STATISTICAL MECHANICS OF MULTI-ANTENNA COMMUNICATIONS: REPLICAS AND CORRELATIONS*

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(Received July 1, 2005)

The use of multi-antenna arrays has been predicted to provide substantial throughput gains for wireless communication systems. However, these predictions have to be assessed in realistic situations, such as correlated channels and in the presence of interference. In this review, we show results obtained using methods borrowed from statistical physics of random media for the average and the variance of the distribution of the mutual information of multi-antenna systems with arbitrary correlations and interferers. Even though the methods are asymptotic in the sense they are valid in the limit of large antenna numbers, the results are accurate even for small arrays. We also optimize over the input signal covariance with channel covariance feedback and calculate closed-loop capacities. This method provides a simple tool to analyze the statistics of throughput for arrays of any size.

PACS numbers: 02.50.–r, 42.25.Dd, 84.40.Ua, 89.70.+c

1. Introduction

In recent years the use of multi-antenna arrays in both transmission and reception has attracted considerable interest. For sufficiently rich scattering environments the Shannon capacity of an n_T-element transmitting and an n_R-element receiving array is roughly proportional to $\min(n_R, n_T)$ for large numbers of antennas [1, 2]. To assess the capacity gains using such MIMO (Multiple Input Multiple Output) technologies in realistic situations, several factors have to be considered. First, the degree of spatial correlations

* Presented at the Conference on Applications of Random Matrices to Economy and Other Complex Systems, Kraków, Poland, May 25–28, 2005.

between antennas at both transmitter and receiver determines the effective number of independent channels. Second, the amount of channel information at both receiver and transmitter also affects the maximum throughput. For example, for interference-limited systems, the accuracy of *instantaneous* channel information of the interference at the receiver will determine how effective the interference suppression will be. Similarly, increased channel information at the transmitter can in some cases substantially increase throughput. Realistically, full channel information at the transmitter is impractical. Instead, partial or statistical channel information, such as the channel covariance may be fed back more efficiently.

For large antenna numbers the analysis of MIMO ergodic capacities (expectation value of mutual information over channel realizations) is greatly facilitated by the use of asymptotic techniques of random matrix theory (RMT) techniques. These methods were introduced in this context by various authors, starting with Foschini [1] and Telatar [2].

The infinite antenna open-loop capacity expression was first derived in the context of CDMA codes in [3] and more recently it was applied to the context of multi-antenna systems [4]. Using methods developed in [5], the infinite antenna capacity with uncorrelated channels and uncorrelated interferers was calculated by [6]. Very recently, the results of [3] were extended by [7] to calculate the open-loop capacity of spatially correlated channels in the infinite antenna limit. In all of these previous studies, the scaling of the capacity with the number of antennas was studied as the number of antennas grows indefinitely. However, for realistic finite antenna systems this scaling may not be adequate to describe the antenna array capacity. We note that more recently, some of these results were obtained in closed form using character expansions [8, 9] and other methods [10, 11]. However, we will not deal with these works here, since the methods discussed here are more general and the results simpler.

In this paper, extending the work of [12] we analyze the distribution of the MIMO capacity and present analytic expressions for the mean and the variance of the capacity distribution in the presence of correlated channels and interferers. Specifically, we find that in the formal limit of large antenna numbers, the first and second moments of the capacity distribution are of order $O(n)$ and $O(1)$, where n is the number of antennas (with higher moments being of $O(1/n)$ [13]). As a result, the capacity distribution tends to a Gaussian for large n. The mathematical framework and the derivation of these expressions is presented elsewhere [13]. Here we verify numerically that the analytic expressions derived in [13] are quite accurate even for just a few (*i.e.* 2–3) antennas in each array and that the capacity distribution is very well represented by our expressions for its first and second moments (*i.e.*, that the distribution is Gaussian). In addition, for large antenna numbers we develop a method to analytically optimize over the input signal covariance subject to the channel covariance. This method enables us to derive analytic expressions for the closed loop capacity, when partial knowledge of the channel is available at the transmitter (covariance feedback) and gives extremely accurate results for all multi-antenna systems, as discussed also in [14]. The analytic methods presented here provide a powerful tool for analyzing array systems with even few antennas. It should be noted that very recently, similar results were verified [11, 15, 16], but for uncorrelated channels and no interference. Also, [17] recently obtained asymptotic results of the average mutual information of a finite antenna MIMO system with a certain type of correlated channels characterized by a single parameter, the richness of the channel.

