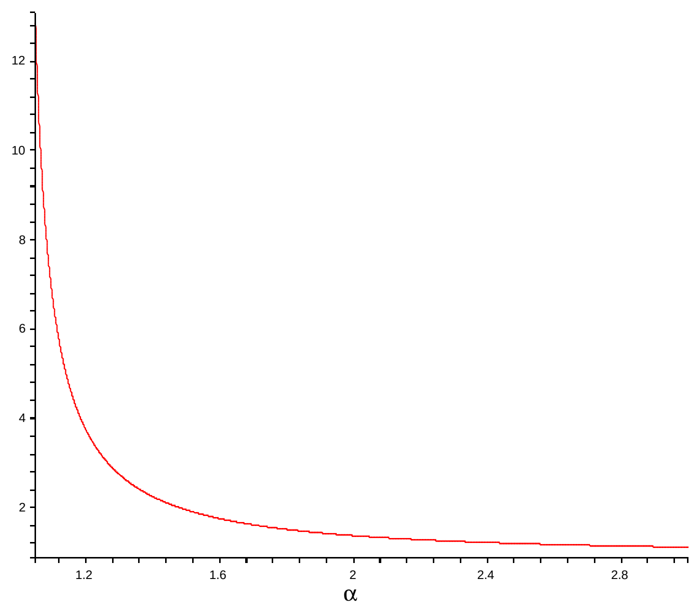

Fig. 2. The critical density curve $\zeta(\alpha)/\zeta(1+\alpha)$.

3. Summary

We have shown that in a simple model of a a "social" economy, condensation may occur if the total wealth of the society exceeds a certain critical value. In our analysis, the system favours the occurrence of a single individual in possession of a finite fraction of the economy's total available wealth, providing a physical mechanism for "corruption". The simple model discussed in this note may be extended to open systems such an economy in interaction with one or more others [10] using similar methods.

We have made no attempt to discuss the dynamics of condensation. This has been investigated in the case of the ZRP [11] for both mean field geometry (which would be the appropriate framework for the model discussed here) and other geometries. Transcribing these results leads one to expect that the wealth condensation would proceed via a concentration of wealth amongst a gradually decreasing number of individuals until one dominated. The timescale for this would go as $\tau \sim N^2$.

It should also be mentioned in closing that there has been a large body of work on more or less realistic agent-based models of economic interactions which include various interactions [12, 13]. It seems that the appearance of power-law distributions for (extreme) wealth may be rather generic in such models. This suggests that although the specific model discussed here is perhaps best thought of as an amusing application of the Balls in Boxes Model/Zero Range Process in an econophysics context by virtue of its naivety, the study of the emergence and behaviour of power law distributions of wealth with more sophisticated models is a worthwhile exercise.

The work of GB was partially supported by the grant OTKA T047050, of IZ by the US-DOE grants DE-FG02-88ER40388 and DE-FG03-97ER4014, and of ZB, JJ and MAN by the Polish State Committee for Scientific Research (KBN) grant 2P03B-08225 (2003-2006), Marie Curie Host Fellowship HPMD-CT-2001-00108 and by EU IST Center of Excellence "COPIRA".

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