In the next section we define the problem and introduce relevant quantities. In Section 3 we present results for the average mutual information. We also show how to optimize over the signal covariance to obtain the closed loop capacity with covariance feedback. In Section 4 we focus on two specific examples: First, we analyze a MIMO system with a multi-antenna interferer whose channel is known at the receiver — a case relevant to multi-user detection. For this case we assume all channels are i.i.d. for simplicity. Second, we consider a MIMO system with correlated transmitting antennas. For both cases, we present results for the mean and the variance of the mutual information distribution and we compare them to simulations.

2. Definitions

We consider the case of single-user transmission from n_T transmit antennas to n_R receive antennas over a narrow band fading channel. We also include a number n_I of interfering transmitters in addition to the noise at each receiver. y_α, the components of the n_R-dimensional received complex signal vector $\boldsymbol{y}$ can be written as

$$y_\alpha = \sqrt{\frac{\rho_S}{n_T}} \sum_{a=1}^{n_T} \boldsymbol{G}^S_{\alpha a} x^S_a + \sqrt{\frac{\rho_i}{n_I}} \sum_{a=1}^{n_I} \boldsymbol{G}^{i}_{\alpha a} x^I_a + z_\alpha \,, \tag{2.1}$$

where $\alpha = 1 \ldots n_R$ and $\boldsymbol{G}^S$ is a complex matrix with the channel coefficients from the transmitting to the receiving arrays. Similarly, $\boldsymbol{G}^{i}$ describes the channel from the interfering antennas to the receiver array. x^S_a and x^I_a are the transmitted and interfering signals, assumed to be both Gaussian. The signal covariance $\boldsymbol{Q}$ with entries $Q_{ab} = E[x^S_a x^{S*}_b]$, is normalized so $\mathrm{Tr}\,\{\boldsymbol{Q}\} = n_T$. The Gaussian noise vector z_α, and the interfering signals x^I_a are assumed to be i.i.d., *i.e.*, $E[z_\alpha z^*_\beta] = \delta_{\alpha\beta}$ and $E[x^I_a x^{I*}_b] = \delta_{ab}$. Here, ρ_S and ρ_{i} are the average signal-to-noise and interference-to-noise ratios. The signal-to-interference-and-noise ratio can also be written as $\mathrm{SINR} = \rho_S/(1+\rho_{i})$.

The associated mutual information can be expressed as [6] $I = I_S - I_I$, where I_S, I_I are given by

$$I_S = \log\det\left(\mathbf{1} + \frac{\rho_i}{n_i}\boldsymbol{G}^i\boldsymbol{G}^{i\dagger} + \frac{\rho_S}{n_T}\boldsymbol{G}^S\boldsymbol{Q}\boldsymbol{G}^{S\dagger}\right),$$
$$I_I = \log\det\left(\mathbf{1} + \frac{\rho_i}{n_i}\boldsymbol{G}^i\boldsymbol{G}^{i\dagger}\right), \tag{2.2}$$

where $\mathbf{1}$ is the unit matrix. The log above (and throughout the whole paper) represents the natural logarithm and thus I, I_S and I_I are all expressed in nats/sec/Hz. It should be noted that this equation for the mutual information can be generalized to wideband multipath channels.

Due to the underlying randomness of $\boldsymbol{G}^S$ and $\boldsymbol{G}^i$, the mutual information I is also a random quantity. To analyze its statistics we assume that $\boldsymbol{G}^S$, $\boldsymbol{G}^i$ are zero mean Gaussian matrices. The covariance for $\boldsymbol{G}^S$ is defined to be

$$\langle G^S_{\alpha a} {G^S_{\beta b}}^* \rangle = T_{ab}\, R_{\alpha\beta}\,. \tag{2.3}$$

The notation $\langle\cdot\rangle$ denotes the ensemble average over channel realizations. $\boldsymbol{R}$ is the $n_R \times n_R$ correlation matrix of the incoming signal at the receiver, respectively, normalized so that $\mathrm{Tr}\,\{\boldsymbol{R}\} = n_R$. $\boldsymbol{T}$ is the $n_T \times n_T$ correlation matrix describing the antenna correlations at the transmitter array with $\mathrm{Tr}\,\{\boldsymbol{T}\} = n_T$. The dependence of these matrices on array properties, polarization and channel properties as angle spread have been described in various references [7, 12, 18]. One might consider similar nontrivial correlations in the covariance of $\boldsymbol{G}^i$. However, for simplicity, in this paper we restrict our attention to the case of $\boldsymbol{G}^i$ being i.i.d. (Generalizations that consider more complicated statistics of $\boldsymbol{G}^i$ cases can also be treated using similar methods [13]).

In Eq. (2.2) it is assumed that the receiver knows the channel matrices $\boldsymbol{G}^S$, $\boldsymbol{G}^i$. The difference between the interference $\left[\boldsymbol{G}^i\boldsymbol{G}^{i\dagger}\right]$ and the noise term (the "$\mathbf{1}$" term inside the log det term) is that in the former term the receiver has detailed knowledge of the interfering channel, while in the latter no such knowledge is available.

The purpose of this paper is to analyze the statistics of the mutual information I in (2.2) for Gaussian channels having correlations given by (2.3). This is done by calculating the moments of the mutual information. We thus introduce the generating function $g(\nu)$ of I

$$g(\nu) = \left\langle \frac{\left[\det\left(\boldsymbol{N} + \frac{\rho_{\mathrm{i}}}{n_I}\boldsymbol{G}^{\mathrm{i}}\boldsymbol{G}^{\mathrm{i}\dagger} + \frac{\rho_S}{n_T}\boldsymbol{G}^S\boldsymbol{Q}\boldsymbol{G}^{S\dagger}\right)\right]^{-\nu}}{\left[\det\left(\boldsymbol{N} + \frac{\rho_{\mathrm{i}}}{n_I}\boldsymbol{G}^{\mathrm{i}}\boldsymbol{G}^{\mathrm{i}\dagger}\right)\right]^{-\nu}} \right\rangle = \left\langle e^{-\nu I} \right\rangle \tag{2.4}$$

$$= 1 - \nu\langle I\rangle + \frac{\nu^2}{2}\langle I^2\rangle + \dots .$$

Assuming that $g(\nu)$ is analytic at least in the vicinity of $\nu = 0$, we can express $\log g(\nu)$ as follows

$$\log g(\nu) = -\nu \langle I\rangle + \sum_{p=2}^{\infty} \frac{(-\nu)^p}{p!}\mathcal{C}_p\,, \tag{2.5}$$

where $\mathcal{C}_p$ is the p-th cumulant moment of I. For example, $\mathcal{C}_2 = \mathrm{Var}(I) = \left\langle (I - \langle I\rangle)^2 \right\rangle$ is the variance and $\mathcal{C}_3 = \mathrm{Sk}(I) = \left\langle (I - \langle I\rangle)^3 \right\rangle$ is the skewness of the distribution. Thus to obtain the moments of the mutual information distribution we need to calculate $g(\nu)$ for ν in the vicinity of $\nu = 0$. This is not necessarily any easier than evaluating the moments $\mathcal{C}_p$ directly, which is a notoriously difficult task, since one has to average products of logarithms of random quantities. In contrast, averaging $g(\nu)$ for integer values of ν involves averages over integer powers of determinants of random quantities, in which case some analytic progress can be made. Once this is done, we will invoke the assumption that we can analytically continue the generating function to $\nu \to 0$. This is the standard replica trick used commonly in statistical mechanics. From the form of (2.4), we see that to properly take into account the different sign of the exponent of the determinants, we need to introduce both complex (bosonic) and Grassman (fermionic) variables [13].

3. Ergodic capacity

In this section we analyze the average mutual information $\langle I\rangle_{\boldsymbol{Q}}$ where the average is taken for fixed $\boldsymbol{Q}$. We will use the general analytical expression derived in [13], which is valid for large antenna numbers. Two ergodic capacities are considered. For the first case no channel information is available at the transmitter. Here the optimal signal covariance $\boldsymbol{Q}$ is unity — there are no preferred antennas or directions at the transmitter. Thus the open loop ergodic capacity is simply $C_{\mathrm{OL}} = \langle I\rangle_{\boldsymbol{Q}=\mathbf{1}} = \langle I\rangle$, where the subscript $\boldsymbol{Q} = \mathbf{1}$ will be dropped for simplicity. In contrast, when channel information is available at the transmitter in the form of channel statistics (*e.g.* $\boldsymbol{R}$, $\boldsymbol{T}$), the optimal $\boldsymbol{Q}$ that maximizes $\langle I\rangle_{\boldsymbol{Q}}$ may in general be non-trivial. Thus the closed loop ergodic capacity will be $C_{\mathrm{CL}} = \max_{\boldsymbol{Q},\mathrm{Tr}\boldsymbol{Q}=n_T}\langle I\rangle_{\boldsymbol{Q}}$. Note that maximization over $\boldsymbol{Q}$ is performed *after* the average, since only partial (statistical) information is available at the transmitter. This is a realistic model for incorporating closed loop feedback, since full and timely channel knowledge at the transmitter is not possible without overloading the reverse link.

3.1. Average mutual information $\langle I \rangle_{\boldsymbol{Q}}$

The average mutual information $\langle I \rangle_{\boldsymbol{Q}}$ in the limit of large antenna numbers is derived in [13] and can be expressed as $\langle I \rangle_{\boldsymbol{Q}} = \langle I_S \rangle_{\boldsymbol{Q}} - \langle I_I \rangle_{\boldsymbol{Q}}$ where the subscript $\boldsymbol{Q}$ will subsequently be dropped for simplicity. Here $\langle I_S \rangle$ is given by

$$\begin{aligned} \langle I_S \rangle \;=\; & \log\det\left(\mathbf{1} + t_S\, \boldsymbol{QT}\right) - t_S r_S n_T \\ & + \log\det\left(\mathbf{1} + \rho_S\, r_S\, \boldsymbol{R} + \rho_i\, r_i\right) \\ & + n_I \log\left(1 + t_i\right) - t_i r_i n_I\,, \end{aligned} \tag{3.1}$$

where the variables r_S, r_i, t_S, and t_i are given by the following system of four equations in four unknowns

$$r_S \;=\; \frac{1}{n_T} \sum_{a=1}^{n_T} \frac{[\boldsymbol{TQ}]_a}{1 + t_S\; [\boldsymbol{TQ}]_a}\,, \tag{3.2}$$

$$t_S \;=\; \frac{1}{n_T} \sum_{\alpha=1}^{n_R} \frac{\rho_S\, R_\alpha}{1 + \rho_S R_\alpha r_S + \rho_i r_i}\,, \tag{3.3}$$

$$r_i \;=\; \left(1 + t_i\right)^{-1}\,, \tag{3.4}$$

$$t_i \;=\; \frac{1}{n_I} \sum_{\alpha=1}^{n_R} \frac{\rho_i}{1 + \rho_S R_\alpha r_S + \rho_i r_i}\,. \tag{3.5}$$

Note that Eqs. (3.2), (3.3), and (3.5) are written in terms of the eigenvalues of $[\boldsymbol{TQ}]$ and $\boldsymbol{R}$, denoted here by $[\boldsymbol{TQ}]_a$ and R_α, respectively. In practice, at most two of the above equations need to be solved numerically.

Similarly $\langle I_I \rangle$ is given by

$$\langle I_I \rangle = n_I \left(\log u - 1 + u^{-1}\right) + n_R \log\left(1 + u^{-1} \rho_i\right) \tag{3.6}$$

with

$$u = \frac{1}{2}\left[1 + \rho_i y + \sqrt{\left(1 + \rho_i y\right)^2 + 4\rho_i}\;\right] \tag{3.7}$$

with $y = n_R/n_I - 1$. As expected from the form of I_I in Eq. (2.2), this result is the open-loop capacity in the large-antenna limit for the $n_R \times n_I$ random i.i.d. matrix G^i [3,13].

As shown in [13] one can also calculate the variance, and higher moments of the distribution of I using methods that are exact in the limit of large antenna number. We note again that successively higher moments of the distribution decrease implying that the distribution hardens to a Gaussian. Although the general expressions for the variance are somewhat complicated, results will be shown below for the two simple cases discussed below. As mentioned above, by making the assumption that the distribution is close to a Gaussian we will be able to make very good approximations of the full distribution of capacity.

3.2. Closed loop capacity: optimizing the signal covariance

As discussed in the beginning of this section, when the transmitter has partial channel information in the form of correlations $\boldsymbol{T}$, the signal covariance $\boldsymbol{Q}$ can be optimized to maximize the average mutual information $\max_{\boldsymbol{Q}}\langle I\rangle_{\boldsymbol{Q}}$. This method was introduced for MIMO systems without interference in [12] and is also discussed in [19], and is discussed in further detail in [13]. It can easily be shown that the $\boldsymbol{Q}$ maximizing $\langle I\rangle_{\boldsymbol{Q}}$ is simultaneously diagonalizable with $\boldsymbol{T}$. This result has been shown [20] to hold in general in the absence of interference for arbitrary antenna numbers (but can be generalized to hold in the presence of interference). It can then be shown [13] that in the large antenna number limit, the optimal eigenvalues q_a of $\boldsymbol{Q}$ (with $a = 1 \ldots n_T$) are given by

$$q_a = \left[\frac{1}{\Lambda} - \frac{1}{T_a t_S}\right]_+ , \tag{3.8}$$

where $[x]_+ = x\Theta(x)$ and T_a is the corresponding eigenvalue of $\boldsymbol{T}$, with $\Theta(x)$ the Heaviside Θ-function. Λ is determined by imposing the power constraint

$$\mathrm{Tr}\,\{\boldsymbol{Q}\} = \sum_{a=1}^{n_T} q_a = n_T\,. \tag{3.9}$$

Essentially, the optimization of $\boldsymbol{Q}$ amounts to waterfilling over the modes of $\boldsymbol{T}$ rather than the instantaneous channel $\boldsymbol{G}^{S\dagger}\boldsymbol{G}^S$ itself. Thus, the statistically waterfilled, closed-loop capacity is found by solving for five unknowns $(t_S, r_S, t_i, r_i, \Lambda)$ from Eqs. (3.2)–(3.5), (3.8), and (3.9) and using them to evaluate Eq. (3.1). Nevertheless, again at most two equations need to be solved numerically. It should be noted that in a network setting this type optimization of $\boldsymbol{Q}$ need not be the optimum for the whole network [21].

4. Applications

We now apply the above capacity equations of the previous section to two representative situations. In addition to calculations of the ergodic capacity, for the current applications we show how to calculate the variance of the capacity distribution. Generally, while the average mutual information is $O(n)$, where n is the order of the number of antennas in each array, the variance is $O(1)$. In [13] we show that the skewness is $O(1/n)$. This suggests that for large antenna numbers the capacity distribution can be described accurately by only its mean and its variance and thus that it approaches a Gaussian. Below, we will make the same assumption for finite antenna arrays to approximate the full capacity distribution.

4.1. Capacity distribution in the presence of interferers whose channel is known at the receiver

In this section we consider the case of a MIMO system in the presence of interfering transmitters whose channel $\boldsymbol{G}^i$ is known at the receiver. For simplicity we assume that all channel coefficients ($\boldsymbol{G}^S$ and $\boldsymbol{G}^i$) are i.i.d.

4.1.1. Ergodic capacity

We have $\langle I\rangle = \langle I_S\rangle - \langle I_I\rangle$, where $\langle I_I\rangle$ is given by Eq. (3.6) and $\langle I_S\rangle$ becomes:

$$\begin{aligned} \langle I_S\rangle &= n_T\left(r_S - \log r_S - 1\right) + n_I\left(r_i - \log r_i - 1\right) \\ &\quad + n_R \log\left[1 + \rho_i\, r_i + \rho_S\, r_S\right], \end{aligned} \tag{4.1}$$

where r_S and r_i are given by

$$\frac{n_R}{1 + \rho_i r_i + \rho_S r_S} = \frac{n_I(1 - r_i)}{r_i \rho_i} = \frac{n_T(1 - r_S)}{r_S \rho_S} \tag{4.2}$$

which can be reduced to a simple cubic equation.

4.1.2. Variance

The variance can be written [13] in a straight forward manner in terms of r_S, r_i and u of Eq. (3.7) as

$$\begin{aligned} \mathrm{Var}(I) &= -\log\left|1 - \frac{n_T}{n_R}\left(1 - r_S\right)^2 - \frac{n_I}{n_R}\left(1 - r_i\right)^2\right| \\ &\quad - \log\left|1 - \frac{n_R}{n_I}\left(\frac{\rho_i}{u + \rho_i}\right)^2\right| + 2\log\left|1 + \frac{\left(1 - r_i\right)\rho_i}{u + \rho_i}\right| . \end{aligned} \tag{4.3}$$

To compare the above equations with simulations for simplicity we set $n_T = n_R = n_I = n$ and consider $n = 2, 3$. Fig. 1 displays the cumulative distribution of I for these two systems, for signal to interference and noise ratio SINR $= \rho_S/(1 + \rho_i) = 10$ and interference to noise ratio $\rho_i = 1$. The analytic results are simply Gaussian distributions with the calculated mean and variance. Also displayed are the simulated distributions for the same cases. The agreement between analytic and simulated distributions is decent for two antennas and impressive for three (less than 1% difference). This analysis suggests that for a few number of antennas our expressions for the mutual information average and its variance are sufficient to describe the full distribution.

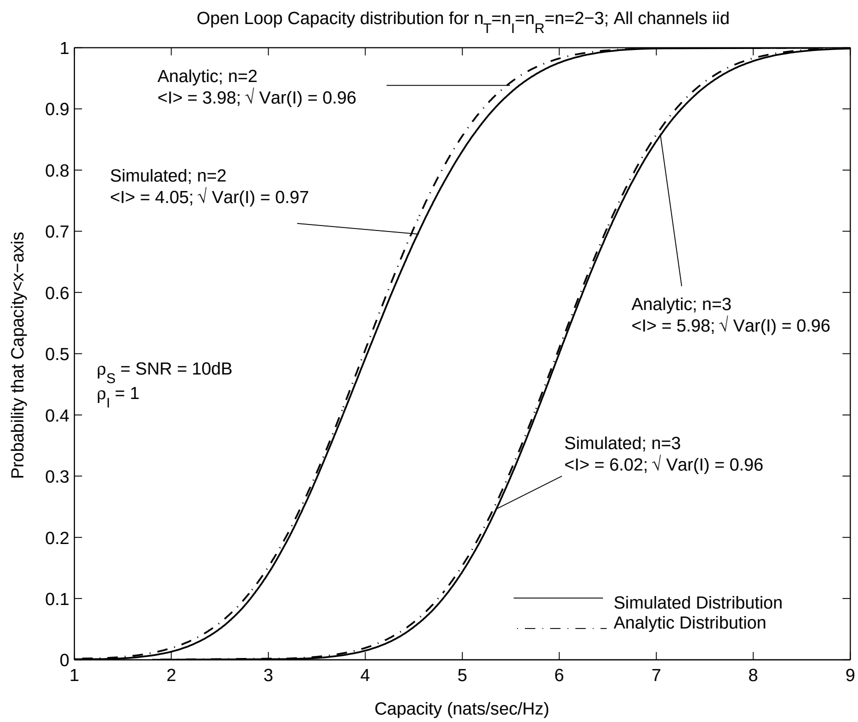

Fig. 1. Cumulative distribution (CDF) of the mutual information for n-transmitter, n-receiver and n-interferer (for $n = 2, 3$) with i.i.d. signal and interfering channels ($\boldsymbol{R} = \boldsymbol{T} = \mathbf{1}$). For both $n = 2$ and $n = 3$ we plot the CDF of the simulated mutual information (solid lines) and a Gaussian distribution with average and variance analytically calculated from Eq. (3.6), (4.1) and Eq. (4.3). We also display the average $\langle I \rangle$ and the square root of the variance $\mathrm{Var}(I)$ of the simulated curves. The error bar in the simulated values of the variance is approximately 2%. The analytical results agree quite well with simulations.

4.2. Closed loop vs. open loop capacity for spatially correlated transmission

We now remove the interfering transmitters by setting $\rho_i = 0$. However, here we will consider a correlated channel at the transmitter. For concreteness, we assume that the antennas form a uniform linear ideal antenna array with $d_\lambda = d_{\min}/\lambda$ the nearest neighbor antenna spacing in wavelengths and we assume a Gaussian power azimuth spectrum (with 2 dimensional propagation), *i.e.* the average incoming power at the antenna array is $P(\theta) \propto \exp[-(\theta/\delta)^2/2]$, [18] where δ is the angle spread measured from the vertical to the array. This results in a $\boldsymbol{T}$ matrix given by

$$T_{ab} = \int_{-180}^{180} \frac{d\phi}{\sqrt{2\pi\delta^2}} e^{2\pi i(a-b)d_\lambda \sin(\phi\pi/180)-\phi^2/(2\delta^2)} \tag{4.4}$$

with $a, b = 1 \ldots n_T$ being the index of transmitting antennas. For simplicity we assume no correlations at the receiver ($\boldsymbol{R} = \mathbf{1}_{n_R}$). This situation corresponds to a receiving mobile array deep inside the clutter and a transmitting base-station array with correlations due to finite angle spread.

4.2.1. Ergodic capacity

In this case we have

$$\langle I \rangle = n_R \log\left(1 + \rho_S r_S\right) - n_T\, t_S\, r_S + \sum_{a=1}^{n_T} \log\left(1 + t_S q_a T_a\right) \,, \tag{4.5}$$

where r_S is calculated using

$$\frac{n_R r_S \rho_S}{1 + r_S \rho_S} = \sum_{i=a}^{n_T} \left[1 - \frac{r_S}{T_a}\right]_+ \tag{4.6}$$

and then t_S, q_a are determined from

$$t_S \;=\; n_R \rho_S \left[n_T \left(1 + \rho_S r_S\right)\right]^{-1} \,, \tag{4.7}$$

$$q_a \;=\; \left[\frac{1}{t_S r_S} - \frac{1}{t_S T_a}\right]_+ \,. \tag{4.8}$$

Interestingly, applying these equations to find when beamforming is optimal (when only one q_a is nonzero) produces extremely accurate results when compared with the exact criterion for optimality of beamforming [14].

For the open loop case, $q_a \equiv 1$, we solve for t_S using

$$\frac{t_S}{\rho_S} = \frac{n_R}{n_T} - 1 + \frac{1}{n_T} \sum_{a=1}^{n_T} \frac{1}{1 + t_S T_a} \tag{4.9}$$

and substitute in Eq. (4.7) to get r_S. We now wish to quantify the throughput gain due to channel feedback to the transmitter in the form of $\boldsymbol{T}$. In Fig. 2 the dependence of the open loop and closed loop capacity is plotted as

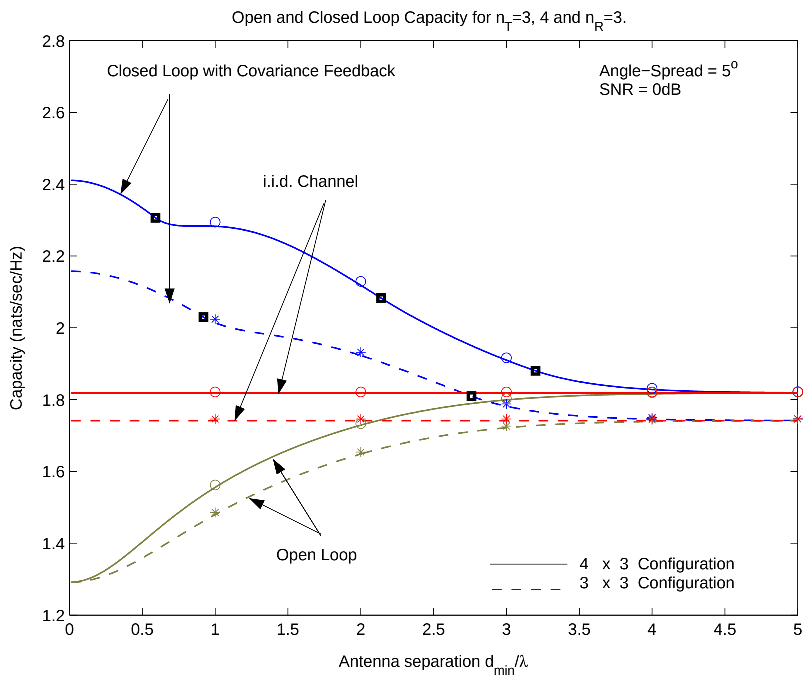

Fig. 2. Plot of capacity per receiving antenna for 2 configurations, namely a 3-element array receiving from a 3-element transmitting array (3×3) and a 4-element transmitting array (4×3), respectively. In all cases the receiving array has no antenna correlations ($\boldsymbol{R} = \mathbf{1}$). The transmitting array is a uniformly spaced array with neighboring antennas spaced by $d_{\min}$. The spatial correlations between antennas are calculated using Eq. (4.4) with $\delta = 5°$. The two lower curves indicated as "Open Loop" depict the open loop capacity per receiving antenna for these two configurations. At small antenna separation the capacities approach each other. The top two curves indicated as "Closed Loop ..." represent the closed loop capacities with covariance feedback: the transmitter adjusts to the *typical* channel. The squares represent the values of antenna separation at which an additional nonzero transmitting mode (q_a in Eq. (4.7)) turns on. Note that these points coincide to apparent kinks in the capacity. The horizontal straight lines are the capacities per antenna in the absence of any antenna correlations ($\boldsymbol{T} = \boldsymbol{R} = \mathbf{1}$). The open circles, and asterisks represent the values of the simulated mean capacities of the 4×3 and 3×3 configurations correspondingly. The agreement with analytical results is very good.

a function of antenna spacing for two transmitting array configurations. The angle-spread is set to $\delta = 5°$. It is interesting to point out that for the low SNR and angle-spread used, the closed loop capacity is substantially higher than the open loop capacity and in fact it is higher than the open loop capacity of uncorrelated channels. Furthermore, we note that the closed loop capacity is not convex and, especially for small arrays, has apparent discontinuities in its slope. This effect is due to the fact that at certain points (marked with squares in the figure) it is optimal for additional transmission modes q_a to become non-zero (see Eq. (4.7)). This non-convex and even non-monotonic behavior of the closed loop capacity as a function of correlations has been shown to be present in exact solutions of multi-antenna arrays with few antennas in [14]. Finally the analytic solutions presented here are in very good agreement with simulated ergodic capacities (depicted with open circles and diamonds in the figure).

4.2.2. Variance

Using t_S, r_S and q_a the variance $\mathrm{Var}(I)$ can be expressed [13] as

$$\mathrm{Var}(I) = -\log|1 - M_t M_r| \,, \tag{4.10}$$

where

$$M_r = \left(\frac{\rho_S n_R}{n_T}\right)(1 + r_S\rho_S)^{-2} \tag{4.11}$$

and

$$M_t = \left(\frac{\rho_S}{n_T}\right)\mathrm{Tr}\left\{(\boldsymbol{QT})^{\mathbf{2}}(\mathbf{1} + t_S(\boldsymbol{QT}))^{\mathbf{-2}}\right\} \,. \tag{4.12}$$

Again, we model the capacity distribution as a Gaussian with the appropriately calculated mean (from Section 4.2 above) and variance. Results are shown in Fig. 3 for antenna spacing $d_\lambda = 1$. We also compare the closed loop capacity with covariance feedback with the full closed loop capacity, in which case the instantaneous channel matrix is known at the transmitter. We see that the mean throughput of the two closed loop schemes are within 5% of each other despite the low signal to noise ratio.

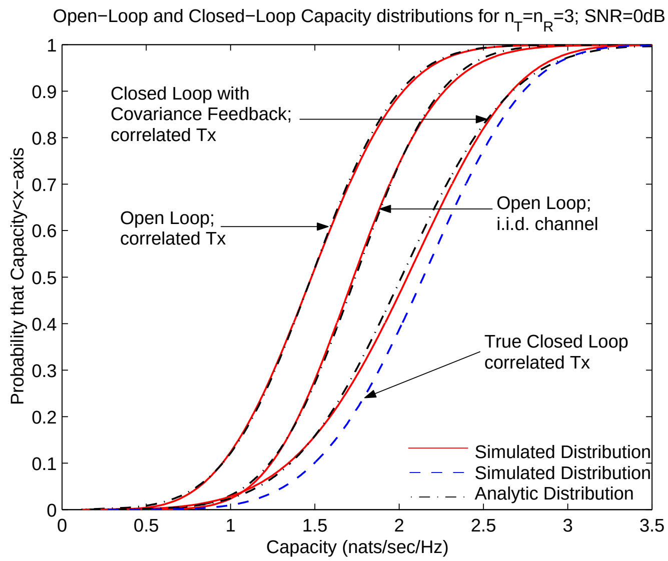

Fig. 3. Cumulative distribution (CDF) of mutual information for 3 antenna transmitting and receiving arrays in the absence of interferers ($\rho_i = 0$). Each closely spaced pair of curves corresponds to a simulated mutual information distribution (solid curves) and a normal distribution (dash-dotted curves) with mean and variance calculated as described in Section 4.2. The agreement between analytic estimate and simulated distributions is impressive. This is especially so given that $n_T = n_R = 3$ while the analytic method described in this paper is formally valid only for large antenna numbers. The three pairs of curves plotted correspond to the three corresponding curves plotted in Fig. 2 with $n_T = n_R = 3$, and SNR $= 0dB$, $d_\lambda = 1$ and $\delta = 5°$. The open loop curve assumes no feedback to the transmitter, while the closed loop curve takes into account covariance feedback. The dashed curve represents the full closed loop capacity distribution, where the full channel matrix is fed back to the transmitter. The mean throughput of the two closed loop schemes are within 5% of each other.

5. Conclusion

In conclusion, we have presented a model to calculate the mean and the variance of the mutual information of MIMO systems in the presence of spatially correlated channels, and with interference known at the receiver. In addition, we have used this method to optimize with respect to the signal covariance $\boldsymbol{Q}$ and thus analytically calculate the closed loop ergodic capacity covariance feedback is available at the transmitter.

We have demonstrated the applicability of this model for arrays with few antennas by comparing (a) the calculated ergodic mutual information as a function of antenna separation to their corresponding simulated averages

(Fig. 2) and (b) the simulated mutual information distribution to a Gaussian distribution obtained by using the calculated mean and variance (Figs. 1, 3). In both cases the agreement is remarkable. This model provides a simple tool to accurately analyze the statistics of throughput of even small arrays in the presence of arbitrary channel correlations, as well as interferers with known channel at the receiver [13].

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